Axiomatic Rewriting Theory III

A factorisation theorem in Rewriting Theory

Paul-André Melliès
LFCS, University of Edinburgh
`<paulm@dcs.ed.ac.uk>`

Abstract. Some computations on a symbolic term M are more judicious than others, for instance the ***leftmost outermost*** derivations in the λ-calculus. In order to characterise generically that kind of judicious computations, [M] introduces the notion of ***external*** derivations in its axiomatic description of Rewriting Systems: a derivation $e : M \longrightarrow P$ is said to be external when the derivation $e; f : M \longrightarrow Q$ is standard whenever the derivation $f : P \longrightarrow Q$ is standard.
In this article, we show that in every Axiomatic Rewriting System [M,1] every derivation $d : M \longrightarrow Q$ can be factorised as an external derivation $e : M \longrightarrow P$ followed by an ***internal*** derivation $m : P \longrightarrow Q$. Moreover, this epi-mono factorisation is functorial (i.e there is a nice diagram) in the sense of Freyd and Kelly [FK].
Conceptually, the factorisation property means that the efficient part of a computation can always be separated from its junk. Technically, the property is the key step towards our illuminating interpretation of Berry's stability (semantics) as a syntactic phenomenon (rewriting). In fact, contrary to the usual Lévy derivation spaces, the external derivation spaces enjoy meets.

1 Motivations on two syntactic λ-calculi

There are algebraic reasons behind the confluence of the λ-calculus: [Lé] shows that the category $[\mathcal{C}_\lambda]$ of ***derivations*** up to ***redex-permutation*** enjoys pushouts. Our first move is to recall the construction of $[\mathcal{C}_\lambda]$.

The construction of $[\mathcal{C}_\lambda]$. The λ-calculus generates a transition graph $\mathcal{G}_\lambda$ whose vertices are the λ-terms up to α-conversion, and whose arrows $M \longrightarrow P$ are the β-redexes from M to P. We recall that a *β-redex* $M \longrightarrow P$ is a couple (M, o, P) where o is the occurrence in M where the β-reduction occurs. The category $\underline{\mathcal{C}}_\lambda$ is the ***free*** category on this graph $\mathcal{G}_\lambda$:

1. its objects are the vertices of $\mathcal{G}_\lambda$,
2. its morphisms from M to P are the sequences $(M_1, r_1, M_2, \cdots, M_n, r_n, M_{n+1})$ where M_j's are vertices and r_j's β-redexes such that $M_1 = M$ and $M_{n+1} = P$ and $r_j : M_j \longrightarrow M_{j+1}$ for every j. A morphism in $\underline{\mathcal{C}}_\lambda$ is called a ***derivation***,
3. composition of derivation is just concatenation. If $d : M \longrightarrow P$ and $e : P \longrightarrow Q$, we write $d; e$ their composite from M to Q. In particular, the notation $r_1; \cdots; r_n$ denotes the derivation $(M_1, r_1, M_2, \cdots, M_n, r_n, M_{n+1})$.

The crux in the construction of $[\mathcal{C}_\lambda]$ is to identify the derivations from M to P with the same computational content but different reduction orders. To do this, Lévy introduces a ***permutation*** equivalence $\equiv$ which identifies the different developments of a set $\{u, v\}$ of coinitial β-redexes. We recall that a ***development*** of a set of coinitial β-redexes is a derivation which sequentialises their simultaneous reduction. In [Lé], the sequentialisation is formalised with a notion of ***residual*** which permits to trace β-redexes in the course of computation.

A ***redex permutation*** in the λ-calculus can be of two species whose paradigms are permutations (1) and (2):

$$
\begin{tikzcd}[column sep=small]
 & P'Q \arrow[dr, "u'"] & \\
PQ \arrow[ur, "v"] \arrow[dr, "u"'] & 1 & P'Q' \\
 & PQ' \arrow[ur, "v'"'] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}[column sep=small]
 & \Delta P' \arrow[dr, "u'"] & \\
\Delta P \arrow[ur, "v"] \arrow[dr, "u"'] & 2 & P'P' \\
 & PP \arrow[ur, "v_1;v_2"'] &
\end{tikzcd}
\qquad
\begin{array}{c}
P \xrightarrow{\beta} P' \\
Q \xrightarrow{\beta} Q' \\
\Delta = (\lambda x.xx)
\end{array}
$$

In the first permutation, the permuted β-redexes u and v are ***disjoint***. In the second one, the β-redex u ***contains*** the other β-redex v. Anticipating Section 2, the permutation (1) from $v; u'$ to $u; v'$ is declared ***disjoint*** because it permutes disjoint redexes, and the permutation (2) from $v; u'$ to $u; v_1; v_2$ ***standardising*** because it permutes the outer redex u before the inner redex v.

Lévy defines the ***permutation equivalence relation*** $\equiv$ as the least equivalence relation

1. which identifies the different developments of $\{u, v\}$ for every couple (u, v) of coinitial β-redexes,
2. which is closed under composition: if $f \equiv g$ for $f, g : P \longrightarrow Q$ then $d_1; f; d_2 \equiv d_1; g; d_2$ for every $d_1 : M \longrightarrow P$ and $d_2 : Q \longrightarrow N$.

In other words, two derivations f and g are identified by $\equiv$ when a sequence of permutations operates on f (to and fro) and transforms it into g.
The category $[\mathcal{C}_\lambda]$ is defined as the quotient of $\underline{\mathcal{C}}_\lambda$ by the equivalence relation $\equiv$:

1. its objects are the λ-terms up to α-conversion,
2. its morphisms $M \longrightarrow P$ are the derivations up to the permutation equivalence $\equiv$.

The category $[\mathcal{C}_\lambda]$ is not just a preorder category. The λ-term $I(Ia)$ for $I = \lambda x.x$ can be rewritten in two different ways to Ia, and these two computations are mirrored in the category $[\mathcal{C}_\lambda]$.

Pushouts. [Lé] shows that $[\mathcal{C}_\lambda]$ enjoys the ***pushout*** property. This result can be read as follows: if $M \xrightarrow{f} P$ and $M \xrightarrow{g} Q$ in $[\mathcal{C}_\lambda]$, then there exists two morphisms f' and g' such that the following diagram commutes in $[\mathcal{C}_\lambda]$:

$$
\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, dotted, "f'"] \\
P \arrow[r, dotted, "g'"'] & N
\end{tikzcd}
$$

and furthermore, if two morphisms $f; h_1$ and $g; h_2$ from M to O are equal in $[\mathcal{C}_\lambda]$ (equivalence by permutation in $\underline{\mathcal{C}}_\lambda$), there exists a unique morphism $h : N \longrightarrow O$ (uniqueness up to $\equiv$ in $\underline{\mathcal{C}}_\lambda$) such that the following diagram commutes in $[\mathcal{C}_\lambda]$:

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "f'"] \arrow[ddr, bend left, "h_2"] & \\
P \arrow[r, "g'"] \arrow[drr, bend right, "h_1"'] & N \arrow[dr, dotted, "h"] & \\
 & & O
\end{tikzcd}$$

Clearly, the confluence property of the λ-calculus (the so-called Church-Rosser property) is a direct consequence of the existence of pushouts in $[\mathcal{C}_\lambda]$.

Placard. We claim that this pushout property is one of the most important results obtained in the field of rewriting theory. From our point of view, it justifies all further attempts to understand rewriting from a structural or algebraic point of view.

Robustness is naturally reinforced by genericity: the same pushout construction is possible on any ***orthogonal*** rewriting system, with an analogous notion of permutation equivalence, see [HL]. For more information on the later developments of this result (standardisation, normalisation, optimality), have a look at [Lé,HL,K,Bo,Ba,GLM,CK,M].

