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Finite semigroups and recognizable languages: an introduction

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1 Foreword

This paper is an attempt to share with a larger audience some modern developments in the theory of finite automata. It is written for the mathematician who has a background in semigroup theory but knows next to nothing on automata and languages. No proofs are given, but the main results are illustrated by several examples and counterexamples.

What is the topic of this theory? It deals with languages, automata and semigroups, although recent developments have shown interesting connections with model theory in logic, symbolic dynamics and topology. Historically, in their attempt to formalize natural languages, linguists such as Chomsky gave a mathematical definition of natural concepts such as words, languages or grammars: given a finite set A, a word on A is simply an element of the free monoid on A, and a language is a set of words. But since scientists are fond of classifications of all sorts, language theory didn't escape to this mania. Chomsky established a first hierarchy, based on his formal grammars. In this paper, we are interested in the *recognizable* languages, which form the lower level of the Chomsky hierarchy. A recognizable language can be described in terms of finite automata while, for the higher levels, more powerful machines, ranging from pushdown automata to Turing machines, are required. For this reason, problems on finite automata are often under-estimated, according to the vague — but totally erroneous — feeling that "if a problem has been reduced to a question about finite automata, then it should be easy to solve".

Kleene's theorem [23] is usually considered as the foundation of the theory. It shows that the class of recognizable languages (i.e. recognized by finite automata), coincides with the class of *rational* languages, which are given by rational expressions. Rational expressions can be thought of as a generalization of polynomials involving three operations: union (which plays the role of addition), product and star operation. An important corollary of Kleene's theorem is that rational languages are closed under complement. In the sixties, several classification schemes for the rational languages were proposed, based on the

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number of nested use of a particular operator (star or product, for instance). This led to the natural notions of star height, extended star height, dot-depth and concatenation level. However, the first natural questions attached to these notions — "do they define strict hierarchies ?", "given a rational language, is there an algorithm for computing its star height, extended star height", etc. ? — appeared to be extremely difficult. Actually, several of them, like the hierarchy problem for the extended star height, are still open.

A break-through was realized by Schützenberger in the mid sixties [53]. Schützenberger established the equivalence between finite automata and finite semigroups and showed that a finite monoid, called the *syntactic monoid*, is canonically attached to each recognizable language. Then he made a non trivial use of this invariant to characterize the languages of extended star height 0, also called *star-free* languages. Schützenberger's theorem states that a language is star-free if and only if its syntactic monoid is aperiodic. Two other "syntactic" characterizations were obtained in the early seventies: Simon [57] proved that a language is of concatenation level 1 if and only if its syntactic monoid is $\mathcal{J}$-trivial and Brzozowski-Simon [9] and independently, McNaughton [29] characterized an important subfamily of the languages of dot-depth one, the locally testable languages. These successes settled the power of the semigroup approach, but it was Eilenberg who discovered the appropriate framework to formulate this type of results [17].

Recall that a variety of finite monoids is a class of monoids closed under the taking of submonoids, quotients and finite direct product. Eilenberg's theorem states that varieties of finite monoids are in one to one correspondence with certain classes of recognizable languages, the varieties of languages. For instance, the rational languages correspond to the variety of all finite monoids, the star-free languages correspond to the variety of aperiodic monoids, and the piecewise testable languages correspond to the variety of $\mathcal{J}$-trivial monoids. Numerous similar results have been established during the past fifteen years and the theory of finite automata is now intimately related to the theory of finite semigroups. This had a considerable influence on both theories: for instance algebraic definitions such as the graph of a semigroup or the Schützenberger product were motivated by considerations of language theory. The same thing can be said for the systematic study of power semigroups. In the other direction, Straubing's wreath product principle has permitted to obtain important new results on recognizable languages. The open question of the decidability of the dot-depth is a good example of a problem that interests both theories (and also formal logic !).

The paper is organized as follows. Sections 2 and 3 present the necessary material to understand Kleene's theorem. The equivalence between finite automata and finite semigroups is detailed in section 4. The various hierarchies of rational languages, based on star height, extended star height, dot-depth and concatenation level are introduced in section 5. The syntactic characterization of star-free, piecewise testable and locally testable languages are formulated in sections 6, 7 and 8, respectively. The variety theorem is stated in section 9 and some examples of its application are given in section 10. Other consequences about the hierarchies are analyzed in section 11 and recent developments are reported in section 12. The last section 13 contains the conclusion of this article.

2 Rational and recognizable sets

The terminology used in the theory of automata originates from various founts. Part of it came from linguistics, some other parts were introduced by physicists or by logicians. This gives sometimes a curious mixture but it is rather convenient in practice.

An *alphabet* is a finite set whose elements are *letters*. Alphabets are usually denoted by capital letters: A, B, ... and letters by lower case letters from the beginning of the latin alphabet: a, b, $c, \ldots$ A *word* (over the alphabet A) is a finite sequence $(a_1, a_2, \ldots, a_n)$ of letters of A; the integer n is the *length* of the word. In practice, the notation $(a_1, a_2, \ldots, a_n)$ is shortened to $a_1a_2 \cdots a_n$. The empty word, which is the unique word of length 0, is denoted by 1. Given a letter a, the number of occurrences of a in a word u is denoted by $|u|_a$. For instance, $|abbab|_a = 2$ and $|abbab|_b = 3$.

The (concatenation) *product* of two words $u = a_1a_2 \cdots a_p$ and $v = b_1b_2 \cdots b_q$ is the word $uv = a_1a_2 \cdots a_pb_1b_2 \cdots b_q$. The product is an associative operation on words. The set of all words on the alphabet A is denoted by A^*. Equipped with the product of words, it is a monoid, with the empty word as an identity. It is in fact the free monoid on the set A. The set of non-empty words is denoted by A^+; it is the free semigroup on the set A.

A *language* of A^* is a set of words over A, that is, a subset of A^*. The *rational operations* on languages are the three operations union, product and star, defined as follows

(1) Union : $L_1 + L_2 = \{u \mid u \in L_1 \text{ or } u \in L_2\}$

(2) Product : $L_1L_2 = \{u_1u_2 \mid u_1 \in L_1 \text{ and } u_2 \in L_2\}$

(3) Star : $L^* = \{u_1 \cdots u_n \mid n \geqslant 0 \text{ and } u_1, \ldots, u_n \in L\}$

It is also convenient to introduce the operator

$$L^+ = LL^* = \{u_1 \cdots u_n \mid n > 0 \text{ and } u_1, \ldots, u_n \in L\}$$

Note that L^+ is exactly the subsemigroup of A^* generated by L, while L^* is the submonoid of A^* generated by L. The set of rational languages of A^* is the smallest set of languages of A^* containing the finite languages and closed under finite union, finite product and star. For instance, $(a + ab)^*ab + (ba^*b)^*$ denotes a rational language on the alphabet $\{a, b\}$.

The set of rational languages of A^+ is the smallest set of languages of A^+ containing the finite languages and closed under finite union, product and plus. It is easy to verify that the rational languages of A^+ are exactly the rational languages of A^* that do not contain the empty word.

