Between Wqo and Bqo, the Space of Ideals of a Wqo

Yann Pequignot

Joint work with Raphaël Carroy

University of Lausanne

Paris VII

Winter School in Abstract Analysis
Section Set Theory and Topology
Hejnice, Czech Republic
January 26 – February 2, 2013

Quasiorder

A **quasiorder** (qo) is a set Q together with a *reflexive* and *transitive* binary relation $\leq$. We write $p < q$ for $p \leq q$ and $p \not\geq q$.

1. Q is **well founded** if it admits no infinite $<$-descending chain;
2. $A \subseteq Q$ is an **antichain** if $p \neq q \rightarrow p \not\leq q$ for all $p, q \in A$;
3. a sequence $(q_n)_{n\in\omega}$ is called
   - **good** if $\exists m, n \in \omega$ with $m < n$ and $q_m \leq q_n$;
   - **perfect** if $\forall m, n \in \omega \ m \leq n$ implies $q_m \leq q_n$;
4. $D \subseteq Q$ is a **downset** if $q \in D$ and $p \leq q$ implies $p \in D$. We write Down(Q) po of downsets of Q with inclusion.
5. for $S \subseteq Q$ we write $\downarrow S = \{p \in Q \mid \exists q \in S \ p \leq q\}$ for the **downward closure** of S.
6. give **upset** and **downward closure** the dual meanings.

Quasiorder

A **well quasiorder** (wqo) is a qo that satisfies one of the following equivalent conditions.

1. Q is well founded and has no infinite antichain;
2. every sequence is good;
3. every sequence admits a perfect subsequence;
4. every upset U admits a finite $F \subseteq Q$ such that $U = \uparrow F$;
5. $(\mathrm{Down}(Q), \subseteq)$ is well founded.

The main tool to show the equivalence is the classical:

Theorem (Ramsey)

Let $k \in \omega$ and let $[\omega]^k = P_0 \cup P_1$ be a partition of the set of sets of natural numbers with cardinality k. There exists an infinite $M \subseteq \omega$ such that

$$\textit{either} \quad [M]^k \subseteq P_0, \quad \textit{or } [M]^k \subseteq P_1.$$

Closure properties of wqo

Here are examples of wqo's:

- Finite qo's ;
- well ordered set, ordinals;
- any subset of a wqo;
- any quotient of a wqo;
- *finite* products of wqo's;
- *finite* unions of wqo's;

For s and t ordinal sequences in Q we define

$$s \leq_{dom} t \quad \text{iff} \quad \begin{array}{c}\text{there exists a strictly increasing map} \\ h : |s| \to |t| \text{ s.t. } s_i \leq t_{h(i)} \text{ for all } i \in |s|\end{array}$$

Theorem

If Q wqo then the qo $(Q^{<\omega}, \leq_{dom})$ of finite sequences in Q is wqo.

Wqo's are stable under finite combination. But if Q is wqo

- Is Q^ω wqo?
- And $(\mathrm{Down}(Q), \subseteq)$?
- and $Q^{\mathbf{ON}}$?

Wqo? Well, we want more

Remember Q is wqo iff $(\mathrm{Down}(Q), \subseteq)$ is well founded.

Question:

What is a witness in Q that $\mathrm{Down}(Q)$ is **not** wqo?

- Let $(D_n)_{n\in\omega}$ is a bad (=not good) sequence in $(\mathrm{Down}(Q), \subseteq)$.
- For all $m, n \in \omega$ and all $m < n$: $D_m \not\subseteq D_n$.
- For all $m \in \omega$ build a sequence $(q_{\{m,n\}})_{m<n}$ by choosing

$$q_{\{m,n\}} \in D_m \quad \text{and} \quad q_{\{m,n\}} \notin D_n.$$

- The sequence of sequences $(q_{\{m,n\}})_{m<n}$ satisfies

$$q_{\{m,n\}} \not\leq q_{\{n,l\}} \quad \text{for all } m < n < l.$$

otherwise $q_{\{m,n\}} \leq q_{\{n,l\}} \in D_n$ implies $q_{\{m,n\}} \in D_n$.

Wqo? Well, we want more

Question:

What does ensure inside Q that $\mathrm{Down}(Q)$ is wqo?

Answer:

Consider sequences of *higher dimension*.