Motivations. This paper is motivated by a negative observation: there seems to be no simple adaptation of the pushout property to ***non orthogonal*** (in particular non confluent) rewriting systems. Consider for instance the λ_+-calculus, a λ-calculus enriched with the operator $+$ and the two rewrite rules:

$$P + Q \longrightarrow P \qquad P + Q \longrightarrow Q$$

The λ_+-calculus is not confluent. Nevertheless, a notion of permutation equivalence $\equiv$ can be defined in the way of [Bo,CK,M] with a permutation on *non-conflicting* redexes. Hence, a category $[\mathcal{C}_+]$ of λ_+-derivations up to $\equiv$ (or Lévy permutation classes) can be constructed. Clearly, by non confluence, this category does not enjoy pushouts. Consequently, it is natural to ask whether $[\mathcal{C}_+]$ enjoys a weaker notion of pushouts: ***bounded pushouts***.

Bounded pushouts. A ***span*** in a category $\mathcal{C}$ is a couple (f, g) of coinitial morphisms. A span $P \xleftarrow{f} M \xrightarrow{g} Q$ is ***bounded*** in $\mathcal{C}$ when there exists two morphisms $h_1 : P \longrightarrow O$ and $h_2 : Q \longrightarrow O$ such that $f; h_1 = g; h_2$:

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, dotted, "h_2"] \\
P \arrow[r, dotted, "h_1"'] & O
\end{tikzcd}$$

A category enjoys ***bounded pushouts*** when every bounded span has a pushout.

A COUNTER-EXAMPLE. We adapt an example of [Lé] and show that the category $[\mathcal{C}_+]$ does not enjoy bounded pushouts. In fact, let $P \xleftarrow{r} M \xrightarrow{s} Q$ be the following span in $[\mathcal{C}_+]$:

$$(\lambda x.Ka(xbR))K \xleftarrow{r} (\lambda x.Ka(xb(R+S)))K \xrightarrow{s} (\lambda x.Ka(xbS))K$$

where R and S are two different normal forms, r and s are the two conflicting $+$-redexes from M, and $K = (\lambda x.\lambda y.x)$. The span $P \xleftarrow{r} M \xrightarrow{s} Q$ is bounded in $[\mathcal{C}_+]$ despite the critical pair formed by r and s. In fact, it is bounded twice. On one hand $r; h_1 \equiv s; h_2$ when h_1 and h_2 are the two following derivations:

$$h_1 : P \longrightarrow (\lambda x.a)K \qquad h_2 : Q \longrightarrow (\lambda x.a)K$$

On the other hand $r; i_1 \equiv s; i_2$ when i_1 and i_2 are the two following derivations:

$$i_1 : P \longrightarrow Ka(KbR) \longrightarrow Kab \qquad i_2 : P \longrightarrow Ka(KbS) \longrightarrow Kab$$

So, we obtain two different ***commutative*** diagrams in $[\mathcal{C}_+]$:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "h_2"] \\ P \arrow[r, "h_1"'] & (\lambda x.a)K \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "i_2"] \\ P \arrow[r, "i_1"'] & Kab \end{tikzcd}$$

Observe that each derivation h_1, h_2, i_1 and i_2 represents a ***minimal*** procedure to erase R or S. What minimal means here is that any (hypothetical) pushout diagram:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "PO"] & Q \arrow[d, "f'"] \\ P \arrow[r, "g'"'] & N \end{tikzcd}$$

would induce commutative diagrams:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "f'"'] \arrow[ddr, bend left, "h_2"] & \\ P \arrow[r, "g'"] \arrow[drr, bend right, "h_1"'] & N \arrow[dr, dotted, "h"] & \\ & & (\lambda x.a)K \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "f'"'] \arrow[ddr, bend left, "i_2"] & \\ P \arrow[r, "g'"] \arrow[drr, bend right, "i_1"'] & N \arrow[dr, dotted, "i"] & \\ & & Kab \end{tikzcd}$$

where h and i would be isos, with the consequence that $(\lambda x.a)K$ and Kab are isomorphic in $[\mathcal{C}_+]$, which is impossible because the identity morphisms are the only isos in $[\mathcal{C}_+]$. We conclude that $P \xleftarrow{r} M \xrightarrow{s} Q$ is a bounded span without a pushout [1] and that the category $[\mathcal{C}_+]$ does not enjoy bounded pushouts.

An aside: The category specialist unhappy with the notion of bounded pushouts should verify that the slice category $([\mathcal{C}_+]\downarrow a)$ does not enjoy pushouts either. This will convince him that the malicious phenomenon above cannot be overcome with a categorically more satisfactory apparatus than bounded pushouts [2].

Directions. Well, observing that the category $[\mathcal{C}_+]$ does not enjoy bounded pushouts is a bit depressing, but there is still some hope that an interesting ***subcategory*** of $[\mathcal{C}_+]$ enjoys them. In particular, our counter-example arises from the possibility of erasing two ***unnecessary*** and ***conflicting*** derivations r and s. What would happen if we restrict our investigation to a class of ***necessary*** derivations in $[\mathcal{C}_+]$?

At this point, we decide to extend our scope (forget the λ_+-calculus!) and develop our analysis in the framework of ***Axiomatic Rewriting Systems*** (ARS). Introduced in [1], ARSs are transition systems (or abstract rewriting systems) equipped with a notion of concurrence and duplication in the spirit of Concurrent Transition Systems [S]. A wide range of orthogonal and non orthogonal Rewriting Systems can be modelled as ARSs, in particular first order term rewriting systems, the λ-calculus, the λ_+-calculus, the call-by-value λ-calculus, the $\lambda\sigma$-calculus, combinatory reduction systems, interaction nets or the π-calculus. Because we work in ARSs from now, all theorems in the sequel apply generically to any of these calculi. In particular, we define in every ARS a subcategory $[\mathcal{E}]$ of external derivations and prove that it enjoys the bounded pushout property — thus solving in every ARS the problem we motivated in the λ_+-calculus.

Structure of the paper. Section 2 introduces Axiomatic Rewriting Systems and explains the derived notions of ***standard*** and ***external*** derivation. The categories $[\mathcal{C}]$, $[\mathcal{E}]$ and $[\mathcal{M}]$ are defined. A summary of the paper's results is provided at the end of the section. Section 3 introduces a 2-categorical notion of ***oriented 2-pushouts***, a precious tool to prove subsequently [section 3.2] that $[\mathcal{E}]$ enjoys bounded pushouts in every Lévy permutation category $[\mathcal{C}]$ constructed from an ARS $(\mathcal{G}, \rhd)$, [section 3.3] that $[\mathcal{E}]$ and $[\mathcal{M}]$ are orthogonal subcategories of $[\mathcal{C}]$. An ingenious characterisation of the ***factorisation systems*** defined in [FK] is given in Section 4 and subsequently applied to prove that $([\mathcal{E}], [\mathcal{M}])$ is a factorisation system in every Lévy permutation category $[\mathcal{C}]$ constructed from an ARS.

[1] Another simpler example suggested by Vincent van Oostrom (private communication) is the bounded span $F(A,B) \longleftarrow F(A,A) \longrightarrow F(A,C)$ in the first order rewriting system whose rules are $A \longrightarrow B$, $A \longrightarrow C$, $F(B,x) \longrightarrow B$ and $F(C,x) \longrightarrow C$.