It may seem a little awkward to have two separate definitions for the rational languages: one for the free monoid A^* and another one for the free semigroup A^+. There are actually two parallel theories and although the difference between them may appear of no great significance at first sight, it turns out to be crucial. The reason is that the algebraic classification of rational languages, as given in the forthcoming sections, rests on the notion of varieties of finite monoids (for languages of the free monoid) or varieties of finite semigroups (for languages of the free semigroup). And varieties of finite semigroups cannot be considered as varieties of finite monoids. The simplest example is the variety of finite nilpotent semigroups, which, as we shall see, characterizes the finite or cofinite languages of the free semigroup. If one tries, in a naive attempt, to add an identity to convert each nilpotent semigroup into a monoid, the variety of finite monoids obtained in this way is the variety of all finite monoids whose idempotents commute with every element. But this variety of monoids does *not* characterize the finite-cofinite languages of the free monoid.

Rational languages are often called *regular* sets in the literature. However, in the author's opinion, this last term should be avoided for two reasons. First, it interferes with the standard use of this word in semigroup theory. Second, the term *rational* has a sound mathematical foundation. Indeed one can extend the theory of languages to series with non commutative variables over a commutative ring or semiring[1] k. Such series can be written as $s = \sum_{u \in A^*} (s, u)u$, where (s, u) is an element of k. In this context, languages appear naturally as series over the boolean semiring. Now the rational series form the smallest set of series $\mathcal{R}$ satisfying the following conditions:

(1) Every polynomial is in $\mathcal{R}$,

(2) $\mathcal{R}$ is a semiring under the usual sum and product of series,

(3) If s is a series in $\mathcal{R}$ such that $(s, 1) = 0$, then $s^* = \sum_{n \geqslant 0} s^n$ belongs to $\mathcal{R}$.

Note that if k is a ring, then $s^* = (1 - s)^{-1}$. In particular, in the one variable case, this definition coincide with the usual definition of rational series, which explains the terminology. We shall not detail any further this nice extension of the theory of languages, but we refer the interested reader to [4] for more details.

3 Finite automata and recognizable sets

A finite (non deterministic) automaton is a quintuple $\mathcal{A} = (Q, A, E, I, F)$ where Q is a finite set (the set of *states*), A is an alphabet, E is a subset of $Q \times A \times Q$, called the set of *transitions* and I and F are subsets of Q, called the set of *initial* and *final* states, respectively. Two transitions (p, a, q) and (p', a', q') are *consecutive* if $q = p'$. A *path* in $\mathcal{A}$ is a finite sequence of consecutive transitions

$$e_0 = (q_0, a_0, q_1), \quad e_1 = (q_1, a_1, q_2), \quad \ldots \quad , \quad e_{n-1} = (q_{n-1}, a_{n-1}, q_n)$$

also denoted

$$q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \quad \cdots \quad q_{n-1} \xrightarrow{a_{n-1}} q_n$$

The state q_0 is the *origin* of the path, the state q_n is its *end*, and the word $x = a_0 a_1 \cdots a_{n-1}$ is its *label*. It is convenient to have also, for each state q, an empty path of label 1 from q to q. A path in $\mathcal{A}$ is *successful* if its origin is in I and its end is in F.

The language *recognized* by $\mathcal{A}$ is the set, denoted $L^*(\mathcal{A})$, of the labels of all successful paths of $\mathcal{A}$. A language X is *recognizable* if there exists a finite automaton $\mathcal{A}$ such that $X = L^*(\mathcal{A})$. Two automata are said to be *equivalent* if they recognize the same language. Automata are conveniently represented by

[1] A semiring is a set k equipped with an addition and a multiplication. It is a commutative monoid with identity 0 for the addition and a monoid with identity 1 for the multiplication. Multiplication is distributive over addition and 0 satisfies $0x = x0 = 0$ for every $x \in k$. The simplest example of a semiring which is not a ring is the boolean semiring $\mathbf{B} = \{0, 1\}$ defined by $0 + 0 = 0$, $0 + 1 = 1 + 1 = 1 + 0 = 1$, $1.1 = 1$ and $1.0 = 0.0 = 0.1 = 0$.

labeled graphs, as in the example below. Incoming arrows indicate initial states and outgoing arrows indicate final states.

Example 3.1 Let $\mathcal{A} = (\{1, 2\}, \{a, b\}, E, \{1\}, \{2\})$ be an automaton, with $E = \{(1, a, 1), (1, b, 1), (1, a, 2)\}$. The path $(1, a, 1)(1, b, 1)(1, a, 2)$ is a successful path of label aba. The path $(1, a, 1)(1, b, 1)(1, a, 1)$ has the same label but is unsuccessful since its end is 1.

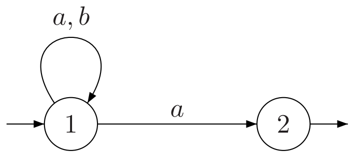

Figure 3.1: An automaton.

The set of words accepted by $\mathcal{A}$ is $L^*(\mathcal{A}) = A^*a$, the set of all words ending with an a.

In the case of the free semigroup, the definitions are the same, except that we omit the empty paths of label 1. In this case, the language recognized by $\mathcal{A}$ is denoted $L^+(\mathcal{A})$. Kleene's theorem states the equivalence between automata and rational expressions.

Theorem 3.1 *A language is rational if and only if it is recognizable.*

In fact, there is one version of Kleene's theorem for the free semigroup and one version for the free monoid. The proof of Kleene's theorem can be found in most books of automata theory [21].

An automaton is *deterministic* if it has exactly one initial state, usually denoted q_0 and if E contains no pair of transitions of the form (q, a, q_1), (q, a, q_2) with $q_1 \neq q_2$.

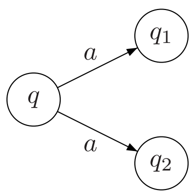

Figure 3.2: The forbidden pattern in a deterministic automaton.

In this case, each letter a defines a partial function from Q to Q, which associates with every state q the unique state $q.a$, if it exists, such that $(q, a, q.a) \in E$. This can be extended into a right action of A^* on Q by setting, for every $q \in Q$, $a \in A$ and $u \in A^*$:

$$q.1 = q$$

$$q.(ua) = \begin{cases} (q.u).a & \text{if } q.u \text{ and } (q.u).a \text{ are defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

One can show that every finite automaton is equivalent to a deterministic one, in the sense that they recognize the same language.

States which cannot be reached from the initial state or from which one cannot access to any final state are clearly useless. This leads to the following definition. A deterministic automaton $\mathcal{A} = (Q, A, E, q_0, F)$ is *trim* if for every state $q \in Q$ there exist two words u and v such that $q_0.u = q$ and $q.v \in F$. It is not difficult to see that every deterministic automaton is equivalent to a trim one.

Deterministic automata are partially ordered as follows. Let

$$\mathcal{A} = (Q, A, E, q_0, F) \text{ and } \mathcal{A}' = (Q', A, E', q_0', F')$$

be two deterministic automata. Then $\mathcal{A} \leqslant \mathcal{A}'$ if there is a surjective function $\varphi : Q \to Q'$ such that $\varphi(q_0) = q_0'$, $\varphi^{-1}(F') = F$ and, for every $u \in A^*$ and $q \in Q$, $\varphi(q.u) = \varphi(q).u$. One can show that, amongst the trim deterministic automata recognizing a given recognizable language L, there is a minimal one for this partial order. This automaton is called the *minimal automaton* of L. Again, there are standard algorithms for minimizing a given finite automaton [21].

