Computational Interpretation for
Natural Deduction plus cut and structural rules

The λlxr-calculus

$(Terms)$	$t, u ::=$		x	$variable$
		$\mid$	$\lambda x.t$	$abstraction$
		$\mid$	$t\ u$	$application$
		$\mid$	$t[x/u]$	$substitution$
		$\mid$	$W_x(t)$	$weakening$
		$\mid$	$C_x^{y,z}(t)$	$contraction$

We only consider *well-formed* terms :

- Linearity
- Compulsory presence
- Barendregt's convention

Typing Rules for the λlxr-calculus

$$\frac{}{x : A \vdash x : A}\ (ax) \qquad \frac{\Delta \vdash u : B \quad \Gamma, x : B \vdash t : A}{\Gamma, \Delta \vdash t[x/u] : A}\ (cut)$$

$$\frac{\Gamma \vdash t : A \to B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash (t\ u) : B}\ (\to e) \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ (\to$$

$$\frac{\Gamma, x : A, y : A \vdash t : B}{\Gamma, z : A \vdash C_z^{x,y}(t) : B}\ (c) \qquad \frac{\Gamma \vdash t : A}{\Gamma, x : B \vdash W_x(t) : A}\ (w)$$

where Γ, Δ is only defined if Γ and Δ do not share variables.

Congruence I

AC of contraction :

$$\begin{array}{lcll} C_w^{x,v}(C_x^{z,y}(t)) & \equiv & C_w^{x,y}(C_x^{z,v}(t)) & \text{if } x \neq y, v \\ C_x^{y,z}(t) & \equiv & C_x^{z,y}(t) & \\ C_{x'}^{y',z'}(C_x^{y,z}(t)) & \equiv & C_x^{y,z}(C_{x'}^{y',z'}(t)) & \text{if } x \neq y', z'\ \&\ x' \neq y, z \end{array}$$

C of weakening :

$$W_x(W_y(t)) \equiv W_y(W_x(t))$$

Congruence II

Commutativity of substitutions :

$$t[x/u][y/v] \equiv t[y/v][x/u] \text{ if } y \notin FV(u) \ \& \ x \notin FV(v)$$

Contraction and substitution have the same status :

$$C_w^{y,z}(t)[x/u] \equiv C_w^{y,z}(t[x/u]) \text{if } x \neq w \ \& \ y, z \notin FV(u)$$

Reduction Rules for the λlxr-calculus

$$(B) \quad (\lambda x.t)\ u \to \ t[x/u]$$

SubSystem x

(Abs)	$(\lambda y.t)[x/u]$	$\to$	$\lambda y.t[x/u]$	
$(App1)$	$(t\ v)[x/u]$	$\to$	$t[x/v]\ v$	if $x \in FV(t)$
$(App2)$	$(t\ v)[x/u]$	$\to$	$t\ v[x/u]$	if $x \in FV(v)$
(Var)	$x[x/u]$	$\to$	u	
$(Weak1)$	$W_x(t)[x/u]$	$\to$	$W_{FV(u)}(t)$	
$(Weak2)$	$W_y(t)[x/u]$	$\to$	$W_y(t[x/u])$	if $x \neq y$
$(Cont1)$	$C_x^{y,z}(t)[x/u]$	$\to$	$C_\Phi^{\Delta,\Pi}(t[y/u_1][z/u_2])$	
			where $\Phi := FV(u)$	
			$u_1 = R_\Delta^\Phi(u)$ and $u_2 = R_\Pi^\Phi(u)$	
$(Comp)$	$t[y/v][x/u]$	$\to$	$t[y/v[x/u]]$ if $x \in FV(v)$	

SubSystem t

$(WAbs)$	$\lambda x.W_y(t)$	$\to$	$W_y(\lambda x.t)$	$x \neq y$
$(WApp1)$	$W_y(u)\ v$	$\to$	$W_y(u\ v)$	
$(WApp2)$	$u\ W_y(v)$	$\to$	$W_y(u\ v)$	
$(WSubs)$	$t[x/W_y(u)]$	$\to$	$W_y(t[x/u])$	
$(Merge)$	$C_w^{y,z}(W_y(t))$	$\to$	$R_w^z(t)$	
$(Cross)$	$C_w^{y,z}(W_x(t))$	$\to$	$W_x(C_w^{y,z}(t))$	$x \neq y, \ \ x \neq z$
$(CAbs)$	$C_w^{y,z}(\lambda x.t)$	$\to$	$\lambda x.C_w^{y,z}(t)$	
$(CApp1)$	$C_w^{y,z}(t\ u)$	$\to$	$C_w^{y,z}(t)\ u$	$y, z \in FV(t)$
$(CApp2)$	$C_w^{y,z}(t\ u)$	$\to$	$t\ C_w^{y,z}(u)$	$y, z \in FV(u)$
$(CSubs)$	$C_w^{y,z}(t[x/u])$	$\to$	$t[x/C_w^{y,z}(u)]$	$y, z \in FV(u)$

