Better-quasi-order: ideals and spaces

Ainsi de suite...

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Ph.D. Thesis

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To Lolita

Bien que les pieds de l'homme
n'occupent qu'un petit coin de la terre,
c'est par tout l'espace qu'il n'occupe pas
que l'homme peut marcher sur la terre immense.

Bien que l'intelligence de l'homme
ne pénètre qu'une parcelle de la vérité totale,
c'est par ce qu'elle ne pénètre pas
que l'homme peut comprendre ce qu'est le ciel.

— *Tchouang-tseu*

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Contents

1 Introduction

Mathematicians have imagined a myriad of objects, most of them infinite, and inevitably followed by an infinite suite.

What does it mean to understand them? How does a mathematician venture to make sense of these infinities he has imagined?

Perhaps, one attempt could be to organise them, to arrange them, to order them. At first, the mathematician can try to achieve this in a relative sense by comparing the objects according to some idea of complexity; this object should be above that other one, those two should be side by side, etc. So the graph theorist may consider the *minor relation* between graphs, the recursion theorist may study the *Turing reducibility* between sets of natural numbers, the descriptive set theorist can observe subsets of the Baire space through the lens of the *Wadge reducibility* or equivalence relations through the prism of the *Borel reducibility*, or the set theorist can organise ultrafilters according to the *Rudin-Keisler ordering*.

This act of organising objects amounts to considering an instance of the very general mathematical notion of a *quasi-order* (qo), namely a transitive and reflexive relation.

As a means of classifying a family of objects, the following property of a quasi-order is usually desired: a quasi-order is said to be *well-founded* if every non-empty sub-family of objects admits a minimal element. This means that there are minimal – or simplest – objects which we can display on a first bookshelf, and then, amongst the remaining objects there are again simplest objects which we can display on a second bookshelf above the previous one, and so on and so forth – most probably into the transfinite.

However, as a matter of fact another concept has been 'frequently discovered' [Kru72] and proved even more relevant in diverse contexts: a *well-quasi-order* (WQO) is a well-founded quasi-order which contains no infinite antichain. Intuitively a well-quasi-order provides a satisfactory notion of hierarchy: as a well-founded quasi-order, it comes naturally equipped with an ordinal rank and there are up to equivalence only finitely many elements of any given rank. To prolong our metaphor, this means that, in addition, every bookshelf displays only finitely many objects – up to equivalence.

The theory of WQOs consists essentially of developing tools in order to show that certain quasi-orders suspected to be WQO are indeed so. This theory exhibits a curious and interesting phenomenon: to prove that a certain quasi-order is WQO, it may very well be easier to show that it enjoys a much stronger property. This observation may be seen as a motivation for considering the complicated but ingenious concept of *better-quasi-order* (BQO) invented by Crispin St. J. A. Nash-Williams [Nas65]. The concept of BQO is weaker than that of well-ordered set but it is stronger than that of WQO. In a sense, WQO is defined by a single 'condition', while uncountably many 'conditions' are necessary to characterise BQO. Still, as Joseph B. Kruskal [Kru72, p.302] observed in 1972: 'all "naturally occurring" WQO sets which are known are BQO'[1].

The first contribution of this thesis is to the theory of WQO and BQO. The main result is the proof of a conjecture made by Maurice Pouzet [Pou78] which states that any WQO whose ideal completion remainder is BQO is actually BQO. Our proof relies on new results with both a combinatorial and a topological flavour concerning maps from a *front* into a compact metric space. We think that these results are of independent interest and hope that they can be applied in other situations where fronts and barriers are used, as in the theory of Banach spaces for example.

Our second contribution is of a more applied nature and deals with topological spaces. We define a quasi-order on the subsets of every second countable T_0 topological space in a way that generalises the Wadge quasi-order on the Baire space, while extending its nice properties to virtually all these topological spaces.

Our starting point is the celebrated Wadge quasi-order – of reducibility by continuous functions – on subsets of the Baire space. This quasi-order is described by Alessandro Andretta and Alain Louveau [AL] as 'the ultimate analysis of the subsets of the Baire space'. The fact that the extremely fine Wadge quasi-order is WQO on Borel sets is doubtless among the most attractive of its properties. The proof of this fact given by Tony Martin, building on previous work by Leonard Monk, is an example of one of the main techniques of BQO theory, namely the use of infinite games and determinacy. Moreover, as we explain in this thesis, this property of the Wadge quasi-order follows from an extension of the idea underlying the very definition of a BQO.

For other important topological spaces the quasi-order of reducibility by continuous functions is however far less satisfactory. For example, the family of Borel subsets of the real line is very far from being WQO under continuous reducibility. While reducibility by discontinuous functions has been studied by

[1] The minor relations on finite graphs, proved to be WQO by Robertson and Seymour [RS04], is to our knowledge the only naturally occurring WQO which is not *yet* known to be BQO.

some authors to remedy this situation, we propose instead to keep continuity but to weaken the notion of function to that of relation. Using the notion of *admissible representation* studied in *Type-2 theory of effectivity*, we define the quasi-order of reducibility by relatively continuous relations. We show that this quasi-order both refines the classical hierarchies of complexity and is WQO on the Borel subsets of virtually every second countable T_0 space.

1.1 From well to better

A quasi-order Q is a WQO if it contains no infinite descending chain nor infinite antichain. However using Ramsey's Theorem this is equivalent to the absence of a so-called *bad sequence*, namely a sequence $(q_n)_{n\in\omega}$ such that $m < n$ in ω implies $q_m \nleq q_n$ in Q.

The concept of *better-quasi-order* was invented by Nash-Williams [Nas65]. Its definition relies on a generalisation of Ramsey's Theorem to transfinite dimension: the notion of *front*. It generalises the definition of WQO given above in the sense that it does not only forbid bad sequences, but also bad sequences of sequences, bad sequences of sequences of sequences and so on and so forth in the transfinite. A front can be thought of as a convenient notion of index sets for these sequences of … of sequences and we call any map from a front into some set a *super-sequence*. A BQO is then a quasi-order which admits no bad super-sequence.

One contribution of this thesis is to show that super-sequences deserve their name since they share significant properties with usual sequences. A crucial property for a sequence in the context of metric spaces is the Cauchy condition. In order to generalise the notion of being Cauchy to super-sequences, we observe that a sequence $(x_n)_{n\in\omega}$ in a metric space $\mathcal{X}$ satisfies the Cauchy condition if and only if the mapping $\omega \to \mathcal{X}$, $n \mapsto x_n$ is uniformly continuous, when ω is identified with a subspace of the Cantor space 2^ω via $n \mapsto 0^n 1 0^\omega$.