- A **sequence of sequences** is a map $f : [\omega]^2 \to Q$ from the pairs in ω.
- Say a sequence of sequences $f : [\omega]^2 \to Q$ is **good** if

$$\text{there exists } m < n < l \text{ s.t. } f(\{m, n\}) \leq f(\{n, l\}).$$

Recall that: every sequence in Q is good $\leftrightarrow$ Q is wqo.

Proposition

Let Q be a qo. Every sequence of sequences in Q is good $\leftrightarrow$ $\mathrm{Down}(Q)$ is wqo.

Richard Rado's Example

Question

Does there exist a wqo Q such that $\mathrm{Down}(Q)$ is not wqo?

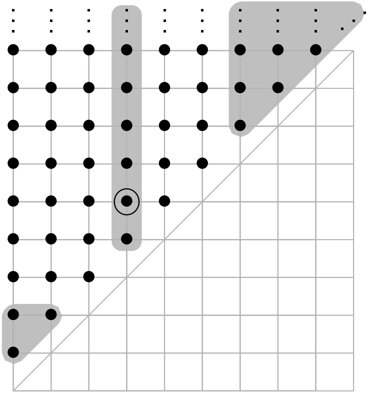
Yes, there is!
Richard Rado
1954

Let $R = ([\omega]^2, \sqsubseteq)$ with

$$\{m, n\} \sqsubseteq \{k, l\}$$

iff

$$\begin{cases} m = k \text{ and } n \leq l, & \text{or} \\ m < n < k < l \end{cases}$$



Wqo? Well, we want better

We want to define a class of quasiorders such that

- Q is wqo
- $\text{Down}(Q)$ is wqo
- $\text{Down}(\text{Down}(Q))$ is wqo
- $\text{Down}^k(Q)$ is wqo
- $\text{Down}^\omega(Q)$ is wqo
- $\text{Down}^\alpha(Q)$ is wqo

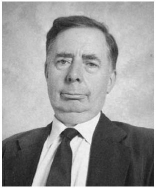
This is done by requiring that

- every sequence is good
- every sequence of sequences is good
- every sequence of sequences of sequences is good
- every sequence of sequences of sequences of sequences... is good
- every ????? is good

We need a transfinite notion of sequence of sequences... supersequences.

Wqo? Well, we want better

Crispin St J. Nash-Williams:
There is a generalisation
of the classical Ramsey theorem
to the transfinite dimension!



A **barrier** is a family B of finite sets of natural numbers such that

1. $\bigcup B$ is infinite;
2. for all $s, t \in B$, $s \subseteq t$ implies $s = t$;
3. every infinite subset of $\bigcup B$ admits an initial segment in B.

Theorem (Nash-Williams,1965)

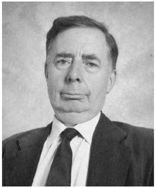
Let B be a barrier and let $B = P_0 \cup P_1$ be a partition of B. Then there exists an infinite $M \subseteq \bigcup B$ such that

$$\text{either} \quad B|M \subseteq P_0, \quad \text{or } B|M \subseteq P_1.$$

where $B|M = \{s \in B \mid s \subset M\}$.

Wqo? Well, we want better

Crispin St J. Nash-Williams:
Wqo? Well, we want better!



For finite set of natural numbers s and t let

$$s \lhd t \quad \text{iff} \quad \begin{array}{c}\text{there exists } u \text{ s.t.}\\ s \sqsubset u \text{ and } t = u \setminus \min u\end{array}$$

- A **supersequence** in Q is a map $f : B \to Q$ from a barrier B.
- A supersequence $f : B \to Q$ is **good** if there is $s, t \in B$ with $s \lhd t$ and $f(s) \leq f(t)$.

Definition (Nash-Williams, 1965)

A qo Q is a **better quasiorder** (bqo) if every supersequence in Q is good.

Cauchy sequences and uniform continuity

Fact

Let $(x_n)_{n\in\omega}$ be a sequence in 2^ω. The following conditions are equivalent:

1. $(x_n)_{n\in\omega}$ *is Cauchy;*
2. $\{n \in \omega \mid x_n \in C\}$ *is finite or cofinite for all clopen C of 2^ω;*
3. *the map $f : [\omega]^1 \to 2^\omega$, $n \mapsto x_n$ is uniformly continuous.*

Where the barrier $[\omega]^1 = \{\{n\} \mid n \in \omega\}$ is equipped with the uniform structure (metric) inherited by 2^ω via the identification:

$$\begin{aligned}[\omega]^{<\infty} &\longrightarrow 2^\omega \\ s = \{2,4,5\} &\longmapsto x_s = 001011000\cdots\end{aligned}$$

Cauchy sequences and uniform continuity

Definition

Let $f : B \to X$ be a supersequence.