The reduction relation $\lambda\mathtt{lxr}$

The reduction relation is generated by the previous rewriting rules and congruence axioms :

$$t \rightarrow_{\lambda\mathtt{lxr}} t' \text{ iff } \exists t_1, t_2 \; t \equiv t_1 \rightarrow_{B+\mathtt{x}+\mathtt{t}} t_2 \equiv t'$$

Example

$$\begin{array}{ll}
(\lambda x.W_u(C_x^{y,z}(y\ z)))\ w & \rightarrow \\
W_u(C_x^{y,z}(y\ z))[x/w] & \rightarrow \\
W_u(C_x^{y,z}(y\ z)[x/w]) & \rightarrow \\
W_u(C_w^{w_1,w_2}((y\ z)[y/w_1][z/w_2])) & \rightarrow \\
W_u(C_w^{w_1,w_2}((y[y/w_1]\ z)[z/w_2])) & \rightarrow \\
W_u(C_w^{w_1,w_2}(y[y/w_1]\ z[z/w_2])) & \rightarrow \\
W_u(C_w^{w_1,w_2}(w_1\ z[z/w_2])) & \rightarrow \\
W_u(C_w^{w_1,w_2}(w_1\ w_2)) &
\end{array}$$

Properties of the $\lambda\mathtt{lxr}$-calculus

1. (Full composition) $t[x/v] \rightarrow^* t\{x = v\}$
   for an *appropiate* notion of meta-substitution and even when t contains non-evaluated substitutions
2. (Free variables are preserved) If $t \rightarrow_{\lambda\mathtt{lxr}} t'$, then $FV(t) = FV(t')$
3. (Subject reduction) If $\Gamma \vdash t : A$ et $t \rightarrow_{\lambda\mathtt{lxr}} t'$, then $\Gamma \vdash t' : A$.
4. (Convergence) $\mathtt{xt} = \mathtt{x} \cup \mathtt{t}$ is convergent (terminating and confluent).
   Which is the form of a term in $\mathtt{xt}$-normal form ?

Connexion with λ-calculus

$\mathcal{B}()$ hides resource control

$$\lambda\mathtt{lxr} \quad \underset{\mathcal{A}()}{\overset{\mathcal{B}()}{\rightleftarrows}} \quad \lambda$$

$\mathcal{A}()$ introduces resource operators

From λlxr to λ

$$
\begin{array}{lcl}
\mathcal{B}(x) & = & x \\
\mathcal{B}(\lambda x.t) & = & \lambda x.\mathcal{B}(t) \\
\mathcal{B}(W_x(t)) & = & \mathcal{B}(t) \\
\mathcal{B}(C_x^{y,z}(t)) & = & \mathcal{B}(t)\{y \leftarrow x\}\{z \leftarrow x\} \\
\mathcal{B}(t\ u) & = & \mathcal{B}(t)\ \mathcal{B}(u) \\
\mathcal{B}(t[x/u]) & = & \mathcal{B}(t)\{x \leftarrow \mathcal{B}(u)\}
\end{array}
$$

Relating λlxr and λ

Lemma

1. If $M \equiv N$, then $\mathcal{B}(M) = \mathcal{B}(N)$.
2. If $M \to_B\ N$, then $\mathcal{B}(M) \to_\beta^*\ \mathcal{B}(N)$.
3. If $M \to_{\mathtt{xt}}\ N$, then $\mathcal{B}(M) = \mathcal{B}(N)$.

Proposition [Projecting λlxr-reductions]
$M \to_{\lambda\mathtt{lxr}}\ N$, then $\mathcal{B}(M) \to_\beta^*\ \mathcal{B}(N)$.

From λ to λlxr

$$
\begin{array}{lcll}
\mathcal{A}(x) & := & x & \\
\mathcal{A}(\lambda x.t) & := & \lambda x.\mathcal{A}(t) & \text{if } x \in FV(t) \\
\mathcal{A}(\lambda x.t) & := & \lambda x.W_x(\mathcal{A}(t)) & \text{if } x \notin FV(t) \\
\mathcal{A}(tu) & := & C_\Phi^{\Delta,\Pi}(R_\Delta^\Phi(\mathcal{A}(t))\ R_\Pi^\Phi(\mathcal{A}(u))) & \text{where } \Phi := FV(t) \cap F
\end{array}
$$

Example : $\mathcal{A}(\lambda x.y\ y) = \lambda x.W_x(C_y^{z,z'}(z\ z'))$

Relating λ and λlxr

Lemma For all λ-terms t and u such that $x \in FV(t)$, we have

$$C_\Phi^{\Delta,\Pi}(R_\Delta^\Phi(\mathcal{A}(t))[x/R_\Pi^\Phi(\mathcal{A}(u))]) \to_{\mathtt{xt}}^*\ \mathcal{A}(t\{x \leftarrow u\})$$

where $\Phi := (FV(t) \setminus \{x\}) \cap FV(u)$.