4 Automata and semigroups

In this section, we turn to a more algebraic definition of the recognizable sets, using semigroups in place of automata. Although this definition is more abstract than the definition using automata, it is more suitable to handle the fine structure of recognizable sets. Indeed, as illustrated in the next sections, semigroups provide a powerful and systematic tool to classify recognizable sets. We treat the case of the free semigroup. For free monoids, just replace every occurrence of "A^+" by "A^*" and "semigroup" by "monoid" in the definitions below.

The abstract definition of recognizable sets is based on the following observation. Let $\mathcal{A} = (Q, A, E, I, F)$ be a finite automaton. To each word $u \in A^+$, there corresponds a boolean square matrix of size $\mathrm{Card}(Q)$, denoted by $\mu(u)$, and defined by

$$\mu(u)_{p,q} = \begin{cases} 1 & \text{if there exists a path from } p \text{ to } q \text{ with label } u \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to see that μ is a semigroup morphism from A^+ into the multiplicative semigroup of square boolean matrices of size $\mathrm{Card}(Q)$. Furthermore, a word u is recognized by $\mathcal{A}$ if and only if $\mu(u)_{p,q} = 1$ for some initial state p and some final state q. Therefore, a word is recognized by $\mathcal{A}$ if and only if $\mu(u) \in \{m \in \mu(A^+) \mid m_{p,q} = 1 \text{ for some } p \in I \text{ and } q \in F\}$. The semigroup $\mu(A^+)$ is called the *transition semigroup* of $\mathcal{A}$, denoted $S(\mathcal{A})$.

Example 4.1 Let $\mathcal{A} = (Q, A, E, I, F)$ be the automaton represented below

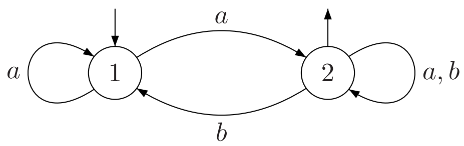

Figure 4.1: A non deterministic automaton.

Here $Q = \{1, 2\}$, $A = \{a, b\}$, $I = \{1\}$, $F = \{2\}$ and

$$E = \{(1, a, 1), (1, a, 2), (2, a, 2), (2, b, 1), (2, b, 2)\}$$

whence

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mu(aa) = \mu(a)$$
$$\mu(ab) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu(ba) = \mu(bb) = \mu(b)$$

Thus $\mu(A^+) = \{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)\}$.

This leads to the following definition. A semigroup morphism $\varphi : A^+ \to S$ *recognizes* a language $L \subset A^+$ if $L = \varphi^{-1}\varphi(L)$, that is, if $u \in L$ and $\varphi(u) = \varphi(v)$ implies $v \in L$. This is also equivalent to saying that there is a subset P of S such that $L = \varphi^{-1}(P)$. By extension, a semigroup S recognizes a language L if there exists a semigroup morphism $\varphi : A^+ \to S$ that recognizes L. As shown by the previous example, a set recognized by a finite automaton is recognized by the transition semigroup of this automaton.

Proposition 4.1 *If a finite automaton recognizes a language L, then $S(\mathcal{A})$ recognizes L.*

The previous computation can be simplified if $\mathcal{A}$ is deterministic. Indeed, in this case, the transition semigroup of $\mathcal{A}$ is naturally embedded into the semigroup of partial functions on Q under composition.

Example 4.2 Let $\mathcal{A}$ be the deterministic automaton represented below.

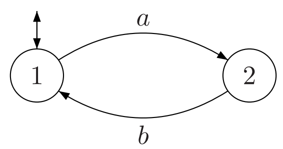

Figure 4.2: A deterministic automaton.

The transition semigroup $S(\mathcal{A})$ of $\mathcal{A}$ contains five elements which correspond to the words a, b, ab, ba and aa. If one identifies the elements of $S(\mathcal{A})$ with these words, one has the relations $aba = a$, $bab = b$ and $bb = aa$. Thus $S(\mathcal{A})$ is the aperiodic Brandt semigroup BA_2. Here is the transition table of $\mathcal{A}$:

	1	2
a	2	-
b	-	1
aa	-	-
ab	1	-
ba	-	2

Conversely, given a semigroup morphism $\varphi : A^+ \to S$ recognizing a subset X of A^+, one can build a finite automaton recognizing X as follows. Denote by S^1 the monoid equal to S if S has an identity and to $S \cup \{1\}$ otherwise. Take the right representation of A on S^1 defined by $s.a = s\varphi(a)$. This defines a deterministic automaton $\mathcal{A} = (S^1, A, E, \{1\}, P)$, where $E = \{(s, a, s.a) \mid s \in S^1, a \in A\}$.

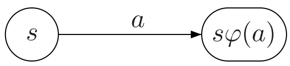

Figure 4.3: The transitions of $\mathcal{A}$.

This automaton recognizes L and thus, the two notions of recognizable sets (by finite automata and by finite semigroups) are equivalent.

Example 4.3 Let $A = \{a, b, c\}$ and let $S = \{1, a, b\}$ be the three element monoid defined by $a^2 = a$, $b^2 = b$, $ab = b$ and $ba = a$. Let $\varphi : A^+ \to S$ be the semigroup morphism defined by $\varphi(a) = a$, $\varphi(b) = b$ and $\varphi(c) = 1$ and let $P = \{a\}$. Then $\varphi^{-1}(P) = A^*ac^*$ and the construction above yields the automaton represented in Figure 4.4:

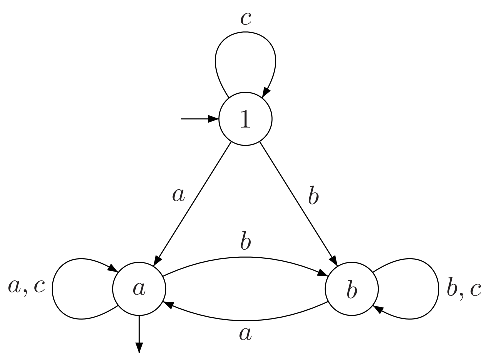

Figure 4.4: The automaton associated with S.

Now, Kleene's theorem can be reformulated as follows.

Theorem 4.2 *Let L be a language of A^+. The following conditions are equivalent.*

(1) *L is recognized by a finite automaton,*

(2) *L is recognized by a finite deterministic automaton,*

(3) *L is recognized by a finite semigroup,*

(4) *L is rational.*

Kleene's theorem has important consequences.

Corollary 4.3 *Recognizable languages are closed under finite boolean operations*[2]*, inverse morphisms and morphisms.*

The trick is that it is easy to prove the last property (closure under morphisms) for rational sets and the other ones for recognizable sets. Here are two examples to illustrate these techniques:

Example 4.4 (Closure of recognizable sets under morphism). Let $\varphi : \{a, b\}^+ \to \{a, b, c\}^+$ be the semigroup morphism defined by $\varphi(a) = aba$ and $\varphi(b) = ca$ and let $L = a^*b + bab$ be a rational set. Then $\varphi(L) = (aba)^*ca + caabaca$ is a rational set.