[2] In posets, bounded pushouts correspond to bounded binary joins $x \vee y$ and pushouts in $([\mathcal{C}_+]\downarrow a)$ to binary joins $x \vee_a y$ in the principal ideal $\{x, x \leq a\}$. The first notion is stronger than the second one since -1- the existence of $x \vee y$ implies the existence of $x \vee_a y$ whenever $x \leq a$ and $y \leq a$, with in that case $x \vee_a y = x \vee y$, and -2- when $x \vee y$ does not exist the existence and value of $x \vee_a y$ may depend on a.

2 Axiomatic Rewriting Theory

In Section 1, we shew that the Lévy pushout construction is difficult to extend from orthogonal to non orthogonal rewriting systems. The example of the λ_+-calculus bears evidence that only a subcategory of well-behaved computations can enjoy bounded pushouts.

Here, we generalise our prospect and introduce Axiomatic Rewriting Systems (ARS) to describe there -1- the generic construction of a category $[\mathcal{C}]$ of derivations up to Lévy permutation equivalence, generalising in this the constructions above of $[\mathcal{C}_\lambda]$ and $[\mathcal{C}_+]$, -2- the generic standardisation theorem obtained in [GLM,M,1], -3- the generic construction of a subcategory $[\mathcal{E}]$ composed of the ***external*** morphisms in $[\mathcal{C}]$.

ARS. An Axiomatic Rewriting System is defined in [1] as a couple $(\mathcal{G}, \triangleright)$ composed of:

1. a graph $\mathcal{G} = (T, R, \partial_0, \partial_1)$ where T is a set of ***terms***, R is a set of ***redexes***, $\partial_0 : R \longrightarrow T$ and $\partial_1 : R \longrightarrow T$ are respectively the source and target functions. We write $M \xrightarrow{u} N$ when $\partial_0 u = M$ and $\partial_1 u = N$,
2. a binary relation $\triangleright$ between coinitial and cofinal paths of $\mathcal{G}$.

We recall that a ***path*** in a graph $\mathcal{G}$ is a sequence

$$(M_1, u_1, M_2, ..., M_m, u_m, M_{m+1})$$

where $M_i \xrightarrow{u_i} M_{i+1}$ for every $i \in [1...m]$. When $m = 0$, the path (M_1) is said to be ***empty***. Two paths $(M_1, u_1, ..., u_m, M_m)$ and $(N_1, v_1, ..., v_n, N_n)$ are coinitial (resp. cofinal) when $M_1 = N_1$ (resp. $M_m = N_n$). In the sequel, paths are also called ***derivations*** to follow the Rewriting terminology.

Many concrete Rewriting Systems can be modelled as an ARS $(\mathcal{G}, \triangleright)$. For instance, the λ-calculus:

- the graph $\mathcal{G}$ is the transition graph of the calculus, in that case $\mathcal{G}_\lambda$,
- the relation $\triangleright$ mirrors the ***oriented*** redex permutations of the calculus. By oriented permutations we mean either (1) disjoint permutations from $u; v'$ to $v; u'$, (2) standardising permutations from $v; u'$ to $u; f$ (where the derivation f reduces the copies of v through u). In this case, (1) and (2) are mirrored by relations $v; u' \triangleright u; v'$ and $v; u' \triangleright u; f$ in the ARS $(\mathcal{G}, \triangleright)$.

Many important concepts in Rewriting Theory are expressed in $(\mathcal{G}, \triangleright)$ without referring to the concrete underlying calculus. For instance, a ***disjoint*** (resp. ***standardising***) permutation is formally defined in $(\mathcal{G}, \triangleright)$ as a couple (f, g) of (coinitial & cofinal) derivations such that $f \triangleright g$ and $g \triangleright f$ (resp. $f \triangleright g$ but not $g \triangleright f$). A ***one-step*** oriented permutation from d to e is defined in $(\mathcal{G}, \triangleright)$ as a quadruple (d_1, f, g, d_2) such that $f \triangleright g$, $d = d_1; f; d_2$ and $e = d_1; g; d_2$.

Thence, every ARS $(\mathcal{G}, \triangleright)$ gives rise to a 2-category $\mathcal{C}$ whose carrier is $\underline{\mathcal{C}}$, the free category on $\mathcal{G}$:

1. $\mathcal{C}$'s objects are the vertices of the graph $\mathcal{G}$,
2. its morphisms $f : P \longrightarrow Q$ are the derivations from P to Q,
3. its 2-cells $\alpha : f \Rightarrow g$ are the sequences of one-step oriented permutations, up to disjoint permutations [3].

Lévy permutation equivalence $\equiv$ can also be expressed in $(\mathcal{G}, \rhd)$ (or in the 2-category $\mathcal{C}$) as the least equivalence relation containing $\Rightarrow$. In fact, the equivalence classes of $\equiv$ correspond exactly to the connected components of the hom-categories $\mathcal{C}(P, Q)$. The category $[\mathcal{C}]$ is thence defined as the quotient of $\mathcal{C}$ by $\equiv$. The canonical functor $[\cdot]$ from the free category $\underline{\mathcal{C}}$ to the category $[\mathcal{C}]$ transports a derivation d to its Lévy class $[d]$:

$$[\cdot] : \underline{\mathcal{C}} \longrightarrow [\mathcal{C}]$$

STANDARD DERIVATIONS. A derivation d in $(\mathcal{G}, \rhd)$ is called ***standard*** when no standardising permutation appears from d after any sequence of disjoint permutations. The derivation d is thus a $\Rightarrow$-normal form up to disjoint permutations: if $\alpha : d \Rightarrow e$ then α is a sequence of disjoint permutations.

Ten elementary axioms are introduced in [1] to establish the ***standardisation*** theorem which states that there exists in every Lévy class $\equiv$ a unique standard derivation, up to disjoint permutations. Every standard derivation is therefore a canonical representative of its Lévy permutation class.

Assumption: The ten axioms are so important that we integrate them in the definition of ARSs and consider from now that every ARS $(\mathcal{G}, \rhd)$ verifies them.

The standardisation theorem can also be expressed as a property of the 2-category $\mathcal{C}$.

Theorem 1 ([M,1]). *Every 2-category $\mathcal{C}$ constructed from an ARS $(\mathcal{G}, \rhd)$ can be enriched to a* ***standardisation*** *2-category $(\mathcal{C}, \Downarrow)$ — see definition 2.*

Definition 2. A ***standardisation*** 2-category $\mathcal{C}$ is a 2-category equipped with an unary operator $\Downarrow$ on morphisms such that:

1. *$\forall e : P \longrightarrow Q$, if $d \equiv e$ then there exists a unique 2-cell $\alpha : e \Rightarrow \Downarrow_d$,*
2. *if there is a 2-cell $\alpha : \Downarrow_d \Rightarrow e$ then α is a 2-iso: $\Downarrow_d \simeq e$.*

To express this in the categorical idiom, $\Downarrow_d$ is a ***strong terminal object*** *in the 2-connected component of $\mathcal{C}(P, Q)$ which contains d.*

We recall from [1] that two derivations $f, g : P \longrightarrow Q$ are 2-isomorphic in $\mathcal{C}$ (i.e there exist 2-isos $\alpha : f \Rightarrow g$ and $\alpha^{-1} : g \Rightarrow f$) if and only if there is a sequence of ***disjoint*** permutations from P to Q. We write $f \simeq g$ in that case. Observe that assertion 1. in Definition 2 implies that $\Downarrow_d$ is unique in its connected component, up to disjoint permutations: if $d \equiv e$ then $\Downarrow_d \simeq \Downarrow_e$.

[3] To be honest, up to disjoint permutations and a bit more, see [1] for details.