Proposition [Simulating β-reductions]
If $t \to_\beta\ t'$, then $\mathcal{A}(t) \to^+{}_{\lambda\mathtt{lxr}}\ W_{FV(t) \setminus FV(t')}(\mathcal{A}(t'))$.

Exemple $t = (\lambda x.y)z \to_\beta\ y = t'$ and
$\mathcal{A}(t) = (\lambda x.W_x(y))z \to^+{}_{\lambda\mathtt{lxr}}\ W_z(\mathcal{A}(y))$.

Connexion with λ-calculus (Summary)

$\mathcal{B}()$ hides resource control

$$t \to^*_{\mathtt{xt}} W_\Pi(\mathcal{A}(\mathcal{B}(t))) \qquad \lambda\mathtt{lxr} \quad \underset{\mathcal{A}()}{\overset{\mathcal{B}()}{\rightleftarrows}} \quad \lambda \qquad t = \mathcal{B}(\mathcal{A}(t))$$

$\mathcal{A}()$ introduces resource operators

Example : $(\lambda x.W_x(y))W_z(z') \to^*_{\mathtt{xt}} W_z(W_{z'}(y))$.

Confluence

Lemma The $\mathtt{xt}$-normal form of t is $W_{FV(t)\setminus FV(\mathcal{B}(t))}(\mathcal{A}(\mathcal{B}(t)))$.

Exemple Let $t = C_x^{x_1,x_2}((\lambda y.x_1\ (x_2\ W_y(z)))\ W_k(w))$. Then $\mathtt{xt}(t) = W_k((\lambda y.W_y(C_x^{x_1,x_2}(x_1\ (x_2\ z))))\ w)$.

Theorem [Confluence modulo] The reduction relation $\lambda\mathtt{lxr}$ is confluent (even on terms with meta-variables).

More Properties of the $\lambda\mathtt{lxr}$-calculus

1. (Preservation of typing)

   (a) If $\Gamma \vdash_\lambda t : A$ then $\Gamma \vdash_{\lambda\mathtt{lxr}} W_{\Gamma\setminus FV(t)}(\mathcal{A}(t)) : A$

   (b) If $\Gamma \vdash_{\lambda\mathtt{lxr}} t : A$ then $\Gamma \vdash_\lambda \mathcal{B}(t) : A$

2. (PSN) If $M \in SN^\beta$ then $\mathcal{A}(M) \in SN^{\lambda\mathtt{lxr}}$.

   breaks Melliès' counter-example of non-termination
   (with $t[y/v][x/u] \to\ t[y/v[x/u]]$ if $x \notin t$)

Strong Normalization and Proof Nets

Translating types :

$$\begin{array}{lcll} A^* & = & A & \text{for atomic types} \\ (A \to B)^* & = & ?((A^*)^\perp) \parr B^* & \text{otherwise} \end{array}$$

Translating terms :

The application $T(B_1, \ldots, B_n \vdash t : A)$ gives a proof-net having wires labelled with $?((B_1^*)^\perp)$, ..., $(?(B_n^*)^\perp)$, A^*.

Details appear in file
`http://www.pps.univ-paris-diderot.fr/~kesner/`

enseignement/mpri/ll/translating-lambdalxr-terms.pdf.

Simulating λlxr with Proof Nets

Lemma Let t, t' be a λlxr-typed terms s.t. $\Gamma \vdash t : A$ and $\Gamma \vdash t' : A$.

- If $t \equiv t'$, then $T(\Gamma \vdash t : A) \sim_E T(\Gamma \vdash t' : A)$.
- If $t \to_B t'$, then $T(\Gamma \vdash t : A) \to^+{}_{R/E} T(\Gamma \vdash t' : A)$.
- If $t \to_{\mathtt{xt}} t'$, then $T(\Gamma \vdash t : A) \to^*_{R/E} T(\Gamma \vdash t' : A)$.

So that λlxr is sound w.r.t proof-nets :
If t is λlxr-typed, then $t \to_{\lambda\mathtt{lxr}} t'$ implies $T(\Gamma \vdash t : A) \to^*_{R/E} T(\Gamma \vdash t' : A)$.

Theorem [Strong Normalisation] The relation λlxr is strongly normalising on well-typed λlxr-terms.

Towards completeness

- Define a congruence $\approx$ for proof-nets.
- Define a congruence $\cong$ for λlxr-terms.
- Show that $T(\Gamma \vdash t_1 : A) \approx T(\Gamma' \vdash t_2 : A')$ implies $t_1 \cong t_2$.

Summary

The λ`lxr`-calculus is a computational interpretation of natural deduction plus cut and structural rules enjoying the following properties :

- Confluence on all the terms.
- Simulation of one-step β-reduction.
- Preservation of β-strong normalization.
- Strong normalization of well-typed terms.
- Full and safe composition.
- Sound and complete with respect to proof-nets.
- Explicit operators for implementation issues.