Example 4.5 (Closure of recognizable sets under complement). Let L be a recognizable set. Then there exists a finite semigroup S, a semigroup morphism $\varphi : A^+ \to S$ and a subset P of S such that $L = \varphi^{-1}(P)$. Now $A^+ \setminus L = \varphi^{-1}(S \setminus P)$ and thus the complement of L is recognizable.

The patient reader can, as an exercise, prove the remaining properties by using either semigroups or automata. The impatient reader may consult [16, 37].

Let L be a recognizable language of A^+. Amongst the finite semigroups that recognize X, there is a minimal one (with respect to division). This finite semigroup is called the *syntactic semigroup* of L. It can be defined directly as the quotient of A^+ under the congruence $\sim_L$ defined by $u \sim_L v$ if and only if, for every $x, y \in A^*$, $xuy \in L \iff xvy \in L$. It is also equal to the transition semigroup of the minimal automaton of L. This last property is especially useful for practical computations. It is a good exercise to take a rational expression at random, to compute the minimal automaton of the language represented by this rational expression and then to compute the syntactic semigroup of the language. See examples 6.1 and 7.2 below for outlines of such computations.

5 Early attempts to classify recognizable languages

Kleene's theorem shows that recognizable languages are closed under complementation. Therefore, every recognizable language can be represented by a *extended rational expression*, that is, a formal expression constructed from the letters by mean of the operations union, product, star and complement. In order to keep concise algebraic notations, we shall denote by L^c the complement of the language L[3], by 0 the empty language and by u the language $\{u\}$, for every word u. In particular, the language $\{1\}$, containing the empty word, is denoted 1. These notations are coherent with the intuitive formulæ

[2]Boolean operations comprise union, intersection, complementation and set difference.

[3]If L is a language of A^*, the complement of L is $A^* \setminus L$; if L is a language of A^+, the complement is $A^+ \setminus L$

$1L = L1 = L$ and $0L = L0 = 0$ which hold for every language L. For instance, if $A = \{a, b\}$, the expression $\Big(\big(0^c(ab + ba)0^c\big)^c + (aba)^*\Big)^c$ represents the language $(A^*abA^* \cup A^*baA^*) \setminus (aba)^*$ of all words containing the factors ab and ba which are not powers of aba.

Thus we have an algebra on A with four operations: $+$, $.$, * and ^{c}. Now a natural attempt to classify recognizable languages is to find a notion analogous with the degree of a polynomial for these extended rational expressions. It is a remarkable fact that all the hierarchies based on these "extended degrees" suggested so far lead to some extremely difficult problems, most of which are still open.

The first proposal concerned the star operation. The *star height* of an extended rational expression is defined inductively as follows:

(1) The star height of the basic languages is 0. Formally

$$sh(0) = 0 \qquad sh(1) = 0 \quad \text{and} \quad sh(a) = 0 \text{ for every letter } a$$

(2) Union, product and complement do not affect star height. If e and f are two extended rational expressions, then

$$sh(e + f) = sh(ef) = \max\{sh(e), sh(f)\} \qquad sh(e^c) = sh(e)$$

(3) Star increases star height. For each extended rational expression e,

$$sh(e^*) = sh(e) + 1$$

Thus the star height counts the number of nested uses of the star operation. For instance

$$\Big((a^* + b^c a^*)^* + (b^* ab^*)^*\Big)^*(b^* a^* + b)^*$$

is an extended rational expression of star height 3. Now, the *extended star height*[4] of a recognizable language L is the minimum of the star heights of the extended rational expressions representing L

$$esh(L) = \min\{sh(e) \mid \ e \text{ is an extended rational expression for } L\ \}$$

The difficulty in computing the extended star height is that a given language can be represented in many different ways by an extended rational expression !

The languages of extended star height 0 (or *star-free languages*) are characterized by a beautiful theorem of Schützenberger that will be presented in section 6. Schützenberger's theorem implies the existence of languages of extended star height 1, such as $(aa)^*$ on the alphabet $\{a\}$, but, as surprising as it may seem, nobody has been able so far to prove the existence of a language of extended star height greater than 1, although the general feeling is that such languages do exist. In the opposite direction, our knowledge of the languages proven to be of extended star height $\leqslant 1$ is rather poor (see [46, 51, 52] for recent advances on this topic).

The *star height* of a recognizable language is obtained by considering rational expressions instead of extended rational expressions [15].

$$sh(L) = \min\{\text{star height}(e) \mid \ e \text{ is a rational expression for } L\ \}$$

[4] also called *generalized star height*

That is, one simply removes complement from the list of the basic operations. This time, the corresponding hierarchy was proved to be infinite by Dejean and Schützenberger [14].

Theorem 5.1 *For each $n \geqslant 0$, there exists a language of star height n.*

It is easy to see that the languages of star height 0 are the finite languages, but the effective characterization of the other levels was left open for several years until Hashiguchi first settled the problem for star height 1 [18] and a few years later for the general case [19].

Theorem 5.2 *There is an algorithm to determine the star height of a given recognizable language.*

Hashiguchi's first paper is now well understood, although it is still a very difficult result, but volunteers are called to simplify the very long induction proof of the second paper.

The second proposal to construct hierarchies was to ignore the star operation (which amounts to working with star-free languages) and to consider the concatenation product or, more precisely, a variation of it, called the *marked concatenation product*. Given languages $L_0, L_1, \ldots, L_n$ and letters $a_1, a_2, \ldots, a_n$, the product of $L_0, \ldots L_n$ marked by $a_1, \ldots a_n$ is the language $L_0a_1L_1a_2 \cdots a_nL_n$. As product is often denoted by a dot, Brzozowski defined the "dot-depth" of languages of the free semigroup [5]. Later on, Thérien (implicitly) and Straubing (explicitly) introduced a similar notion (often called the *concatenation level* in the literature) for the languages of the free monoid. The languages of dot-depth 0 are the finite or cofinite languages, while the languages of concatenation level 0 are A^* and the empty language 0. Otherwise, the two hierarchies are constructed in the same way and count the number of alternations in the use of the two different types of operations: boolean operations and marked product. More precisely, the languages of dot-depth (resp. concatenation level) $n+1$ are the finite boolean combinations of marked products of the form

$$L_0a_1L_1a_2 \cdots a_kL_k$$

where $L_0, L_1, \ldots, L_k$ are languages of dot-depth (resp. concatenation level) n and $a_1, \ldots, a_k$ are letters.

Note that a language of dot-depth (resp. concatenation level) m is also a language of dot-depth (resp. concatenation level) n for every $n \geqslant m$. Brzozowski and Knast [8] have shown that the hierarchy is strict: if A contains at least two letters, then for every n, there exist some languages of dot-depth (resp. level) $n+1$ that are not of level n.