We usually call $\underline{\mathcal{C}}$ the derivation category, $\mathcal{C}$ the 2-category and $[\mathcal{C}]$ the Lévy permutation category of the ARS $(\mathcal{G}, \triangleright)$ they mirror.

EXTERNAL DERIVATIONS. That a standard derivation stands among the very best computations in its Lévy permutation class does not mean that it is judicious at all. In the λ-calculus, the derivation $Ka(Ix) \longrightarrow Kax$ is standard but cannot be judicious because its Lévy class itself is not judicious. We have to find a stronger criterion than standardness to characterise the good computations of a calculus. [M,2] propounds the following criterion. A derivation $e : M \longrightarrow P$ is called ***external*** when the derivation $e; f : M \longrightarrow Q$ is ***standard*** whenever $f : P \longrightarrow Q$ is ***standard***.

In the λ-calculus for instance, the head-redex $(\lambda x.M)N \longrightarrow M[x := N]$ is external but not the redex $u : I((\lambda x.M)N) \longrightarrow I(M[x := N])$ because the derivation $u; f$ where $f : I(M[x := N]) \longrightarrow M[x := N]$ is standard:

$$u; f : \ I((\lambda x.M)N) \longrightarrow I(M[x := N]) \xrightarrow{f} M[x := N]$$

can be standardised to:

$$\Downarrow_{u;f} : \ I((\lambda x.M)N) \longrightarrow (\lambda x.M)N \longrightarrow M[x := N]$$

We observe in [M,2] that the composite $d; e : M \longrightarrow Q$ of two external derivations $d : M \longrightarrow P$ and $e : P \longrightarrow Q$ is also external (immediate from the definition). Consequently, the external derivations form a subcategory $\mathcal{E}$ of the derivation category $\underline{\mathcal{C}}$. Another point: In every Axiomatic Rewriting System, external derivations are standard, therefore all standard normalising derivations are in $\mathcal{E}$ (very easy). For these two reasons, we claim in [M,2] that $\mathcal{E}$ is ***the*** category of well-behaved computations (but are you convinced?)

EXTERNAL VS INTERNAL. However, the category $\mathcal{E}$ is a subcategory of $\underline{\mathcal{C}}$ and some translation is required to transport it to a subcategory of $[\mathcal{C}]$. The category $[\mathcal{E}]$ image of $\mathcal{E}$ under $[\cdot]$ is called the subcategory of *external* morphisms in $[\mathcal{C}]$.

The notion of external derivation has a dual. A derivation $m : M \longrightarrow Q$ is ***internal*** when the derivation $[e] : M \longrightarrow P$ is iso whenever $e \in \mathcal{E}$ and $m \equiv e; f$. To express this another way, a derivation is internal when it contains no external derivation up to $\equiv$. $\mathcal{M}$ denotes the ***class*** of internal derivations (unfortunately we do not know yet that $\mathcal{M}$ is a category, this will be proved in the sequel). Of course, the image $[\mathcal{M}]$ of $\mathcal{M}$ under $[\cdot]$ is a class of morphisms in $[\mathcal{C}]$.

SUMMARY OF THE RESULTS. In this paper, we show that $[\mathcal{E}]$ enjoys bounded pushouts in every ARS and consequently solve the problem opened at the end of Section 1. To speak the truth, we prove something more fundamental perhaps. We show that $([\mathcal{E}], [\mathcal{M}])$ forms a ***factorisation system*** in the sense of Freyd and Kelly [FK]. This robust property means:

1. that $[\mathcal{E}]$ and $[\mathcal{M}]$ are categories,
2. that every morphism f can be factored as $f = e; m$ with $e \in [\mathcal{E}]$ and $m \in [\mathcal{M}]$,

3. that this factorisation is functorial: if $(e;m);v = u;(f;n)$ where $e, f \in [\mathcal{E}]$ and $m, n \in [\mathcal{M}]$, there is a unique w rendering commutative the diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & N \arrow[r, "m"] \arrow[d, dotted, "w"'] & P \arrow[d, "v"] \\ M' \arrow[r, "f"] & N' \arrow[r, "n"] & P' \end{tikzcd}$$

We mention some important consequences of the factorisation theorem:

- if $e_1; e_2 \in [\mathcal{E}]$ then $e_2 \in [\mathcal{E}]$,
- if $m_1; m_2 \in [\mathcal{M}]$ then $m_1 \in [\mathcal{M}]$,
- $[\mathcal{E}]$ is closed under pushouts and $[\mathcal{M}]$ is closed under pullbacks,
- the fibered coproduct of $e_\alpha : A \longrightarrow B_\alpha$ is in $[\mathcal{E}]$ if each e_α is in $[\mathcal{E}]$.

ORTHOGONALITY. One guiding idea in [FK] is that two morphisms can be ***orthogonal*** in a category $\mathcal{C}$. A morphism e is orthogonal to a morphism m when for every commutative diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & Q \arrow[d, "v"] \\ P \arrow[r, "m"'] & N \end{tikzcd}$$

there is a ***unique*** morphism w rendering the diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & Q \arrow[d, "v"] \arrow[dl, dotted, "w"'] \\ P \arrow[r, "m"'] & N \end{tikzcd}$$

commutative: $e; w = u$ and $w; f = v$. In that case, we write $e \perp m$. Beware: the relation $\perp$ is not symmetric.

If $\mathcal{X}$ is a class of morphim in a category $\mathcal{C}$, we write $\mathcal{X}^\downarrow$ the set of morphisms m such that $x \perp m$ for every $x \in \mathcal{X}$, and $\mathcal{X}^\uparrow$ the set of morphisms e such that $e \perp x$ for every $x \in \mathcal{X}$. We say that two classes $\mathcal{X}$ and $\mathcal{Y}$ of morphisms of $\mathcal{C}$ are orthogonal when $\mathcal{X}^\downarrow \supset \mathcal{Y}$ or (equivalently) when $\mathcal{X} \subset \mathcal{Y}^\uparrow$.

3 A 2-categorical proof that $[\mathcal{E}]$ enjoys bounded pushouts

This section is concerned with a proof that $[\mathcal{E}]$ enjoys bounded pushouts. The result is obtained from 2-categorical considerations on ***oriented*** pushout diagrams in the 2-category $\mathcal{C}$ constructed from $(\mathcal{G}, \rhd)$. This detour through 2-categorical techniques should not be a surprise. It testifies that the standardisation structures behind the construction of the Lévy category $[\mathcal{C}]$ cannot be neglected during the analysis of non orthogonal ARSs.

3.1 Oriented 2-pushouts

Oriented pushouts. An ***oriented*** (f,g)-pushout diagram (written OPO in the diagrams) in a 2-category is a diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "f'"] \\ P \arrow[r, "g'"'] & N \end{tikzcd}$$

such that:

1. there is a cell $\alpha : g; f' \Rightarrow f; g'$,
2. for every two morphisms $h_1 : P \longrightarrow O$ and $h_2 : Q \longrightarrow O$, if there is a cell $\beta : g; h_2 \Rightarrow f; h_1$, then there exists a morphism $h : N \longrightarrow O$ and a cell $\gamma : h_2 \Rightarrow f'; h$ such that $h_1 \equiv g'; h$.

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "f'"'] \arrow[ddr, bend left, "h_2"] & \\ P \arrow[r, "g'"] \arrow[drr, bend right, "h_1"'] & N \arrow[dr, dotted, "h"] & \\ & & O \end{tikzcd}$$

3. Moreover, if h' is another morphism such that $h_1 \equiv g'; h'$ and $h_2 \equiv f'; h'$, then $h \equiv h'$. We call this last requirement the ***universality*** condition.