- A **sub-supersequence** of f is a restriction of f to some barrier $B' \subseteq B$.

Remark: sub-supersequences of f are exactly the $f : B|N \to X$ for an infinite $N \subseteq \bigcup B$. Recall $B|N = \{s \in B \mid s \subset N\}$.

Definition

For a metric space X, say a supersequence $f : B \to X$ is **Cauchy** if it is uniformly continuous when B is equipped with the uniform structure (metric) induced by 2^ω.

Every sequence in 2^ω has a Cauchy (convergent) subsequence and

Theorem (Carroy R. and P.)

Every supersequence in 2^ω (i.e. in any 0-dim compact Polish space) has a Cauchy sub-supersequence.

Cauchy sequences and uniform continuity

A Cauchy $f : [\omega]^1 \to 2^\omega$ converges and thus extends uniquely to a continuous map

$$\begin{aligned} \overline{f} : \overline{[\omega]^1} &\longrightarrow 2^\omega \\ \{n\} &\longmapsto f(\{n\}) \\ \emptyset = 0^\omega &\longmapsto \lim_n f(\{n\}). \end{aligned}$$

Similarly if $f : B \to 2^\omega$ is Cauchy (i.e. uniformly continuous) then it extends uniquely to a continuous

$$\overline{f} : \overline{B} \longrightarrow 2^\omega$$

Example

The closure of $[\omega]^2 = [\omega]^{\leq 2}$. A sequence of sequences $f : [\omega]^2 \to X$ in a complete metric space X is Cauchy iff

- for each n we have $f(\{n, m\})_{n<m} \to \overline{f}(\{n\})$, and
- $\overline{f}(\{n\})_{n\in\omega} \to f(\emptyset)$.

The space of ideals of a wqo

A *non empty* subset I of a qo Q is an **ideal** if

- I is a downset;
- I is directed, i.e. for all $p, q \in I$ there is $r \in I$ with $p \leq r$ and $q \leq r$.

Let $\mathrm{Idl}(Q)$ be the po of ideals of Q under inclusion.

Let 2^Q be the generalised Cantor space of subsets of Q.

Any qo Q is naturally mapped into 2^Q via

$$Q \longrightarrow 2^Q$$
$$q \mapsto \downarrow q.$$

We identify Q (the po quotient of Q) with its image in 2^Q.

Proposition (M. Pouzet and N. Sauer, 2005)

If Q is wqo then the closure of Q in 2^Q equals $I(Q)$.

- *$I(Q)$ is compact;*
- *$I(Q)$ is scattered;*
- *the set of isolated points of $I(Q)$ equals Q.*

Cauchy supersequence in a wqo

Theorem (Carroy R. and P.)

Every supersequence in 2^ω (i.e. in any 0-dim compact Polish space) has a Cauchy sub-supersequence.

makes essential use of the metrisability of 2^ω. However, since

Proposition

Let Q be wqo and $S \subseteq \mathrm{Idl}(Q)$ be countable. Then $\overline{S}$ is countable and metrisable.

the theorem applies to supersequences in a wqo:

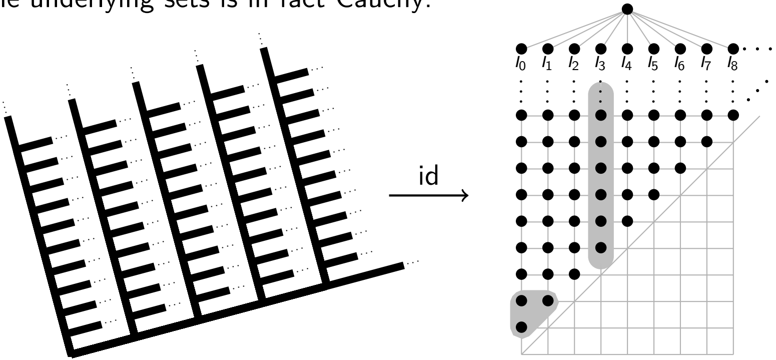
Corollary (Carroy R. and P.)