As observed notably by Todorčević [AT05; Tod10], fronts can naturally be seen as subsets of the Cantor space. Being a compact Hausdorff space, the Cantor space admits a unique uniformity that is compatible with its topology. Even though a front is a discrete topological subspace of 2^ω, we observe that it inherits a non-trivial uniformity from 2^ω. Let us say that a super-sequence in a metric space is Cauchy when it is uniformly continuous. We show the following theorem, which generalises the usual sequential compactness for metric spaces.

Theorem 1.1 (with R. Carroy)**.** *Every super-sequence in a compact metric space has a Cauchy sub-super-sequence.*

This combinatorial result should be compared with Erdös-Rado Theorem [ER50] and Pudlak-Rödl Theorem [PR82] as a Ramsey theorem for partitions into infinitely many classes. We also note also that this result subsumes Nash-Williams' Theorem.

Given a complete metric space $\mathcal{X}$, every Cauchy sequence $f : \omega \to \mathcal{X}$ converges, and thus extends to a continuous map $\overline{f} : \overline{\omega} \to \mathcal{X}$, where $\overline{\omega}$ is the one point compactification of ω. The same is true about Cauchy super-sequences: any uniformly continuous super-sequence $f : F \to \mathcal{X}$ from a front F into a complete metric space $\mathcal{X}$ continuously extends to the uniform completion $\overline{F}$ of F, which coincides with the topological closure of the front inside the Cantor space, to yield a continuous map $\overline{f} : \overline{F} \to X$.

We also study the continuous extension of Cauchy super-sequences. In full generality, we are concerned with continuous maps from the topological closure of a front into some topological space.

Recall that a point x in a topological space $\mathcal{X}$ is called *isolated* if the singleton $\{x\}$ is open in $\mathcal{X}$, and *limit* otherwise. The following simple fact exhibits a property of converging sequences that can always be achieved by going to a subsequence: If $(x_n)_{n\in\omega}$ is a sequence converging in a topological space $\mathcal{X}$ to some point x, then there is a subsequence $(x_n)_{n\in N}$ such that

1. if x is isolated, then $(x_n)_{n\in N}$ is constant equal to x;
2. if x is limit, then

   either x_n is isolated for all $n \in N$;

   or x_n is limit for all $n \in N$.

We generalise this fact to super-sequences by notably showing the following result.

Theorem 1.2 (with R. Carroy)**.** *Let $\overline{f} : \overline{F} \to \mathcal{X}$ be a continuous extension of a super-sequence f in a topological space $\mathcal{X}$. Then there exists a sub-super-sequence $f' : F' \to \mathcal{X}$ of f such that*

either *$\overline{f'} : \overline{F'} \to X$ is constant and equal to an isolated point;*

or *$\{s \in \overline{F'} \mid f(s) \text{ is limit}\} = \overline{G}$ for some front G.*

We then apply these theorems to the theory of BQO. The main result is a proof of a conjecture made by Pouzet [Pou78] in his *thèse d'état*. By an *ideal* of a quasi-order Q we mean a downward closed and up-directed subset of Q. To every element $q \in Q$ corresponds the *principal ideal* $\downarrow q = \{p \in Q \mid p \leqslant q\}$.

The *ideal completion* of Q is defined as the set of ideals of Q partially ordered by inclusion, and it is denoted by $\mathrm{Id}(Q)$. Notice that Q embeds into $\mathrm{Id}(Q)$ via the map $e : q \mapsto \downarrow q$. We denote by $\mathrm{Id}^*(Q)$ the set $\mathrm{Id}(Q) \setminus e(Q)$ of non-principal ideals of Q, this is the remainder of the ideal completion of Q.

The statement of the conjecture made by Pouzet [Pou78] is the following:

Theorem 1.3 (with R. Carroy)**.** *If Q is* WQO *and* $\mathrm{Id}^*(Q)$ *is* BQO*, then Q is* BQO*.*

Pouzet and Sauer [PS06] advanced a proof of this statement, but their proof contains a gap, as clearly revealed by Alberto Marcone and acknowledged by Pouzet and Sauer. While the approach of Pouzet and Sauer [PS06] is purely combinatorial, we follow a completely different line and make essential use of the fact that the ideal completion of WQO admits a natural compact topology.

We view the importance of the ideal completion as the coincidence in the case of a WQO of several notions of completions of a quasi-order. In fact, gathering many results and facts which certainly belong to the folklore we obtain the following:

Theorem 1.4. *For a* WQO *P the following completions coincide:*

(i) *the ideal completion* $\mathrm{Id}(P)$ *equipped with the Lawson topology,*

(ii) *the Cauchy ideal completion of P,*

(iii) *the Nachbin order-compactification, or ordered Stone-Čech compactification, of P with the discrete topology.*

The different properties of these three different completions combine to give what we call the *ideal space* of a WQO. This enables us to show that Theorem 1.1 admits the following nice corollary in the context of WQO theory.

Theorem 1.5 (with R. Carroy)**.** *Every super-sequence $f : F \to Q$ into a* WQO *Q admits a Cauchy sub-super-sequence $f' : F' \to Q$, which therefore extends to a continuous map $\overline{f'} : \overline{F'} \to \mathrm{Id}(Q)$ into the ideal space of Q.*

As a matter of fact, the ideal space of a WQO is a scattered compact space whose limit points are exactly the non principal ideals. Applying Theorem 1.2, this allows us to prove that any bad super-sequence in a WQO Q yields a bad super-sequence into the non principal ideals of Q. Therefore proving Pouzet's conjecture.

1.2 A well-quasi-order on the subsets of a topological space

The versatile concept of a topological space has proved valuable in various areas of mathematics. In many cases of interest, the spaces are second countable, i.e. their topology admits a countable base. While separable metrisable spaces are of primary importance to Analysis [Kec95], topological spaces that do not satisfy the Hausdorff separation property are central to Algebraic Geometry [EH00] and to Computer Science [Gou13]. We consider without distinction all second countable spaces which satisfy the weakest separation property T_0, namely every two points which have exactly the same neighbourhoods are equal.

We are interested in finding a way to quasi-order the subsets of a topological space according to their complexity. Among the properties of such a quasi-ordering the following are arguably desired.