Observe that the definition (point 2.) is not symmetric. This justifies our taxonomy of *oriented* pushouts: in general, a (f,g)-pushout is not a (g,f)-pushout.

Lemma 3 (Horizontal Pasting). .

$$\text{If } \begin{tikzcd} M_1 \arrow[r, "g_1"] \arrow[d, "f"'] \arrow[dr, phantom, "OPO"] & M_2 \arrow[d, "f'"] \\ P_1 \arrow[r, "g_1'"'] & P_2 \end{tikzcd} \text{ and } \begin{tikzcd} M_2 \arrow[r, "g_2"] \arrow[d, "f'"'] \arrow[dr, phantom, "OPO"] & M_3 \arrow[d, "f''"] \\ P_2 \arrow[r, "g_2'"'] & P_3 \end{tikzcd} \text{ then } \begin{tikzcd} M_1 \arrow[r, "g_1;g_2"] \arrow[d, "f"'] \arrow[dr, phantom, "OPO"] & M_3 \arrow[d, "f''"] \\ P_1 \arrow[r, "g_1';g_2'"'] & P_3 \end{tikzcd}$$

Proof. We use the traditional technique of diagram chasing. Let us call (A) and (B) the two first O-pushout diagrams.

First of all, observe (by cell composition) that $g_1; g_2; f'' \Rightarrow f; g_1; g_2$.

Then, let there be two morphisms $h_1 : P_1 \longrightarrow O$ and $h_2 : M_3 \longrightarrow O$ and a cell $\beta : g_1; g_2; h_2 \Rightarrow f; h_1$. We will show the existence of a morphism $h : P_3 \longrightarrow O$ and a cell $\gamma : h_2 \Rightarrow f''; h$ such that $h_1 \equiv g_1'; g_2'; h$. By diagram chasing on (A), there is a morphism $i : P_2 \longrightarrow O$ and a cell $\gamma_1 : g_2; h_2 \Rightarrow f'; i$ such that $h_1 \equiv g_1'; i$. The existence of γ_1 permits to chase on (B) and deduce that there is a morphism $h : P_3 \longrightarrow O$ and a cell $\gamma_2 : h_2 \Rightarrow f''; h$ such that $i \equiv g_2'; h$. The morphism h then verifies the two expected conditions: $h_1 \equiv g_1'; g_2'; h$ and $\gamma = \gamma_2 : h_2 \Rightarrow f''; h$.

$$
\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[d, "f"'] \arrow[dr, phantom, "(A)"] & M_2 \arrow[d, "f'"] \arrow[ddr, bend left, "g_2;h_2"] & \\
P_1 \arrow[r, "g_1'"] \arrow[drr, bend right, "h_1"'] & P_2 \arrow[dr, dotted, "i"] & \\
& & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M_2 \arrow[r, "g_2"] \arrow[d, "f'"'] \arrow[dr, phantom, "(B)"] & M_3 \arrow[d, "f''"] \arrow[ddr, bend left, "h_2"] & \\
P_2 \arrow[r, "g_2'"] \arrow[drr, bend right, "i"'] & P_3 \arrow[dr, dotted, "h"] & \\
& & O
\end{tikzcd}
$$

We show the universality condition on h. Let h' be another morphism such that $h_1 \equiv g_1'; g_2'; h'$ and $h_2 \equiv f''; h'$. The cell $\beta : g_1; g_2; h_2 \Rightarrow f; h_1$ permits to apply the universality condition on (A) and deduce $g_2'; h \equiv i \equiv g_2'; h'$ from the relations:

$$
\begin{gathered}
g_1'; (g_2'; h) \equiv h_1 \equiv g_1'; (g_2'; h') \\
f'; (g_2'; h) \equiv g_2; (f''; h) \equiv g_2; h_2 \equiv g_2; (f''; h') \equiv f'; (g_2'; h')
\end{gathered}
$$

Because $g_2; h_2 \Rightarrow f'; i$, we can apply the universality condition on (B) and deduce from

$$
\begin{array}{rl}
g_2'; h \equiv g_2'; h' & \text{proved above} \\
f''; h \equiv h_2 \equiv f''; h' & \text{hypothesis}
\end{array}
$$

that $h \equiv h'$. We conclude. ■

Lemma 4 (Vertical Pasting). *Suppose that $(\mathcal{C}, \Downarrow)$ is a standardisation 2-category whose morphism f_2 is external.*

$$
\textit{If}\
\begin{tikzcd}
M_1 \arrow[r, "g"] \arrow[d, "f_1"'] \arrow[dr, phantom, "OPO"] & P_1 \arrow[d, "f_1'"] \\
M_2 \arrow[r, "g'"'] & P_2
\end{tikzcd}
\ \textit{and}\
\begin{tikzcd}
M_2 \arrow[r, "g'"] \arrow[d, "f_2"'] \arrow[dr, phantom, "OPO"] & P_2 \arrow[d, "f_2'"] \\
M_3 \arrow[r, "g''"'] & P_3
\end{tikzcd}
\ \textit{then}\
\begin{tikzcd}
M_1 \arrow[r, "g"] \arrow[d, "f_1;f_2"'] \arrow[dr, phantom, "OPO"] & P_1 \arrow[d, "f_1';f_2'"] \\
M_3 \arrow[r, "g''"'] & P_3
\end{tikzcd}
$$

Proof. Less traditional than lemma 3 because we use the fact that f_2 is external. Let us call (A) and (B) the two first O-pushout diagrams.

First of all, observe that $g; f_1'; f_2' \Rightarrow f_1; f_2; g''$ by vertical composition of the cells underlying (A) and (B).

Suppose the existence of two morphisms $h_1 : M_3 \longrightarrow O$ and $h_2 : P_1 \longrightarrow O$ and of a cell $\beta : g; h_2 \Rightarrow f_1; f_2; h_1$. We construct a morphism $h : P_3 \longrightarrow O$ and a cell $\gamma : h_2 \Rightarrow f_1'; f_2'; h$ such that $h_1 \equiv g''; h$. By chasing on (A), there is a morphism $i : P_2 \longrightarrow O$ and a cell $\gamma_1 : h_2 \Rightarrow f_1'; i$ such that $f_2; h_1 \equiv g'; i$. Because f_2 is external, the morphism $f_2; \Downarrow_{h_1}$ is standard, hence $g'; i \Rightarrow f_2; \Downarrow_{h_1}$. This allows to chase on (B) and deduce that there is a morphism $h : P_3 \longrightarrow O$ and a cell $\gamma_2 : i \Rightarrow f_2'; h$ such that $\Downarrow_{h_1} \equiv g''; h$. We deduce from the equivalence $h_1 \equiv \Downarrow_{h_1}$ that h verifies the existential conditions of oriented 2-pushouts: $h_1 \equiv g''; h$ and $\gamma : h_2 \Rightarrow f_1'; f_2'; h$, with γ the vertical composite of the cells γ_1 and γ_2.

$$\begin{tikzcd}
M_1 \arrow[r, "g"] \arrow[d, "f_1"'] \arrow[dr, phantom, "(A)"] & P_1 \arrow[d, "f_1'"] \arrow[ddr, bend left, "h_2"] & \\
M_2 \arrow[r, "g'"] \arrow[drr, bend right, "f_2;h_1"'] & P_2 \arrow[dr, dotted, "i"] & \\
 & & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M_2 \arrow[r, "g'"] \arrow[d, "f_2"'] \arrow[dr, phantom, "(B)"] & P_2 \arrow[d, "f_2'"] \arrow[ddr, bend left, "i"] & \\
M_3 \arrow[r, "g''"] \arrow[drr, bend right, "\Downarrow_{h_1}"'] & P_3 \arrow[dr, dotted, "h"] & \\
 & & O
\end{tikzcd}$$