Every supersequence g in a wqo Q has a Cauchy sub-supersequence $f : B \to Q$. This Cauchy supersequence extends to a continuous

$$\overline{f} : \overline{B} \to \mathrm{Idl}(Q).$$

Back to Rado's example

The bad sequence of sequences in $\mathcal{R}$ given by the identity map on the underlying sets is in fact Cauchy:



The barrier $[\omega]^2$

Rado's poset

Its continuous extension restricts to a bad sequence in the non principal ideals:

$$\begin{aligned} [\omega]^1 &\longrightarrow \mathrm{Idl}^*(Q) \\ \{n\} &\longmapsto I_n \end{aligned}$$

Continuous extensions of supersequences

A point x in a topological space $\mathcal{X}$ is **isolated** if $\{x\}$ is open.
A non isolated point is said to be **limit**.

If $x_n \to x$ in a topological space $\mathcal{X}$ there is $M \in [\omega]^\infty$ such that

either x is isolated and for all $m \in M$ $x_m = x$;

or x is limit and $\begin{cases} \text{either} & x_m \text{ is isolated for all } m \in M; \\ \text{or} & x_m \text{ is limit for all } m \in M. \end{cases}$

For a continuous extension $\overline{f} : \overline{B} \to \mathcal{X}$
of a supersequence $f : B \to \mathcal{X}$ let $\Lambda_{\overline{f}} = \{s \in \overline{B} \mid \overline{f}(s) \text{ is limit}\}$.

Theorem (Carroy R. and P.)

Let $\overline{f} : \overline{B} \to \mathcal{X}$ be a continuous extension of a supersequence f in a topological space $\mathcal{X}$. Then there exists a sub-supersequence $g : B' \to \mathcal{X}$ of f s.t.

either $\Lambda_{\overline{g}}$ *is empty;*

or $\Lambda_{\overline{g}} = \overline{C}$ *for some barrier C.*

A new proof of a result on bqo

Let $\mathsf{Idl}^*(Q)$ denote the po of non principal ideals of Q under inclusion.
We have $\mathsf{Idl}(Q) = \mathsf{Idl}^*(Q) \cup Q$.

Theorem (M. Pouzet and N. Sauer, 2005)

Let Q be wqo. If $\mathsf{Idl}^*(Q)$ is bqo, then Q is bqo.

We can give a new topological proof of this result.

The space of ideals of a wqo

Last ingredient for the proof

Let $(E_n)_n$ be a sequence in 2^Q.

$$\bigcap_{n\in\omega} E_n \subseteq \bigcup_{i\in\omega} \bigcap_{j\geq i} E_j \subseteq \bigcap_{i\in\omega} \bigcup_{j\geq i} E_j \subseteq \bigcup_{n\in\omega} E_n.$$

Recall : $E_n \to E$ in 2^Q iff $\bigcup_{i\in\omega} \bigcap_{j\geq i} E_j = \bigcap_{i\in\omega} \bigcup_{j\geq i} E_j = E$

The following trick we took in a proof by R. Rado (1954).

Lemma (Rado's trick)

Let Q be wqo. For all sequence $(D_n)_{n\in\omega}$ of downsets of Q there exists $M \in [\omega]^\infty$ s.t.

$$\bigcup_{i\in N} \bigcap_{j\in N/i} I_j = \bigcup_m I_m.$$

Corollary

Let $(I_n)_{n\in\omega}$ be a sequence in $\mathrm{Idl}(Q)$. Then there exists an infinite $N \subseteq \omega$ such that $(D_n)_{n\in N}$ converges to $\bigcup_{n\in N} I_n$ in 2^Q.

A new proof of a result on bqo

Theorem (M. Pouzet and N. Sauer, 2005)

Let Q be wqo. If $\mathrm{Idl}^*(Q)$ *is bqo, then Q is bqo.*

Sketch of our proof.

- Let $f : B \to Q$ be a supersequence (to see: f is good);
- Go to a Cauchy sub-supersequence $g : B' \to Q$;
- Extend it continuously to $\overline{g} : \overline{B'} \to \mathrm{Idl}(Q)$;
- Go to a sub-supersequence indexed by B'' s.t.

$$\Lambda = \left\{ s \in \overline{B''} \mid f(s) \in \mathrm{Idl}^*(Q) \right\} = \begin{cases} \text{is either empty, or} \\ \overline{C} \text{ for some barrier } C. \end{cases}$$

- three cases:

$\Lambda = \emptyset$	Then f has a constant sub-supersequence.
$\Lambda = \overline{C} = \{\emptyset\}$	Q wqo $\Rightarrow f$ is good.
$\Lambda = \overline{C}$ is non trivial	$\mathrm{Idl}^*(Q)$ bqo $\Rightarrow f$ is good.

□