- It should agree with an a priori idea of topological complexity, in particular it should refine the classical hierarchies.
- It should be as fine as possible.
- It should be WQO or even BQO – at least on Borel subsets.

The very act of defining a topology on a set of objects consists in specifying simple, easily observable properties: the open sets. We are then interested in understanding the complexity of the other subsets relatively to the open sets.

Already at the turn of the twentieth century, the French analysts – Baire, Borel and Lebesgue – stratified the Borel sets of a metric space into a transfinite hierarchy: the Baire classes $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ and $\boldsymbol{\Delta}^0_\alpha$. These classes are well-known to exhibit the following pattern:

$$\begin{array}{ccccccccccccccc}
\boldsymbol{\Sigma}^0_1 & \subsetneq & & \subsetneq & \boldsymbol{\Sigma}^0_2 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \boldsymbol{\Sigma}^0_\alpha & \subsetneq & & \subsetneq & \\
 & & \boldsymbol{\Delta}^0_2 & & & & \boldsymbol{\Delta}^0_3 & & \cdots & & \boldsymbol{\Delta}^0_\alpha & & & & \boldsymbol{\Delta}^0_{\alpha+1} & & \cdots \\
\boldsymbol{\Pi}^0_1 & \subsetneq & & \subsetneq & \boldsymbol{\Pi}^0_2 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \boldsymbol{\Pi}^0_\alpha & \subsetneq & & \subsetneq &
\end{array}$$

Borel sets are thus classified according to the complexity of their definition from open sets along this transfinite ladder. This classification was further refined by Hausdorff, and later by Kuratowski, by identifying what is now called the difference hierarchies, consisting of the Hausdorff–Kuratowski classes $D_\xi(\boldsymbol{\Sigma}^0_\alpha)$. Since for every map $f : \mathcal{X} \to \mathcal{X}$, the preimage function $f^{-1} : \mathcal{P}(\mathcal{X}) \to$

$\mathcal{P}(\mathcal{X})$ is a complete Boolean homomorphism, it directly follows from their definition that the Borel classes and the Hausdorff–Kuratowski classes are *closed under continuous preimages*[2].

Wadge in his Ph.D. thesis [Wad82] was the first to investigate the quasi-order of *continuous reducibility* on the subsets of the Baire space ω^ω: for $A, B \subseteq \omega^\omega$ we say that A is *Wadge reducible* to B, $A \leqslant_{\mathrm{W}} B$, if and only if there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}(B) = A$. This quasi-order called the *Wadge quasi-order* relates to the complexity of the subsets of the Baire space in the sense that $A \leqslant_{\mathrm{W}} B$ if and only if one can continuously reduce the membership problem for A to the membership problem for B. This quasi-order is remarkable. By considering suitable infinite games and using the determinacy of these games, which follows from Borel determinacy, this quasi-order turns out to be well-founded and to admit antichains of size at most 2 on the Borel sets. As Andretta and Louveau [AL] describe in their introduction to [KLS12]: 'The Wadge hierarchy is the ultimate analysis of $\mathcal{P}(\omega^\omega)$ in terms of topological complexity'. While the Borel classes and the Hausdorff–Kuratowski classes are closed under continuous preimages, and therefore represent initial segments for $\leqslant_{\mathrm{W}}$, there are in fact many more initial segments, so that the Wadge qo refines greatly these classical hierarchies.

Of course the *quasi-order of continuous reducibility* can be defined in any topological space $\mathcal{X}$ in the obvious way, for $A, B \subseteq \mathcal{X}$ let $A \leqslant_{\mathrm{W}} B$ if and only if there exists a continuous function $f : \mathcal{X} \to \mathcal{X}$ such that $A = f^{-1}(B)$. The nice properties of the Wadge quasi-order extend easily to all zero-dimensional Polish spaces, or even to all Luzin – or Borel absolute – zero-dimensional spaces. The restriction to Luzin spaces and their Borel subsets comes from the use of determinacy in the proof, but it can be weaken if one is willing to assume the determinacy of a larger class of games. In particular assuming the Axiom of Determinacy, the same holds for the quasi-order of continuous reducibility on all subsets of any zero-dimensional second countable space.

However the restriction to zero-dimensional spaces is of a different nature. In fact when the space is not zero-dimensional there may be very few continuous functions, independently of any determinacy hypothesis. Hertling in his Ph.D. thesis [Her96] shows that the qo of continuous reducibility of the Borel subsets of the real line $\mathbb{R}$ exhibits a more complicated pattern than in the case of the Baire space. For example, Ikegami showed in his Ph.D. thesis [Ike10] (see also [IST]) that the powerset $\mathcal{P}(\omega)$ of ω partially ordered by inclusion modulo finite – and hence any partial order of size $\aleph_1$ – embeds in the qo of continuous

[2] i.e. for every $A \subseteq \mathcal{X}$ in the class and every continuous $f : \mathcal{X} \to \mathcal{X}$ the set $f^{-1}(A)$ belongs to the class.

reducibility of Borel sets of the real line (cf. Subsection 5.6.1). In a more general setting, Schlicht showed in [Sch] that in any non zero-dimensional metric space there is an antichain for the qo of continuous reducibility of size continuum consisting of Borel sets. Selivanov [Sel06, and references there] and also Becher and Grigorieff [BG15] studied continuous reducibility in non Hausdorff spaces, where the situation is in general much less satisfactory than in the case of the Baire space.

In search for a useful notion of hierarchy outside Polish zero-dimensional spaces, Motto Ros, Schlicht, and Selivanov [MSS15] consider reductions by discontinuous functions. For example they obtain that the Borel subsets of the real line are well-founded with antichains of size at most 2 when quasi-ordered by reducibility via functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $A \in \boldsymbol{\Sigma}^0_3(\mathbb{R})$ we have $f^{-1}(A) \in \boldsymbol{\Sigma}^0_3$. They leave open the question whether $\boldsymbol{\Sigma}^0_3$ can be replaced by $\boldsymbol{\Sigma}^0_2$ in the above statement. Arguably one defect of this qo is that it does not refine the low level Borel classes, nor does it respect the Hausdorff hierarchy of the $\boldsymbol{\Delta}^0_2$.

Instead of considering reduction by discontinuous functions, we propose to keep continuity but to release the second concept at stake, namely that of function. In the abstract, our first remark is that *total relations* account perfectly for the idea of reducibility and in fact generalise the framework of reductions as functions.