We show that the morphism h is unique up to $\equiv$. Let $h' : P_3 \longrightarrow O$ be another morphism $h' : P_3 \longrightarrow O$ such that $g''; h \equiv h_1 \equiv g''; h'$ and $f_1'; f_2'; h \equiv h_2 \equiv f_1'; f_2'; h'$. The existence of a cell $g; h_2 \Rightarrow f_1; f_2; h_1$ permits to apply the universality condition on (A) and derive the equivalence $f_2'; h \equiv i \equiv f_2'; h'$ from

$$g'; (f_2'; h) \equiv f_2; (g''; h) \equiv f_2; (g''; h') \equiv g'; (f_2'; h') \quad \text{and} \quad f_1'; (f_2'; h) \equiv f_1'; (f_2'; h')$$

Because $g'; i \Rightarrow f_2; \Downarrow_{h_1}$, we can apply the universality condition on (B) to establish $h \equiv h'$ from

$$g''; h \equiv h_1 \equiv g''; h' \quad \text{and} \quad f_2'; h \equiv f_2'; h'$$

We conclude. ∎

Lemma 5. *Let $(\mathcal{C}, \Downarrow)$ be a standardisation 2-category. If $e : M \longrightarrow P$ is external in an oriented (e, f)-pushout diagram of the form*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "e'"] \\
P \arrow[r, "f'"'] & N
\end{tikzcd}$$

then $e' : Q \longrightarrow N$ is external too.

Proof. Easy. Let $h : N \longrightarrow O$ be any morphism. We will show that $e'; \Downarrow_h \simeq \Downarrow_{e';h}$ and conclude. Consider the following diagram:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "e'"] \arrow[ddr, bend left, "\Downarrow_{e';h}"] & \\
P \arrow[r, "f'"] \arrow[drr, bend right, "\Downarrow_{f';h}"'] & N & \\
 & & O
\end{tikzcd}$$

The series of equivalence

$$f; \Downarrow_{e';h} \equiv f; e'; h \equiv e; f'; h$$

implies the existence of a cell $f; \Downarrow_{e';h} \Rightarrow \Downarrow_{e;f';h}$. Because the derivation e is external, $\Downarrow_{e;f';h} \simeq e; \Downarrow_{e';h}$ and we deduce the existence of a cell

$$\gamma : f; \Downarrow_{e';h} \Rightarrow e; \Downarrow_{f';h}$$

The existence of γ allows to chase on the O-pushout diagram and deduce the existence of a morphism i such that

$$\Downarrow_{e';h} \Rightarrow e'; i \qquad \Downarrow_{f';h} \equiv f'; i \tag{1}$$

The equivalence $e';h \equiv e';i$ and $f';h \equiv f';i$ follow (1). By universality, the equivalence $h \equiv i$ follows. The definition of a standardisation 2-category tells that $\Downarrow_h$ is ***terminal*** in its connected component in $\mathcal{C}(N,O)$, in particular that there exists a cell $i \Rightarrow \Downarrow_h$. We deduce from this and (1) that

$$\Downarrow_{e';h} \Rightarrow e'; i \Rightarrow e'; \Downarrow_h$$

The definition of a standardisation 2-category tells also that $\Downarrow_{e';h}$ is ***strong*** in its connected component in $\mathcal{C}(Q,O)$, thus that $\Downarrow_{e';h} \simeq e'; \Downarrow_h$. We conclude. ■

3.2 Consequence 1 on ARSs: The subcategory $[\mathcal{E}]$ enjoys bounded pushouts

We apply the 2-categorical results of section 3.1 to ARSs. First of all, we import a lemma from [M]:

Lemma 6. *Let $\mathcal{C}$ be the 2-category of an ARS $(\mathcal{G}, \rhd)$. Suppose that $P \xleftarrow{r} M \xrightarrow{f} Q$ is a span in $\mathcal{C}$ and that r is a redex. If there exists two derivations h_1, h_2 and a cell $\gamma : f; h_2 \Rightarrow r; h_1$, then there exists an oriented 2-pushout diagram:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "r"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "r'"] \\ P \arrow[r, "f'"'] & N \end{tikzcd}$$

where r' is either a redex or an empty derivation.

Proof. The property is a consequence of two results of [M]: the ***transitivity*** lemma (see lemma 4.21) and the ***left-simplification*** or ***epi*** theorem (see theorem 4.58). ■

We use the Vertical Pasting lemma 4 and lemma 6 to prove the following theorem, a key step towards theorem 8.

Theorem 7. *Let $\mathcal{C}$ be the 2-category of an ARS $(\mathcal{G}, \rhd)$, and let $P \xleftarrow{e} M \xrightarrow{f} Q$ be a span in $\mathcal{C}$. Suppose that e is external: $e \in \mathcal{E}$, and that there exists two derivations h_1 and h_2 such that $e; h_1 \equiv f; h_2$. Then, there is an oriented 2-pushout diagram in $\mathcal{C}$:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "e'"] \\ P \arrow[r, "f'"'] & N \end{tikzcd}$$

Moreover, the morphism e' is external: $e' \in \mathcal{E}$.

Proof. By induction on the number of rewrite steps in the external derivation e. First of all, the equivalence $e;h_1 \equiv f;h_2$ and $e \in \mathcal{E}$ implies that $f;h_2 \Rightarrow e;\Downarrow_{h_1}$. Suppose that $e = r;E$ for a redex r. The existence of a cell $f;h_2 \Rightarrow r;(E;\Downarrow_{h_1})$ permits to apply lemma 6 and deduce that there is an oriented (r,f)-pushout. In particular, there exists a morphism $h : N' \longrightarrow O$ such that $h_2 \Rightarrow r';h$ and $h \equiv E;\Downarrow_{h_1}$:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "r"'] & Q \arrow[d, "r'"] \arrow[dd, bend left, "h_2"] \\
P' \arrow[r, "F"] \arrow[dr, bend right, "E;\Downarrow_{h_1}"'] & N' \arrow[d, dotted, "h"] \\
 & O
\end{tikzcd}$$

Because $e \in \mathcal{E}$, the ***right decomposition*** lemma, see [M,2], establishes that $E \in \mathcal{E}$. By externality of E, $F;h \Rightarrow E;\Downarrow_{h_1}$. By induction hypothesis, there is an oriented (E,F)-pushout diagram, which by lemma 4 can be pasted to the OPO-diagram above. We conclude.

Externality of e' is a corollary of lemma 5. ∎

Theorem 8 (bounded pushouts). *Let $[\mathcal{C}]$ be the Lévy permutation category of an ARS $(\mathcal{G}, \rhd)$ and let $P \xleftarrow{e} M \xrightarrow{f} Q$ be a bounded span in $[\mathcal{C}]$. If e is external: $e \in [\mathcal{E}]$, then there exists a pushout diagram of the form:*

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "PO"] & Q \arrow[d, "e'"] \\
P \arrow[r, "f'"'] & N
\end{tikzcd}$$

More over, the morphism $e' : Q \longrightarrow N$ is external: $e' \in [\mathcal{E}]$.