It is still an outstanding open problem to know whether there is an algorithm to compute the dot-depth (resp. concatenation level) of a given star-free language. The problem has been solved positively, however, for the dot-depth (resp. concatenation level) 1: there is an algorithm to decide whether a language is of dot-depth (resp. concatenation level) 1. These results are detailed in sections 7 and 11. The other partial results concerning these hierarchies are briefly reviewed in section 11. Another remarkable fact about these hierarchies is their connections with some hierarchies of formal logic. See the article of W. Thomas in this volume or the survey article [41].

But it is time for us to hark back to Schützenberger's theorem on star-free sets.

6 Star-free languages

The set of *star-free* subsets of A^* is the smallest set of subsets of A^* containing the finite sets and closed under finite boolean operations and product. For instance, A^* is star-free, since A^* is the complement of the empty set. More generally, if B is a subset of the alphabet A, the set B^* is also star-free since B^* is the complement of the set of words that contain at least one letter of $B' = A \setminus B$. This leads to the following star-free expression

$$B^* = A^* \setminus A^*(A \setminus B)A^* = (0^c(A \setminus B)0^c)^c = (0^c(A^c \cup B)^c 0^c)^c$$

If $A = \{a, b\}$, the set $(ab)^*$ is star-free, since $(ab)^*$ is the set of words not beginning with b, not finishing by a and containing neither the factor aa, nor the factor bb. This gives the star-free expression

$$(ab)^* = A^* \setminus \big(bA^* \cup A^*a \cup A^*aaA^* \cup A^*bbA^*\big) = \big(b0^c + 0^c a + 0^c aa0^c + 0^c bb0^c\big)^c$$

Readers may convince themselves that the sets $\{ab, ba\}^*$ and $\big(a(ab)^*b\big)^*$ also are star-free but may also wonder whether there exist any non star-free rational sets. In fact, there are some, for instance the sets $(aa)^*$ and $\{b, aba\}^*$, or similar examples that can be derived from the algebraic approach presented below.

Let S be a finite semigroup and let s be an element of S. Then the subsemigroup of S generated by s contains a unique idempotent, denoted s^ω. Recall that a finite semigroup M is *aperiodic* if and only if, for every $x \in M$, $x^\omega = x^{\omega+1}$. This notion is in some sense "orthogonal" to the notion of group. Indeed, one can show that a semigroup is aperiodic if and only if it is $\mathcal{H}$-trivial, or, equivalently, if it contains no non-trivial subgroup. The connection between aperiodic semigroups and star-free sets was established by Schützenberger [53].

Theorem 6.1 *A recognizable subset of A^* is star-free if and only if its syntactic monoid is aperiodic.*

There are essentially two techniques to prove this result. The original proof of Schützenberger [53, 37, 22], slightly simplified in [32], is by induction on the $\mathcal{J}$-depth of the syntactic semigroup. The second proof [11, 31] makes use of a weak form of the Krohn-Rhodes theorem: every aperiodic finite semigroup divides a wreath product of copies of the monoid $U_2 = \{1, a, b\}$, given by the multiplication table $aa = a$, $ab = b$, $ba = b$ and $bb = b$.

Corollary 6.2 *There is an algorithm to decide whether a given[5] recognizable language is star-free.*

Given the minimal automaton $\mathcal{A}$ of the language, the algorithm consists to check whether the transition monoid of M is aperiodic. The complexity of this algorithm is analyzed in [10, 58].

[5] A recognizable set can be given either by a finite automaton, by a finite semigroup or by a rational expression since there are standard algorithms to pass from one representation to the other.

Example 6.1 Let $A = \{a, b\}$ and consider the set $L = (ab)^*$. Its minimal automaton is represented below:

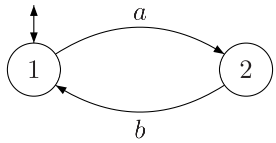

Figure 6.1: The minimal automaton of $(ab)^*$.

The transitions and the relations defining the syntactic monoid M of L are given in the following tables

1	1	2
a	2	$-$
b	$-$	1
aa	$-$	$-$
ab	1	$-$
ba	$-$	2

$$a^2 = b^2 = 0$$
$$aba = a$$
$$bab = b$$

Since $a^2 = a^3$, $b^2 = b^3$, $(ab)^2 = ab$ and $(ba)^2 = ba$, M is aperiodic and thus L is star-free. Consider now the set $L' = (aa)^*$. Its minimal automaton is represented below:

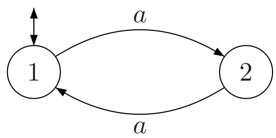

Figure 6.2: The minimal automaton of $(aa)^*$.

The transitions and the relations defining the syntactic monoid M' of L' are given in the following tables

1	1	2
a	2	1
b	$-$	$-$
aa	1	2

$$a^3 = a$$
$$b = 0$$

Thus M' is not aperiodic and hence L' is not star-free.

7 Piecewise testable languages

Recall that the languages of concatenation level 0 of A^* are A^* and 0. According to the general definition, the languages of concatenation level 1 are the finite boolean combinations of the languages of the form $A^*a_1A^*a_2A^* \cdots A^*a_kA^*$, where $k \geqslant 0$ and $a_i \in A$. The languages of this form are called *piecewise testable*. Intuitively, such a language can be recognized by an automaton that one could call a *Hydra automaton*.

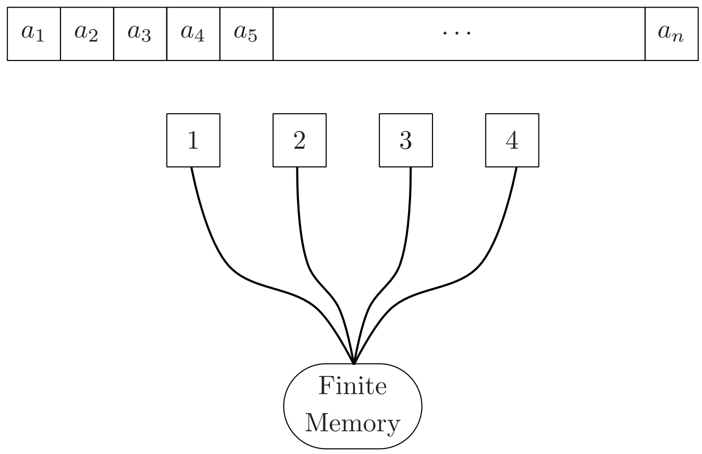

Figure 7.1: A Hydra automaton with four heads.

Such an automaton has a finite number h of heads, each of which can read a letter of the input word. The heads are ordered, so that together they permit to read a subword (in the sense of a subsequence of non necessarily consecutive letters) of the input word. The automaton computes in this way the set of all subwords of length $\leqslant h$ of the input word. This set is then compared to the finite collection of sets of words contained in the memory. If it occurs in the memory, the word is accepted, otherwise it is rejected. For instance, for the language $(A^*aA^*bA^*aA^* \cap A^*bA^*bA^*aA^*) \setminus (A^*aA^*bA^*bA^* \cup A^*bA^*bA^*bA^*)$, the memory would contain the collection of all sets of words of length 3 containing aba and bba but containing neither abb nor bbb. Piecewise testable languages are characterized by a deep result of I. Simon [57].