The notion of continuity for relations that fits our purpose is called *relative continuity*. It relies on the simple and fundamental concept of *admissible representation* of a topological space which is the starting point of the development of computable analysis from the point of view of Type-2 theory of effectivity [Wei00].

The basic idea is to represent the points of a topological space $\mathcal{X}$ by means of infinite sequences of natural numbers. Given such a representation of $\mathcal{X}$, i.e. a partial surjective function $\rho :\subseteq \omega^\omega \to \mathcal{X}$, an $\alpha \in \omega^\omega$ is a *name* for a point $x \in \mathcal{X}$ when $\rho(\alpha) = x$. A function $f : \mathcal{X} \to \mathcal{X}$ is then said to be *relatively continuous* (resp. computable) with respect to ρ if the function f is continuous (or computable) in the ρ-names, i.e. there exists a continuous (resp. computable) $F : \operatorname{dom}\rho \to \operatorname{dom}\rho$ such that $f \circ \rho = \rho \circ F$. Of course the notion of relatively continuous function depends on the considered representation. However, for every second countable T_0 space $\mathcal{X}$ there exists – up to equivalence – a greatest representation (cf. Theorem 5.17) among the continuous ones, called an *admissible representation* of $\mathcal{X}$. The importance of admissible representations resides in the following fact (cf. Theorem 5.24): for an admissible representation ρ of $\mathcal{X}$, a function $f : \mathcal{X} \to \mathcal{X}$ is relatively continuous with respect to ρ if and only if f is continuous. Notice however that in general as long as the representation is *not* injective, many continuous transformations of the names exist which do not induce a map on the space $\mathcal{X}$. Indeed different names α, β of some point x can be sent by a continuous function F onto names $F(\alpha)$, $F(\beta)$ representing different points, i.e. $\rho(F(\alpha)) \neq \rho(F(\beta))$. Such transformations are called *relatively continuous relations* (cf. Definition 5.29) and they were first investigated in a systematic manner by Brattka and Hertling [BH94].

We propose to consider reducibility by total relatively continuous relations. When we fix an admissible representation ρ of a second countable T_0 space $\mathcal{X}$, it is natural to think of reductions by relatively continuous relations as 'reductions in the names': if $A, B \subseteq \mathcal{X}$, then A reduces to B, in symbols $A \preccurlyeq_{\mathrm{W}} B$, if and only if there exists a continuous function F from the names to the names such that for every name α, $\rho(\alpha) \in A \leftrightarrow \rho(F(\alpha)) \in B$. In other words, for every point x and every name α for x, $F(\alpha)$ is the name of a point that belongs to B if and only if x belongs to A.

We wish to mention that in 1981 Tang [Tan81] worked with an admissible representation of the Scott domain $\mathcal{P}\omega$ and studied on this particular space the exact same notion of reduction that we propose here in a more general setting. But firstly, this study is antecedent to the introduction by Kreitz and Weihrauch [KW85] of the concept of admissible representation and Tang does not notice that his representation of $\mathcal{P}\omega$ is admissible. This remark is indeed important since it allows one to see that his notion of reduction is actually topological, namely it depends only on the topology of the space $\mathcal{P}\omega$. Secondly, even though his paper is often cited, no author seem to notice his particular approach to reducibility on $\mathcal{P}\omega$.

To confront the quasi-order $\preccurlyeq_{\mathrm{W}}$ of reducibility by relatively continuous relations to our expectations, we show the following results.

Firstly, we show that reducibility by relatively continuous relations is a generalisation of Wadge reducibility outside zero-dimensional spaces.

Proposition 1.6. *On every zero-dimensional space, the reducibility by relatively continuous relations coincides with the continuous reducibility.*

Notice however that using a result of Schlicht [Sch] we show that it differs from the continuous reducibility in every separable metrisable space that is not zero-dimensional.

Secondly, using a result by Saint Raymond [Sai07] extended by de Brecht [deB13] we obtain that that reducibility by relatively continuous relations refines the classical hierarchies of Borel and Hausdorff–Kuratowski.

Proposition 1.7. *Let $\mathcal{X}$ be a second countable T_0 spaces and A and B be subsets of $\mathcal{X}$. For every $1 \leqslant \alpha, \xi < \omega_1$,*

(i) *if* $B \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})$ *and* $A \preccurlyeq_W B$*, then* $A \in \mathbf{\Sigma}^0_\alpha(\mathcal{X})$*,*

(ii) *if* $B \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$ *and* $A \preccurlyeq_W B$*, then* $A \in D_\xi(\mathbf{\Sigma}^0_\alpha(\mathcal{X}))$*.*

Finally, we show that the quasi-order $\preccurlyeq_{\mathrm{W}}$ is as well behaved on the Borel sets of a very large class of second countable T_0 spaces as the Wadge quasi-order is on the Borel subsets of the Baire space. The use of Borel determinacy naturally leads us to define the class of *Borel representable spaces*, which contains every Borel subspace of the Scott domain $\mathcal{P}\omega$, and in particular every Borel subspace of a Polish space.

Theorem 1.8. *Let* $\mathcal{X}$ *be a Borel representable space. Then the reducibility by relatively continuous relations* $\preccurlyeq_W$ *is well-founded on the Borel subsets of* $\mathcal{X}$*. Moreover the Wadge Lemma holds, namely for every Borel subset* A *and* B *of* $\mathcal{X}$

$$\text{either } A \preccurlyeq_W B \text{ or } B \preccurlyeq_W A^{\complement}.$$

As in the case of the Baire space, this structural result depends on the determinacy of the games under consideration. In particular, under the Axiom of Determinacy, the above theorem extends to all subsets of every second countable T_0 space.

1.3 Organisation of the thesis

Chapter 2: Sequences in sets and orders Several articles – notably [Mil85; Kru72; Sim85; Lav71; Lav76; For03] – contains valuable introductory material to the theory of better-quasi-orders. However, a book entitled 'Introduction to better-quasi-order theory' is yet to be written. This chapter represents our attempt to give the motivated introduction to the deep definition of Nash-Williams we wished we had when we began studying the theory two years before.

In Section 2.1 we prove a large number of characterisations of well-quasi-orders, all of them are folklore except the one stated in Proposition 2.14 which benefits from both an order-theoretical and a topological flavour.