Proof. Fairly simple with theorem 7. The equality $e;h_1 = f;h_2$ in $[\mathcal{C}]$ implies that $E;H_1 \equiv F;H_2$ for derivations E, F, H_1 and H_2 in the permutation classes e, f, h_1 and h_2. Moreover, since $e \in [\mathcal{E}]$, the derivation E can be chosen external: $E \in \mathcal{E}$. This establishes the existence of a cell $F;H_2 \Rightarrow E;\Downarrow_{H_1}$. Henceforth, there is an oriented (E,F)-pushout diagram in $\mathcal{C}$:

$$\begin{tikzcd}
M \arrow[r, "F"] \arrow[d, "E"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "E'"] \\
P \arrow[r, "F'"'] & N
\end{tikzcd}$$

By externality of E, it should be clear that the diagram:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "OPO"] & Q \arrow[d, "e'"] \\
P \arrow[r, "f'"'] & N
\end{tikzcd}$$

is a pushout in $[\mathcal{C}]$ for e' and f' the permutation classes of E' and F'. Observe that e' is external: $e' \in [\mathcal{E}]$, because, by theorem 7, $E' \in \mathcal{E}$. We conclude. ∎

3.3 Consequence 2 on ARSs: the classes $[\mathcal{E}]$ and $[\mathcal{M}]$ are orthogonal in $[\mathcal{C}]$

We use theorem 8 to prove that external morphisms and internal morphisms are orthogonal in $[\mathcal{C}]$.

Corollary 9 (orthogonality). *Let $[\mathcal{C}]$ be the Lévy permutation category of an ARS $(\mathcal{G}, \rhd)$. If $e \in [\mathcal{E}]$ and $m \in [\mathcal{M}]$ then $e \perp m$.*

Proof. Let the following diagram be commutative in $[\mathcal{C}]$:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & Q \arrow[d, "v"] \\ P \arrow[r, "m"'] & O \end{tikzcd}$$

By theorem 8 there is a pushout diagram $e; u' = u; e'$, and therefore a commutative diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & Q \arrow[d, "u'"] \arrow[ddr, bend left, "v"] & \\ P \arrow[r, "e'"] \arrow[drr, bend right, "m"'] & N \arrow[dr, dotted, "h"] & \\ & & O \end{tikzcd}$$

with $e' \in [\mathcal{E}]$ and $m = e'; h$. Because m is internal and e' is external by theorem 8, the morphism e' is an iso. This establishes that there is a pushout diagram of the form (we write $w = u'; (e')^{-1}$).

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] \arrow[dr, phantom, "PO"] & Q \arrow[d, "w"] \\ P \arrow[r, equal, "\mathbf{id}_P"'] & P \end{tikzcd}$$

Observe that $w; m = v$ and that $w = w'$ whenever $e; w' = u$. The property $e \perp m$ follows. ∎

4 The factorisation theorem

4.1 Three equivalent definitions

This section is concerned with various (three) equivalent definitions of a factorisation system. The two first definitions appear in a seminal paper by Freyd and

Kelly, see [FK]. The third definition was specially designed [4] to apply on ARSs and solve our specific rewriting problem: to prove that $([\mathcal{E}], [\mathcal{M}])$ is a factorisation system of the Lévy permutation category $[\mathcal{C}]$.

REMARK. Sections 4.1 and 4.2 use $\mathcal{C}$ to denote a category, $\mathcal{E}$ and $\mathcal{M}$ to denote two classes of morphisms in $\mathcal{C}$. In fact, we reuse the notations of [FK] and forget for some time (except when explicitly mentionned) our rewriting theoretic meanings of $\mathcal{E}$ and $\mathcal{M}$.

FACTORISATION SYSTEM. A factorisation system on a category $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ such that:

1. *every morphism f in $\mathcal{C}$ can be factored as $f = e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,*
2. *$\mathcal{E} \subset \mathcal{M}^{\uparrow}$ and $\mathcal{M} \subset \mathcal{E}^{\downarrow}$,*
3. *both $\mathcal{E}$ and $\mathcal{M}$ contain the isos and are closed under composition.*

SECOND DEFINITION. Unfortunately, we cannot use directly the definition above: we do not know yet that our class $[\mathcal{M}]$ of internal morphisms in $[\mathcal{C}]$ is a category. There is another possibility: to use the following lemma, also appearing in [FK].

Lemma 10 ([FK]). *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. $(\mathcal{E}, \mathcal{M})$ is a factorisation system if and only if:*

1. *every morphism f in $\mathcal{C}$ can be factored as $f = e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,*
2. *$\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$.*

THIRD DEFINITION. Again, we cannot apply lemma 10 on ARSs, this time because we do not know if the equalities $[\mathcal{E}] = [\mathcal{M}]^{\uparrow}$ and $[\mathcal{M}] = [\mathcal{E}]^{\downarrow}$ hold in our rewriting categories. So, we take the opportunity to establish another characterisation of factorisation systems, see section 4.2 for a proof.

Lemma 11 (characterisation). *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. $(\mathcal{E}, \mathcal{M})$ is a factorisation system if and only if:*

1. *every morphism f in $\mathcal{C}$ can be factored as $f = e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,*
2. *$\mathcal{E} \subset \mathcal{M}^{\uparrow}$ and $\mathcal{M} \subset \mathcal{E}^{\downarrow}$,*
3. *$e \in \mathcal{E}$ and j iso imply that $e; j \in \mathcal{E}$,*
4. *$m \in \mathcal{M}$ and j iso imply that $j; m \in \mathcal{M}$.*

To our knowledge, lemma 11 is the weakest characterisation of factorisation systems in the literature. Section 4.3 applies the characterisation to establish that $([\mathcal{E}], [\mathcal{M}])$ is a factorisation system of the Lévy permutation category $[\mathcal{C}]$.

[4] I believed for some time that the third characterisation was mine. In fact, the result was already known to some people as I could check later on the category mailing list, look at [AHS,DT,T] for more information.

4.2 A useful characterisation of factorisation systems

This section is devoted to the proof of lemma 11. First of all, we prove that simple and nice result:

Lemma 12. *Let e be epi or m mono or $e \perp m$. Then*

$$e; f \perp m \Rightarrow f \perp m$$

Proof. suppose that $e; f \perp m$. Let the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y_1 \arrow[d, "v"] \\ Y_2 \arrow[r, "m"'] & Z \end{tikzcd}$$

be commutative. We may left compose it with e to obtain a diagram:

$$\begin{tikzcd} X_0 \arrow[r, "e"] & X \arrow[r, "f"] \arrow[d, "u"'] & Y_1 \arrow[d, "v"] \\ & Y_2 \arrow[r, "m"'] & Z \end{tikzcd}$$

By $e; f \perp m$, there exists a morphism $w : Y_1 \longrightarrow Y_2$ such that $e; f; w = e; u$ and $w; m = v$. We initiate a case study to show that $f; w = u$:

1. if the morphism e is epi: the two arrows $f; w$ and u are equal,
2. if the morphism m is mono: $f; (w; m) = f; v = u; m$ implies that $f; w = u$,
3. if the morphism e is orthogonal to m, then consider the commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "e"] \arrow[dr, "e;u"'] & X \arrow[d, "w'"] \arrow[dr, "f;v"] & \\ & Y_2 \arrow[r, "m"'] & Z \end{tikzcd}$$

   and conclude that $u = w' = f; w$ from the unicity of the arrow w'.

We show the unicity of w: suppose that $f; x = u$ and $x; m = v$ for a morphism $x : X \longrightarrow Y_2$; then $e; f; w' = e; u$ and therefore $x = w$ by definition of $e; f \perp m$. Consequently the unicity of w is trivial.