Theorem 7.1 *A language of A^* is piecewise testable if and only if its syntactic monoid is $\mathcal{J}$-trivial, or, equivalently, if it satisfies the equations $x^\omega = x^{\omega+1}$ and $(xy)^\omega = (yx)^\omega$.*

Corollary 7.2 *There is an algorithm to decide whether a given star-free language is of concatenation level* 1*.*

Given the minimal automaton $\mathcal{A}$ of the language, the algorithm consists in checking whether the transition monoid of M is $\mathcal{J}$-trivial. Actually, this condition can be directly checked on $\mathcal{A}$ in polynomial time [10, 58].

There exist several proofs of Simon's theorem [2, 57, 69, 58]. The central argument of Simon's original proof [57] is a careful study of the combinatorics of the subword relation. Stern's proof [58] borrows some ideas from model theory. The proof of Straubing and Thérien [69] is the only one that avoids totally combinatorics on words. In the spirit of the proof of Schützenberger, it works by induction on the cardinality of the syntactic monoid. The proof of Almeida [2] is based on implicit operations (see the papers of J. Almeida and P. Weil in this volume for more details).

Example 7.1 Let $A = \{a, b, c\}$ and let $L = A^*abA^*$. The minimal automaton of L is represented below

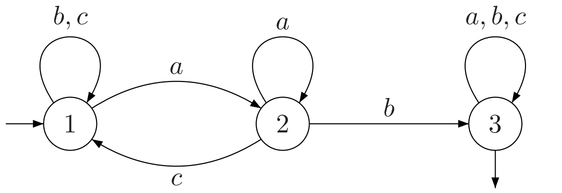

Figure 7.2: The minimal automaton of L.

The transitions and the relations defining the syntactic monoid M of L are given in the following tables

	1	2	3
1	1	2	3
a	2	2	3
b	1	3	3
c	1	1	3
ab	3	3	3
ba	2	3	3

$$\begin{aligned} a^2 &= a \\ ab &= 0 \\ ac &= c \\ b^2 &= b \\ bc &= b \\ ca &= a \\ cb &= c \\ c^2 &= c \end{aligned}$$

The $\mathcal{J}$-class structure of M is represented in the following diagram.

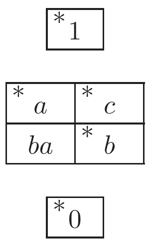

Figure 7.3: The $\mathcal{J}$-classes of M.

In particular, $a \mathrel{\mathcal{J}} c$ and thus M is not $\mathcal{J}$-trivial. Therefore L is not piecewise testable.

Example 7.2 Consider now the language $L' = A^*abA^*$ on the alphabet $A = \{a, b\}$. Then the minimal automaton of L' is obtained from that of L by erasing the transitions with label c (see Figure 7.4).

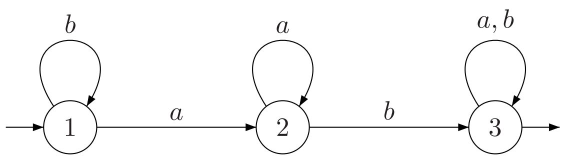

Figure 7.4: The minimal automaton of L'.

The transitions and the relations defining the syntactic monoid M' of L' are given in the following tables

1	1	2	3
a	2	2	3
b	1	3	3
ab	3	3	3
ba	2	3	3

$$a^2 = a$$
$$ab = 0$$
$$b^2 = b$$

The $\mathcal{J}$-class structure of M' is represented in the following diagram.

$$\boxed{{}^*1}$$

$$\boxed{{}^*a} \qquad \boxed{{}^*b}$$

$$\boxed{ba}$$

$$\boxed{{}^*0}$$

Figure 7.5: The $\mathcal{J}$-classes of M'.

Thus M' is $\mathcal{J}$-trivial and L' is piecewise testable. In fact $L' = A^*aA^*bA^*$.

Simon's theorem also has some nice consequences of pure semigroup theory. An *ordered monoid* is a monoid equipped with a stable order relation. An ordered monoid $(M, \leqslant)$ is called 1-*ordered* if, for every $x \in M$, $x \leqslant 1$. A finite 1-ordered monoid is always $\mathcal{J}$-trivial. Indeed, if $u \mathrel{\mathcal{J}} v$, there exist $x, y, z, t \in M$ such that $u = xvy$ and $v = zyt$. Now $x \leqslant 1$, $y \leqslant 1$ and thus $u = xvy \leqslant v$ and similarly, $v \leqslant u$ whence $u = v$. The converse is not true: there exist finite $\mathcal{J}$-trivial monoids which cannot be 1-ordered.

Example 7.3 Let M be the monoid with zero presented on $\{a, b, c\}$ by the relations $aa = ac = ba = bb = ca = cb = cc = 0$. Thus $M = \{1, a, b, c, ab, bc, abc, 0\}$ and M is $\mathcal{J}$-trivial. However, M is not a 1-ordered monoid. Otherwise, one would have on the one hand, $b \leqslant 1$, whence $abc \leqslant ac = 0$ and on the other hand, $0 \leqslant 1$, whence $0 = 0.abc \leqslant 1.abc = abc$, a contradiction since $abc \neq 0$.

However, Straubing and Thérien [69] proved that 1-ordered monoids generate all the finite $\mathcal{J}$-trivial monoids in the following sense.

Theorem 7.3 *A monoid is $\mathcal{J}$-trivial if and only if it is a quotient of a* 1*-ordered monoid.*

Actually, it is not difficult to establish that this result is equivalent to Simon's theorem. But Straubing and Thérien also gave an ingenious direct proof of their result by induction on the cardinality of the monoid. This gives in turn a proof of Simon's theorem. Straubing [63] also observed the following connection with semigroups of relations.

Theorem 7.4 *A monoid is $\mathcal{J}$-trivial if and only if it divides a monoid of reflexive relations on a finite set.*

8 Locally testable languages

A language of A^+ is *locally testable* if it is a boolean combination of languages of the form uA^*, A^*v or A^*wA^* where $u, v, w \in A^+$. For instance, if $A = \{a, b\}$, the language $(ab)^+$ is locally testable since $(ab)^+ = (aA^* \cap A^*b) \setminus (A^*aaA^* \cup A^*bbA^*)$. These languages occur naturally in the study of the languages of dot-depth one. Actually they form the first level of a natural subhierarchy of the languages of dot-depth one (see [36] for more details). Locally testable languages also have a natural interpretation in terms of automata. They are recognized by *scanners*. A scanner is a machine equipped with a finite memory and a window of size n to scan the input word.

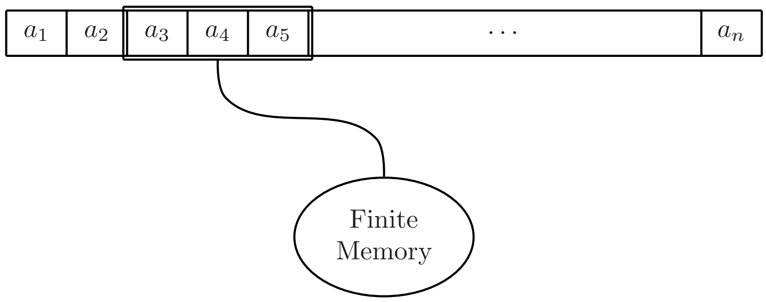

Figure 8.1: A scanner.