We make our way towards the definition of better-quasi-orders in Section 2.2. One of the difficulties we encountered when we began studying better-quasi-order is due to the existence of two main different definitions – obviously equivalent to experts – and along with them two different communities, the graph theorists and the descriptive set theorists, who only rarely cite each other in their contributions to the theory. The link between the original approach of

Nash-Williams (graph theoretic) with that of Simpson (descriptive set theoretic) is merely mentioned by Argyros and Todorčević [AT05] alone. We present basic observations in order to remedy this situation in Subsection 2.2.3. Building on an idea due to Forster [For03], we introduce the definition of better-quasi-order in a new way, using insight from one of the great contributions of descriptive set theory to better-quasi-order theory, namely the use of games and determinacy.

Finally in Section 2.3 we put the definition of better-quasi-order into perspective. This last section contains original material which have not yet been published by the author.

Chapter 3: Sequences in spaces Building on the previous chapter, we study super-sequences in metric spaces. After making some simple observations on Cauchy sequences, we define Cauchy super-sequences in Section 3.1 and collect some basic facts about the closure of a front inside 2^ω. The main result of this chapter is that any super-sequence into a compact metric space $\mathcal{X}$ admits a Cauchy sub-super-sequence. This general result actually follows easily from the particular case where $\mathcal{X}$ is the Cantor space. Our reason to focus on the Cantor space lies in the fact that uniform continuity admits of a nice characterisation in the zero-dimensional setting as showed in Section 3.2. In particular the uniform structure of a front F essentially consists in a distinguished countable Boolean algebra of subsets of F (a characterisation of which is given in Proposition 3.20), that we call the blocks of the front.

In Section 3.3 we prove that any countable family of subsets of a front can be turned into blocks by eventually going to a sub-front in Theorem 3.24. From this combinatorial result we deduce that every super-sequence in 2^ω admits a Cauchy super-sequence.

Of course, when $\mathcal{X}$ is a complete metric space and $f : F \to \mathcal{X}$ is a Cauchy super-sequence, then f extends to a continuous map $\bar{f} : \overline{F} \to \mathcal{X}$, where $\overline{F}$ is the topological closure of F inside 2^ω. We study continuous map from the closure $\overline{F}$ of a front into an arbitrary topological space in Section 3.4.

In particular we define a certain 'normal form' for the continuous functions $f : \overline{F} \to \mathcal{Y}$ from the closure of a front F into a topological space $\mathcal{Y}$ and we prove that this normal form can always be achieved by a restriction to some $\overline{H}$ where H is a sub-front of F.

Importantly, these results are applied in the next chapter to prove the conjecture suggested by Pouzet [Pou78].

While the exposition given in this chapter is new, the results are published in an article [CP14] by the author and R. Carroy in *Fundamenta Mathematicae.*

Chapter 4: The ideal space of a well-quasi-order The main result of this chapter is the proof of a conjecture made by Pouzet [Pou78] which relates the BQO character of a given WQO with the BQO character of the remainder of the ideal completion of the WQO. Our approach relies on the fact that the ideal completion of a WQO is actually a compactification. This can be explained by the coincidence in the case of a WQO of the ideal completion with two other important completions of a quasi-order, the properties of which combine to yield what we call the ideal space of a WQO. We do not attempt to be comprehensive on the tentacular topic of completions of partial orders. Most of the results of Section 4.1 certainly belongs to the folklore but while most authors focus mainly on lattice theoretic or domain theoretic aspects, we concentrate on the WQO property.

In Subsection 4.1.2 we supply the definition of the so-called Cauchy ideal completion of a partial order which is studied by Erné and Palko [EP98] with a different approach. We give a characterisation of the partial orders in which the Cauchy ideals coincide with the ideals. They are the partial orders which enjoy the so-called property M – well-known in domain theory. Notably the two notions coincide in the case of a WQO, since every WQO trivially satisfies property M. Moreover for the partial orders with property M, we show that the ideal completion when equipped with the Lawson topology coincides with the Cauchy ideal completion.

Next we present the Cauchy ideal completion of a partial order P as the Priestley dual of a certain lattice of subsets of P. Following Bekkali, Pouzet, and Zhani [BPZ07] we view this as a particular case of a duality result relating the 'taking of the topological closure' with the 'algebraic generation of a lattice'. We also provide a new proof of this duality result. In particular, it turns out that the Cauchy ideal completion of a WQO is the Priestley dual of the lattice of downsets. This observation leads us in Subsection 4.1.4 to consider the profinite completion of a partial order. This is also the Nachbin order-compactification of the partial order considered with the discrete topology.

We see the coincidence of these various completions in the case of a WQO as a sign of the importance of the space of ideals of a WQO.

We then utilise the results of Chapter 3 to prove Pouzet's conjecture in Section 4.2. A slightly different proof was published by the author and R. Carroy [CP14].

We close this chapter by discussing some applications of Pouzet's conjecture in Section 4.3. We notably obtain as a corollary that an *interval order* is WQO if and only if it is BQO, as observed by Pouzet and Sauer [PS06].

Chapter 5: A Wadge Hierarchy for second countable spaces This chapter is based on an article [Peq15] published by the author in *Archive for Mathematical Logic*.

The fact that the Wadge quasi-order is well-founded on Borel subsets of ω^ω relies on the determinacy of certain infinite games and this result is actually best seen as an immediate corollary of a theorem on BQOs obtained by van Engelen, Miller, and Steel [vEMS87]. Elaborating on Chapter 2, we present in Section 5.1 a slight generalisation of this theorem (cf Theorems 5.7 and 5.9) in a way which makes it appear as an extension of the idea underlying the very definition of BQO.

From these results, we get that the quasi-order of continuous reducibility on the Borel subsets of any zero-dimensional Luzin[3] space $\mathcal{X}$ is a WQO – in fact a BQO – which satisfies the Wadge Lemma, namely for every Borel $A, B \subseteq \mathcal{X}$ either $A \leqslant_{\mathrm{W}} B$ or $B \leqslant_{\mathrm{W}} \mathcal{X} \setminus A$. In particular antichains have size at most 2.

The main idea of this chapter is to generalise the Wadge quasi-order to a large class of spaces while maintaining the nice properties it enjoys on the Borel subsets of the Baire space. To do this we move from reductions by continuous functions to reductions by 'continuous' relations. To begin with, we observe in Section 5.2 that total relations account perfectly for the idea of reducibility in the abstract and in fact generalise the framework of reductions as functions.