We conclude that $e \perp m$. ■

Lemma 12 may be dualised:

Lemma 13. *Let m be mono or e be epi or $e \perp m$. Then*

$$e \perp f; m \Rightarrow e \perp f$$

Lemma 14 (Characterisation lemma 11). *Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms in a category $\mathcal{C}$. The couple $(\mathcal{E}, \mathcal{M})$ is a factorisation system if and only if:*

1. *every morphism f in $\mathcal{C}$ can be factored as $f = e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,*
2. *$\mathcal{E} \subset \mathcal{M}^{\uparrow}$ and $\mathcal{M} \subset \mathcal{E}^{\downarrow}$,*
3. *$e \in \mathcal{E}$ and j iso imply that $e; j \in \mathcal{E}$,*
4. *$m \in \mathcal{M}$ and j iso imply that $j; m \in \mathcal{M}$.*

Proof. If $(\mathcal{E}, \mathcal{M})$ is a factorisation system then the four assertions are true. We will prove that $\mathcal{M}^{\uparrow} \subset \mathcal{E}$ and $\mathcal{E}^{\downarrow} \subset \mathcal{M}$ to apply lemma 10 and show the converse. Let $t \in \mathcal{M}^{\uparrow}$. This morphism may be factored in $t = e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Since $e \in \mathcal{E}$ and $\mathcal{E} \subset \mathcal{M}^{\uparrow}$ we obtain that $e \in \mathcal{M}^{\uparrow}$ and hence e is orthogonal to any morphism in $\mathcal{M}$. That $m \in \mathcal{M}^{\uparrow}$ follows lemma 12.

In particular, the morphism m is orthogonal to itself and therefore m is iso. By hypothesis 4., the morphism $t = e; m$ is in $\mathcal{E}$. We conclude that $\mathcal{E} = \mathcal{M}^{\uparrow}$. The other equation $\mathcal{M} = \mathcal{E}^{\downarrow}$ is proved dually. Lemma 10 shows that $(\mathcal{E}, \mathcal{M})$ is a factorisation system. ∎

4.3 Consequence: The external-internal factorisation theorem in every ARS

It is now easy to apply lemma 14 and prove the factorisation theorem 15 on every Lévy permutation category $[\mathcal{C}]$ given by an ARS $(\mathcal{G}, \triangleright)$. Point 2. of lemma 14 is corollary 9. Point 3. and 4. are immediate consequences of the fact that the only isos in $[\mathcal{C}]$ are the identities.

Thus, we only need to prove the first point of the lemma, which means here that every morphism F in $[\mathcal{C}]$ can be factored as $E; M$ with $E \in [\mathcal{E}]$ and $M \in [\mathcal{M}]$. Because every morphism F in $[\mathcal{C}]$ is of the form $[f]$ for f in $\underline{\mathcal{C}}$, we only need to prove that every derivation f in $\underline{\mathcal{C}}$ is Lévy equivalent to some $e; m$ for $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

The ***length*** of a derivation d is its number of rewrite steps. Let n be the length of $\Downarrow_f$. We say that a derivation $e : M \longrightarrow Q$ is a ***prefix*** of $d : M \longrightarrow Q$ (up to $\simeq$) when there exists $h : P \longrightarrow Q$ such that $d \simeq e; h$. The length of a prefix of $\Downarrow_f$ being less than n, there exists among its prefixes an external derivation e of ***maximal length***. We show that the derivation g which verifies $f \simeq e; g$ is ***internal***. Suppose that $[g] = [e']; [h]$ for some external derivation e'. From that follows a series of equivalence $g \equiv e'; h$ and $f \equiv e; g \equiv (e; e'; h)$ which induces by theorem 1 and $e, e' \in \mathcal{E}$ that

$$\Downarrow_f \simeq \Downarrow_{e;e';h} \simeq e; \Downarrow_{e';h} \simeq (e; e'); \Downarrow_h$$

But $\mathcal{E}$ is a subcategory of $\underline{\mathcal{C}}$ and therefore $e; e'$ is an external prefix of $\Downarrow_f$. By definition of e, the derivations e and $e; e'$ have the same length, hence e' is iso (=empty). Thus, $g \in \mathcal{M}$. Point 1. follows the $[\mathcal{E}][\mathcal{M}]$-factorisation $F = [e]; [g]$ of $F = [f]$.

Theorem 15 (factorisation). *The couple $([\mathcal{E}], [\mathcal{M}])$ is a factorisation system in every Lévy permutation category $[\mathcal{C}]$ constructed from an Axiomatic Rewriting System $(\mathcal{G}, \triangleright)$.*

In particular, every Axiomatic Rewriting System $(\mathcal{G}, \triangleright)$ (and related categories $\underline{\mathcal{C}}$, $[\mathcal{C}]$, $\mathcal{E}$, $[\mathcal{E}]$, $\mathcal{M}$, $[\mathcal{M}]$) verifies the three following properties:

1. the class $\mathcal{M}$ of internal derivations is closed under composition,
2. every morphism f in $[\mathcal{C}]$ (= permutation class) can be factored as $f = e; m$ with $e \in [\mathcal{E}]$ and $m \in [\mathcal{M}]$,

3. this factorisation is functorial: if $(e;m);v = u;(f;n)$ where $e, f \in [\mathcal{E}]$ and $m, n \in [\mathcal{M}]$, there is a unique w in $[\mathcal{C}]$ rendering commutative the diagram:

$$\begin{tikzcd}
M \arrow[r, "e"] \arrow[d, "u"'] & N \arrow[r, "m"] \arrow[d, dotted, "w"'] & P \arrow[d, "v"] \\
M' \arrow[r, "f"] & N' \arrow[r, "n"] & P'
\end{tikzcd}$$

5 Conclusion

Duplication is the source of many conceptual and technical difficulties in rewriting theory. To use an ***external-internal*** factorisation and establish a non-duplicating realm of ***external*** computations is a new[5] and promising idea. With standardisation, it is one of the very few generic techniques available on non confluent rewriting systems.

The factorisation theorem is the first explicit borrowing from category theory in the author's axiomatic exploration of rewriting systems. The robustness of the property should convince other authors that categorical concepts can be fruitfully imported to rewriting theory. It also justifies the abstract approach to rewriting developed in [M], in particular the definition of external derivations.

Let us conclude the paper with a short survey of the interesting properties of the category $[\mathcal{E}]$ of external derivations, interwoven with future directions.

- two external derivations e_1 and e_2 equivalent by permutation: $e_1 \equiv e_2$ are also equivalent by ***disjoint*** permutation: $e_1 \simeq e_2$,
- in the category $[\mathcal{E}]$, every morphism e is an epi: if $e; e_1 \simeq e; e_2$, then $e_1 \simeq e_2$; and a mono: if $e_1; e \simeq e_2; e$, then $e_1 \simeq e_2$,
- every slice category $(M{\downarrow}[\mathcal{E}])$ is a partial order with bounded joins ***and*** bounded meets. Reckon that the slice categories $(M{\downarrow}[\mathcal{C}_\lambda])$ called ***derivation spaces*** in [Lé,HL] are only $\sqcup$-semi-lattices because they do not have meets. The existence of bounded meets in $(M{\downarrow}[\mathcal{E}])$ shall be used in [4] to establish a generic ***syntactic*** stability theorem.
- in important calculi like orthogonal first-order rewriting systems [HL], the λ-calculus [Lé] and the $\lambda\sigma$-calculus [M,2], every external strategy normalises. In particular, the ***external*** derivations from a ***normalising*** term M form a ***complete*** rewriting system (i.e. confluent and strongly normalising). This observation should help to apply homological techniques, see [La,LP], to these frameworks.

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[5] But ideas of that nature already appear in a paper written by Boudol in the mid'80s, see [Bo], where every computation is projected to its ***needed*** component.

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