The window can also be moved beyond the first and last letter of the word, so that the prefixes and suffixes of length $< n$ can be read. For instance, if $n = 3$, and $u = abbaaabab$, the different positions of the window are represented on the following diagrams:

$$\boxed{\ \ \ a}bbaaabab \qquad \boxed{\ ab}baaabab \qquad \boxed{abb}aaabab \qquad a\boxed{bba}aabab \qquad \cdots \qquad abbaaaba\boxed{b\ \ \ }$$

At the end of the scan, the scanner memorizes the prefixes and the suffixes of length $< n$ and the set of factors of length n of the input word, but does not count the multiplicities. That is, if a factor occurs several times, it is memorized just once. This information is then compared to a collection of permitted sets of prefixes, suffixes and factors contained in the memory. The word is accepted or rejected, according to the result of this test.

The algebraic characterization of locally testable languages is slightly more involved than for star-free or piecewise testable languages. Recall that a finite semigroup S is said to have a property *locally*, if, for every idempotent e of S, the subsemigroup $eSe = \{ese \mid s \in S\}$ has the property. In particular, a semigroup is *locally trivial* if, for every idempotent e of S, $eSe = e$ and is *locally idempotent and commutative* if, for every idempotent e of S, eSe is idempotent and commutative. Equivalently, S is locally idempotent and commutative if, for every $e, s, t \in S$ such that $e = e^2$, $(ese)^2 = (ese)$ and $(ese)(ete) = (ete)(ese)$. The following result was proved independently by Brzozowski and Simon [9] and by McNaughton [29].

Theorem 8.1 *A recognizable language of A^+ is locally testable if and only if its syntactic semigroup is locally idempotent and commutative.*

This result, or more precisely the proof of this result, had a strong influence on pure semigroup theory. The reason is that Theorem 8.1 can be divided into two separate statements.

Proposition 8.2 *A recognizable language of A^+ is locally testable if and only if its syntactic semigroup divides a semidirect product of a semilattice by a locally trivial semigroup.*

Proposition 8.3 *A semigroup divides a semidirect product of a semilattice by a locally trivial semigroup if and only if it is locally idempotent and commutative.*

The proof of Proposition 8.2 is relatively easy, but Proposition 8.3 is much more difficult and relies on an interesting property. Given a semigroup S, form a graph $G(S)$ as follows: the vertices are the idempotents of S and the edges from e to f are the elements of the form esf. Then one can show that a semigroup divides a semidirect product of a semilattice by a locally trivial semigroup if and only if its graph is locally idempotent and commutative in the following sense: if p and q are loops around the same vertex, then $p = p^2$ and $pq = qp$. We shall encounter another condition on graphs in Theorem 11.1. This type of graph conditions is now well understood, although numerous problems are still pending. The graph of a semigroup is a special instance of a derived category and is deeply connected with the study of the semidirect product (see Straubing [68] and Tilson [71]).

9 Varieties, another approach to recognizable languages

In 1974, the syntactic characterizations of the star-free, piecewise testable and locally testable languages had already established the power of the semigroup approach. However, these theorems were still isolated. In 1976, Eilenberg presented in his book a unified framework for these three results. The cornerstone of this approach is the concept of variety.

Recall that a *variety of finite semigroups* (or *pseudovariety*) is a class of semigroups $\mathbf{V}$ such that:

(1) if $S \in \mathbf{V}$ and if T is a subsemigroup of S, then $T \in \mathbf{V}$,

(2) if $S \in \mathbf{V}$ and if T is a quotient of S, then $T \in \mathbf{V}$,

(3) if $(S_i)_{i\in I}$ is a finite family of semigroups of $\mathbf{V}$, then $\prod_{i\in I} S_i$ is also in $\mathbf{V}$.

Varieties of finite monoids are defined in the same way.

Condition (3) can be replaced by the conjunction of conditions (4) and (5):

(4) the trivial semigroup 1 belongs to $\mathbf{V}$,

(5) if S_1 and S_2 are semigroups of $\mathbf{V}$, then $S_1 \times S_2$ is also in $\mathbf{V}$.

Indeed, condition (4) is obtained by taking $I = \emptyset$ in (3).

Recall that a semigroup T *divides* a semigroup S if T is a quotient of a subsemigroup of S. Division is a transitive relation on semigroups and thus conditions (1) and (2) can be replaced by condition (1′)

(1′) if $S \in \mathbf{V}$ and if T divides S, then $T \in \mathbf{V}$.

Given a class $\mathcal{C}$ of semigroups, the intersection of all varieties containing $\mathcal{C}$ is still a variety, called the variety generated by $\mathcal{C}$, and denoted by $\langle \mathcal{C} \rangle$. In a more constructive way, $\langle \mathcal{C} \rangle$ is the class of all semigroups that divide a finite product of semigroups of $\mathcal{C}$.

Example 9.1

(1) The class $\mathbf{M}$ of all finite monoids forms a variety of finite monoids.
(2) The smallest variety of finite monoids is the trivial variety, denoted by $\mathbf{I}$, consisting only of the monoid 1.
(3) The class **Com** of all finite commutative monoids form a variety of finite monoids.
(4) The class $\mathbf{J_1}$ of all finite idempotent and commutative monoids (or semilattices) forms a variety of finite monoids.
(5) The class $\mathbf{A}$ of all finite aperiodic monoids forms a variety of finite monoids.
(6) The class $\mathbf{J}$ of all finite $\mathcal{J}$-trivial monoids forms a variety of finite monoids.
(7) The class of **LI** of all finite locally trivial semigroups forms a variety of finite semigroups.
(8) The class $\mathbf{LJ_1}$ of all finite locally idempotent and commutative semigroups forms a variety of finite semigroups.

Equations are a convenient way to define varieties. For instance, the variety of finite commutative semigroups is defined by the equation $xy = yx$, the variety of aperiodic semigroups is defined by the equation $x^\omega = x^{\omega+1}$. Of course, $x^\omega = x^{\omega+1}$ is not an equation in the usual sense, since ω is not a fixed integer... However, one can give a rigorous meaning to this "pseudoequation". Since J. Almeida and P. Weil present this topic in great detail in this volume, we refer the reader to their article for more information. For our purpose, it suffices to remember that equations (or pseudoequations) give an elegant description of the varieties of finite semigroups, but are sometimes very difficult to determine. We shall now extend this purely algebraic approach to recognizable languages.

If $\mathbf{V}$ is a variety of semigroups, we denote by $\mathcal{V}(A^+)$ the set of recognizable languages of A^+ whose syntactic semigroup belongs to $\mathbf{V}$. This is also the set of languages of A^+ recognized by a semigroup of $\mathbf{V}$.

A $+$-*class* of recognizable languages is a correspondence which associates with every finite alphabet A, a set $\mathcal{C}(A^+)$ of recognizable languages of A^+. Similarly, a $*$-*class* of recognizable languages is a correspondence which associates with every finite alphabet A, a set $\mathcal{C}(A^*)$ of recognizable languages of A^*. In particular, the correspondence $\mathbf{V} \to \mathcal{V}$ associates with every variety of semigroups a $+$-class of recognizable languages. Eilenberg gave a combinatorial description of the classes of languages that occur in this way.