The notion of continuity for relations that fits our purpose is called *relative continuity*. It relies on the concept of *admissible representation of a topological space*. While this concept is fundamental to Type-2 Theory of Effectivity (see the textbook by Weihrauch [Wei00]), we do not expect our reader to be familiar with the simple and interesting underpinning of this approach to computable analysis. We therefore review the basic definitions and provide proofs for his convenience in Section 5.3. This Section ends with the definition of the quasi-order $\preccurlyeq_{\mathrm{W}}$ of reducibility by relatively continuous relations.

We prove in Section 5.4 that the quasi-order $\preccurlyeq_{\mathrm{W}}$ refines the classical hierarchies of Borel and Hausdorff–Kuratowski.

We define a general reduction game for represented spaces in Section 5.5 as a simple adaptation of the game we used in Section 5.1. This allows us to show that the quasi-order $\preccurlyeq_{\mathrm{W}}$ satisfies the Wadge Lemma on Borel subsets of Borel representable spaces. Also, moving from continuous functions to relatively continuous relations we extend our version of the theorem by van Engelen, Miller, and Steel [vEMS87] from Luzin zero-dimensional spaces to all Borel representable spaces. This yields in particular that the reducibility by relatively continuous relations is well-founded – in fact BQO – on the Borel

[3] Luzin spaces are also called Borel absolute spaces.

subsets of every Borel representable space.

Finally in Section 5.6 we exemplify the difference between the continuous reducibility and the reducibility by relatively continuous relations in two major examples: the real line $\mathbb{R}$ and the Scott domain $\mathcal{P}\omega$.

Bibliography

[AL] Alessandro Andretta and Alain Louveau. 'Wadge Degrees and Pointclasses: Introduction to Part III'. In: *Wadge Degrees and Projective Ordinals: The Cabal Seminar Volume II*, p. 3 (cit. on pp. 2, 7).

[AT05] Spiros A. Argyros and Stevo Todorčević. *Ramsey methods in analysis*. Birkhäuser Basel, 2005 (cit. on pp. 3, 11).

[Bez+06] Guram Bezhanishvili, Mai Gehrke, Ray Mines, and Patrick J Morandi. 'Profinite completions and canonical extensions of Heyting algebras'. In: *Order* 23.2-3 (2006), pp. 143–161.

[BG15] Verónica Becher and Serge Grigorieff. 'Wadge hardness in Scott spaces and its effectivization'. In: *Mathematical Structures in Computer Science* FirstView (Jan. 2015), pp. 1–26. DOI: 10.1017/S0960129513000248 (cit. on p. 8).

[BH94] Vasco Brattka and Peter Hertling. *Continuity and computability of relations*. Fernuniv., Fachbereich Informatik, 1994 (cit. on p. 9).

[Bir37] Garrett Birkhoff. 'Rings of sets'. In: *Duke Mathematical Journal* 3.3 (Sept. 1937), pp. 443–454. DOI: 10.1215/S0012-7094-37-00334-X.

[Bog93] Kenneth P Bogart. 'An obvious proof of Fishburn's interval order theorem'. In: *Discrete Mathematics* 118.1 (1993), pp. 239–242.

[Bou06] Nicolas Bourbaki. *Topologie générale: Chapitres 1 à 4*. Springer, 2006.

[BPZ07] Mohamed Bekkali, Maurice Pouzet, and Driss Zhani. 'Incidence structures and Stone–Priestley duality'. In: *Annals of Mathematics and Artificial Intelligence* 49.1 (2007), pp. 27–38 (cit. on p. 12).

[Bra99] Vasco Brattka. 'Recursive and computable operations over topological structures'. PhD thesis. FernUniversität Hagen, 1999.

[Car13] Raphaël Carroy. 'A quasi-order on continuous functions'. In: *Journal of Symbolic Logic* 78.2 (2013), pp. 633–648.

[CP14] Raphaël Carroy and Yann Pequignot. 'From well to better, the space of ideals'. In: *Fundamenta Mathematicae* 227.3 (2014), pp. 247–270 (cit. on pp. 11–12).

[CS06] Florence D. Conway and Jim Siegelman. *Dark Hero of the Information Age: In Search of Norbert Wiener - The Father of Cybernetics*. Basic Books, 2006.

[deB13] Matthew de Brecht. 'Quasi-polish spaces'. In: *Annals of Pure and Applied Logic* 164.3 (2013), pp. 356–381 (cit. on p. 9).

[Del80] Claude Dellacherie. 'Un cours sur les ensembles analytiques'. In: *Analytic sets*. Ed. by Claude Ambrose Rogers. Academic Press, London, 1980, pp. 183–316.

[DF] Jacques Duparc and Kevin Fournier. 'The Baire space and reductions by relatively continuous functions'. unpublished.

[Dil50] Robert P Dilworth. 'A decomposition theorem for partially ordered sets'. In: *Annals of Mathematics* (1950), pp. 161–166.

[DP02] Brian A Davey and Hilary A Priestley. *Introduction to lattices and order*. Cambridge university press, 2002.

[DT06] Carlos A Di Prisco and Stevo Todorčević. 'Canonical forms of shift-invariant maps on $[\mathbb{N}]^\omega$'. In: *Discrete mathematics* 306.16 (2006), pp. 1862–1870.

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*. English. New York, NY: Springer, 2000, pp. x + 294. DOI: 10.1007/b97680 (cit. on p. 6).

[EP98] Marcel Erné and Vladimír Palko. 'Uniform ideal completions'. In: *Mathematica Slovaca* 48.4 (1998), pp. 327–335 (cit. on p. 12).

[ER50] Paul Erdös and Richard Rado. 'A combinatorial theorem'. In: *Journal of the London Mathematical Society* 1.4 (1950), pp. 249–255 (cit. on p. 4).

[Ern01] Marcel Erné. 'Ideal completions and compactifications'. In: *Applied Categorical Structures* 9.3 (2001), pp. 217–243.

[Ern93] Marcel Erné. 'Algebraic Ordered Sets and Their Generalizations'. English. In: *Algebras and Orders*. Ed. by Ivo G Rosenberg and Gert Sabidussi. Vol. 389. NATO ASI Series. Springer Netherlands, 1993, pp. 113–192. DOI: 10.1007/978-94-017-0697-1_3.

[Fis70] Peter C Fishburn. 'Intransitive indifference with unequal indifference intervals'. In: *Journal of Mathematical Psychology* 7.1 (1970), pp. 144–149.

[FM92] Peter Fishburn and Bernard Monjardet. 'Norbert Wiener on the theory of measurement 1914 1915 1921'. In: *Journal of Mathematical Psychology* 36.2 (1992), pp. 165–184.

[For03] Thomas Forster. 'Better-quasi-orderings and coinduction'. In: *Theoretical computer science* 309.1 (2003), pp. 111–123 (cit. on pp. 10–11).

[Fra00] Roland Fraïssé. *Theory of Relations*. Studies in Logic and the Foundations of Mathematics. Elsevier Science, 2000.

[GGP10] Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin. 'A Topological Approach to Recognition'. English. In: *Automata, Languages and Programming*. Ed. by Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis. Vol. 6199. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2010, pp. 151–162. DOI: 10.1007/978-3-642-14162-1_13.

[Gie+03] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D Lawson, Michael Mislove, and Dana S Scott. *Continuous Lattices and Domains*. Cambridge Books Online. Cambridge University Press, 2003.

[Gou13] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*. Vol. 22. New Mathematical Monographs. Cambridge University Press, Mar. 2013 (cit. on p. 6).

[GP08] Mai Gehrke and Hilary A Priestley. 'Canonical extensions and completions of posets and lattices'. In: *Reports on Mathematical Logic* 43 (2008), pp. 133–152.

[GP73] Fred Galvin and Karel Prikry. 'Borel sets and Ramsey's theorem'. In: *The Journal of Symbolic Logic* 38 (02 June 1973), pp. 193–198. DOI: 10.2307/2272055.

[GV11] Mai Gehrke and Jacob Vosmaer. 'A View of Canonical Extension'. English. In: *Logic, Language, and Computation*. Ed. by Nick Bezhanishvili, Sebastian Löbner, Kerstin Schwabe, and Luca Spada. Vol. 6618. Lecture Notes in Computer Science. Springer

Berlin Heidelberg, 2011, pp. 77–100. DOI: 10.1007/978-3-642-22303-7_6.

[Hal11] Lorenz J Halbeisen. *Combinatorial Set Theory: With a Gentle Introduction to Forcing.* Springer, 2011.

[Her96] Peter Hertling. 'Unstetigkeitsgrade von Funktionen in der effektiven Analysis'. PhD thesis. Fernuniv., Fachbereich Informatik, 1996 (cit. on p. 7).

[Hof79] Rudolf-Eberhard Hoffmann. 'Sobrification of partially ordered sets'. In: *Semigroup Forum.* Vol. 17. 1. Springer. 1979, pp. 123–138.

[Ike10] Daisuke Ikegami. 'Games in Set Theory and Logic'. PhD thesis. Institute for Logic, Language and Computation, Universiteit von Amsterdam, 2010 (cit. on p. 7).

[IST] Daisuke Ikegami, Philipp Schlicht, and Hisao Tanaka. 'Continuous reducibility for the real line'. unpublished notes (cit. on p. 7).

[Jec03] Thomas J. Jech. *Set Theory: The Third Millennium Edition, Revised and Expanded.* Springer Monographs in Mathematics. Springer, 2003.

[Joh86] Peter Tennant Johnstone. *Stone spaces.* Cambridge University Press, 1986.

[Kec95] Alexander S Kechris. *Classical descriptive set theory.* Vol. 156. Springer-Verlag New York, 1995 (cit. on p. 6).

[KLS12] Alexander S. Kechris, Benedikt Löwe, and John R. Steel, eds. *Wadge Degrees and Projective Ordinals: The Cabal Seminar Volume II.* Vol. 37. Lecture Notes in Logic. Cambridge University Press, 2012 (cit. on p. 7).

[Kru72] Joseph Bernard Kruskal. 'The theory of well-quasi-ordering: A frequently discovered concept'. In: *Journal of Combinatorial Theory, Series A* 13.3 (1972), pp. 297–305 (cit. on pp. 1–2, 10).

[KST99] Alexander S. Kechris, Sławomir Solecki, and Stevo Todorčević. 'Borel Chromatic Numbers'. In: *Advances in Mathematics* 141.1 (1999), pp. 1–44. DOI: 10.1006/aima.1998.1771.

[KW85] Christoph Kreitz and Klaus Weihrauch. 'Theory of representations'. In: *Theoretical computer science* 38 (1985), pp. 35–53 (cit. on p. 9).

[Lav71] Richard Laver. 'On Fraïssé's order type conjecture'. In: *The Annals of Mathematics* 93.1 (1971), pp. 89–111 (cit. on p. 10).

[Lav76] Richard Laver. 'Well-quasi-orderings and sets of finite sequences'. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 79. 01. Cambridge Univ Press. 1976, pp. 1–10 (cit. on p. 10).

[LS11] Alain Louveau and Jean Saint Raymond. 'The strength of Borel Wadge determinacy'. In: *Wadge Degrees and Projective Ordinals: The Cabal Seminar Volume II*. Vol. 37. Cambridge University Press. 2011, p. 74.

[LS82] Alain Louveau and Stephen G. Simpson. 'A separable image theorem for Ramsey mappings'. In: *Bulletin de l'Académie Polonaise des Sciences, Série Mathématique* 20 (1982), pp. 105–108.

[LS90] Alain Louveau and Jean Saint Raymond. 'On the quasi-ordering of Borel linear orders under embeddability'. In: *Journal of Symbolic Logic* 55.2 (1990), pp. 537–560.

[Mar07] Alberto Marcone. 'Interval Orders and Reverse Mathematics'. In: *Notre Dame J. Formal Logic* 48.3 (July 2007), pp. 425–448. DOI: `10.1305/ndjfl/1187031412`.

[Mar75] Donald A Martin. 'Borel determinacy'. In: *Annals of Mathematics* (1975), pp. 363–371.

[Mar94] Alberto Marcone. 'Foundations of bqo theory'. In: *Transactions of the American Mathematical Society* 345.2 (1994), pp. 641–660.

[Mil85] Eric Charles Milner. 'Basic wqo-and bqo-theory'. In: *Graphs and order*. Springer, 1985, pp. 487–502 (cit. on p. 10).

[MSS15] Luca Motto Ros, Philipp Schlicht, and Victor Selivanov. 'Wadge-like reducibilities on arbitrary quasi-Polish spaces'. In: *Mathematical Structures in Computer Science* FirstView (Jan. 2015), pp. 1–50. DOI: `10.1017/S0960129513000339` (cit. on p. 8).

[Nac65] Leopoldo Nachbin. *Topology and order*. Trans. from the Portuguese by Lulu Bechtolsheim. D. Van Nostrand Company, 1965.

[Nas65] Crispin St. John Alvah Nash-Williams. 'On well-quasi-ordering infinite trees'. In: *Proc. Cambridge Phil. Soc.* Vol. 61. Cambridge Univ Press. 1965, p. 697 (cit. on pp. 2–3).

[Pag11] Warren Page. *Topological Uniform Structures*. Dover Books on Mathematics Series. Dover Publications, Incorporated, 2011.

[Par93] Ivan Ivanovich Parovičenko. 'On a universal bicompactum of weight $\aleph$'. In: *The Mathematical Legacy of Eduard Čech*. Springer, 1993, pp. 93–96.

[Peq15] Yann Pequignot. 'A Wadge hierarchy for second countable spaces'. In: *Archive for Mathematical Logic* 54.5-6 (2015), pp. 659–683. DOI: `10.1007/s00153-015-0434-y` (cit. on p. 13).

[Pou78] Maurice Pouzet. 'Sur la théorie des relations'. Thèse d'état. Université Claude Bernard, 1978 (cit. on pp. 2, 4–5, 11–12).

[PR82] Pavel Pudlák and Vojtěch Rödl. 'Partition theorems for systems of finite subsets of integers'. In: *Discrete Mathematics* 39.1 (1982), pp. 67–73 (cit. on p. 4).

[Pri72] Hilary A Priestley. 'Ordered topological spaces and the representation of distributive lattices'. In: *Proceedings of the London Mathematical Society* 3.3 (1972), pp. 507–530.

[Pri94] Hilary A Priestley. 'Spectral sets'. In: *Journal of Pure and Applied Algebra* 94.1 (1994), pp. 101–114.

[PS06] Maurice Pouzet and Norbert Sauer. 'From well-quasi-ordered sets to better-quasi-ordered sets'. In: *Journal of Combinatorics* 13.4 (2006), p. 101 (cit. on pp. 5, 12).

[PV86] Hans Jürgen Prömel and Bernd Voigt. 'Hereditary attributes of surjections and parameter sets'. In: *European Journal of Combinatorics* 7.2 (1986), pp. 161–170.

[PV92] Hans Jürgen Prömel and Bernd Voigt. 'From wqo to bqo, via Ellentuck's theorem'. In: *Discrete mathematics* 108.1-3 (1992), pp. 83–106.

[PZ13] Arno M Pauly and Martin A Ziegler. 'Relative computability and uniform continuity of relations'. In: *Journal of Logic and Analysis* 5 (2013).

[Rad54] Richard Rado. 'Partial well-ordering of sets of vectors'. In: *Mathematika* 1.02 (1954), pp. 89–95.

[Ram30] Frank Plumpton Ramsey. 'On a Problem of Formal Logic'. In: *Proceedings of the London Mathematical Society* 2.1 (1930), pp. 264–286.

[RS04] Neil Robertson and P.D. Seymour. 'Graph Minors. XX. Wagner's conjecture'. In: *Journal of Combinatorial Theory, Series B* 92.2 (2004). Special Issue Dedicated to Professor W.T. Tutte, pp. 325–357. DOI: `http://dx.doi.org/10.1016/j.jctb.2004.08.001` (cit. on p. 2).

[Sai07] Jean Saint Raymond. 'Preservation of the Borel class under countable-compact-covering mappings'. In: *Topology and its Applications* 154.8 (2007), pp. 1714–1725 (cit. on p. 9).

[Sch] Philipp Schlicht. 'Continuous reducibility and dimension'. unpublished notes (cit. on pp. 8–9).

[Sel06] Victor L Selivanov. 'Towards a descriptive set theory for domain-like structures'. In: *Theoretical computer science* 365.3 (2006), pp. 258–282 (cit. on p. 8).

[She82] Saharon Shelah. 'Better quasi-orders for uncountable cardinals'. In: *Israel Journal of Mathematics* 42.3 (1982), pp. 177–226.

[Sim85] Stephen G. Simpson. 'Bqo theory and Fraïssé's conjecture'. In: Richard Mansfield and Galen Weitkamp. *Recursive aspects of descriptive set theory.* 1985, pp. 124–138 (cit. on p. 10).

[Sol13] Sławomir Solecki. 'Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem'. In: *Advances in Mathematics* 248 (2013), pp. 1156–1198.

[Spe72] Terrence Paul Speed. 'Profinite posets'. In: *Bulletin of the Australian Mathematical Society* 6 (02 Apr. 1972), pp. 177–183. DOI: `10.1017/S0004972700044415`.

[Spu10] Jiří Spurný. 'Borel sets and functions in topological spaces'. In: *Acta Mathematica Hungarica* 129.1 (2010), pp. 47–69.

[Sto36] Marshall Harvey Stone. 'The theory of representation for Boolean algebras'. In: *Transactions of the American Mathematical Society* 40.1 (1936), pp. 37–111.

[Tan79] Adrian Tang. 'Chain properties in Pω'. In: *Theoretical Computer Science* 9.2 (1979), pp. 153–172.

[Tan81] Adrian Tang. 'Wadge reducibility and Hausdorff difference hierarchy in Pω'. In: *Continuous Lattices*. Springer, 1981, pp. 360–371 (cit. on p. 9).

[Tch80] Tchouang-tseu. 'L'Œuvre complète'. In: *Philosophes Taoïstes.* Bibliothèque de la Pléiade. Gallimard, Paris, 1980. Chap. XXIV, pp. 282–283.

[Tod10] Stevo Todorčević. *Introduction to Ramsey spaces.* Princeton Univ Pr, 2010, p. 174 (cit. on p. 3).

[vEMS87] Fons van Engelen, Arnold W Miller, and John Steel. 'Rigid Borel Sets and Better Quasiorder Theory'. In: *Contemporary mathematics* 65 (1987) (cit. on p. 13).

[VM44] John Von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior.* Science Editions. Princeton University Press, 1944.

[Wad82] William Wilfred Wadge. 'Reducibility and determinateness on the Baire space'. PhD thesis. 1982 (cit. on p. 7).

[Wei00] Klaus Weihrauch. *Computable analysis: an introduction.* Springer, 2000 (cit. on pp. 8, 13).

[Wie14] Norbert Wiener. 'A contribution to the theory of relative position'. In: *Proc. Camb. Philos. Soc.* Vol. 17. 1914, pp. 441–449.