If X is a language of A^+ and if $u \in A^*$, the *left quotient* (resp. *right quotient*) of X by u is the language

$$u^{-1}X = \{v \in A^+ \mid uv \in X\} \qquad (\text{resp. } \; Xu^{-1} = \{v \in A^+ \mid vu \in X\})$$

Left and right quotients are defined similarly for languages of A^* by replacing A^+ by A^* in the definition.

A $+$-*variety* is a class of recognizable languages such that

(1) for every alphabet A, $\mathcal{V}(A^+)$ is closed under finite boolean operations (finite union and complement),
(2) for every semigroup morphism $\varphi : A^+ \to B^+$, $X \in \mathcal{V}(B^+)$ implies $\varphi^{-1}(X) \in \mathcal{V}(A^+)$,
(3) If $X \in \mathcal{V}(A^+)$ and $u \in A^+$, then $u^{-1}X \in \mathcal{V}(A^+)$ and $Xu^{-1} \in \mathcal{V}(A^+)$.

Similarly, a $*$-*variety* is a class of recognizable languages such that

(1) for every alphabet A, $\mathcal{V}(A^*)$ is closed under finite boolean operations,
(2) for every monoid morphism $\varphi : A^* \to B^*$, $X \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(X) \in \mathcal{V}(A^*)$,
(3) If $X \in \mathcal{V}(A^*)$ and $u \in A^*$, then $u^{-1}X \in \mathcal{V}(A^*)$ and $Xu^{-1} \in \mathcal{V}(A^*)$.

We are ready to state Eilenberg's theorem.

Theorem 9.1 *The correspondence* $\mathbf{V} \to \mathcal{V}$ *defines a bijection between the varieties of semigroups and the* $+$*-varieties.*

The variety of finite semigroups corresponding to a given $+$-variety is the variety of semigroups generated by the syntactic semigroups of all the languages $L \in \mathcal{V}(A^+)$, for every finite alphabet A. There is, of course, a similar statement for the $*$-varieties.

Theorem 9.2 *The correspondence* $\mathbf{V} \to \mathcal{V}$ *defines a bijection between the varieties of monoids and the* $*$*-varieties.*

Varieties of finite semigroups or monoids are usually denoted by boldface letters and the corresponding varieties of languages are denoted by the corresponding cursive letters.

We already know four instances of Eilenberg's variety theorem.

(1) By Kleene's theorem, the $*$-variety corresponding to $\mathbf{M}$ is the $*$-variety of rational languages.
(2) By Schützenberger's theorem, the $*$-variety corresponding to $\mathbf{A}$ is the $*$-variety of star-free languages.
(3) By Simon's theorem, the $*$-variety corresponding to $\mathbf{J}$ is the $*$-variety of piecewise testable languages.
(4) By Theorem 8.1, the $+$-variety corresponding to $\mathbf{LJ_1}$ is the $+$-variety of locally testable languages.

To clear up any possible misunderstanding, note that the four theorems mentioned above (Kleene, Schützenberger, etc.) are *not* corollaries of the variety theorem. For instance, the variety theorem indicates that the languages corresponding to the finite aperiodic monoids form a $*$-variety; it doesn't say that this $*$-variety is the variety of star-free languages... Actually, it is often a difficult problem to find an explicit description of the $*$-variety of languages corresponding to a given variety of finite monoids, or, conversely, to find the variety of finite monoids corresponding to a given $*$-variety.

However, the variety theorem provided a new direction to classify recognizable languages. Systematic searches for the variety of monoids (resp. languages) corresponding to a given variety of languages (resp. monoids) were soon undertaken. A partial account of these results is given into the next section.

10 Bestiary

We review in this section a few examples of correspondence between varieties of finite monoids (or semigroups) and varieties of languages. A *boolean algebra* is a set of languages containing the empty language and closed under finite union, finite intersection and complement.

Let us start our visit of the zoo with the subvarieties of the variety **Com** of all finite commutative monoids: the variety **Acom** of commutative aperiodic monoids, the variety **Gcom** of commutative groups, the variety $\mathbf{J_1}$ of idempotent and commutative monoids (or semilattices) and the trivial variety **I**.

Proposition 10.1 *For every alphabet A, $\mathcal{I}(A^*) = \{0, A^*\}$.*

Proposition 10.2 *For every alphabet A, $\mathcal{J}_1(A^*)$ is the boolean algebra generated by the languages of the form A^*aA^* where a is a letter. Equivalently, $\mathcal{J}_1(A^*)$ is the boolean algebra generated by the languages of the form B^* where B is a subset of A.*

Proposition 10.3 *For every alphabet A, $\mathcal{G}com(A^*)$ is the boolean algebra generated by the languages of the form*

$$L(a, k, n) = \{u \in A^* \mid |u|_a \equiv k \bmod n\}$$

where $a \in A$ and $0 \leqslant k < n$.

Proposition 10.4 *For every alphabet A, $\mathcal{A}com(A^*)$ is the boolean algebra generated by the languages of the form*

$$L(a, k) = \{u \in A^+ \mid |u|_a = k\}$$

where $a \in A$ and $k \geqslant 0$.

Proposition 10.5 *For every alphabet A, $\mathcal{C}om(A^*)$ is the boolean algebra generated by the languages of the form*

$$L(a, k) = \{u \in A^+ \mid |u|_a = k\} \quad \text{or} \quad L(a, k, n) = \{u \in A^+ \mid |u|_a \equiv k \bmod n\}$$

where $a \in A$ and $0 \leqslant k < n$.

Consider now the variety **LI** of all locally trivial semigroups and its subvarieties $\mathbf{L^rI}$, $\mathbf{L^\ell I}$ and **Nil**. A finite semigroup S belongs to **LI** if and only if, for every $e \in E(S)$ and every $s \in S$, $ese = e$. The asymmetrical versions of this condition define the varieties $\mathbf{L^rI}$ and $\mathbf{L^\ell I}$. Thus $\mathbf{L^rI}$ (resp. $\mathbf{L^\ell I}$) is the variety of all finite semigroups S such that $se = e$ (resp. $es = e$). Equivalently, a semigroup belongs to **LI** (resp. $\mathbf{L^rI}$, $\mathbf{L^\ell I}$) if it is a nilpotent extension of a rectangular band (resp. a right rectangular band, a left rectangular band). Finally **Nil** is the variety of nilpotent semigroups, defined by the condition $es = se = e$ for every $e \in E(S)$ and every $s \in S$. Recall that a subset F of a set E is cofinite if its complement in E is finite.

Proposition 10.6 *For every alphabet A, $\mathcal{N}il(A^+)$ is the set of finite or cofinite languages of A^+.*

Proposition 10.7 *For every alphabet A, $\mathcal{L}^r\mathcal{I}(A^+)$ (resp. $\mathcal{L}^\ell\mathcal{I}(A^+)$) is the set of languages of the form $A^*X \cup Y$ (resp. $XA^* \cup Y$), where X and Y are finite subsets of A^+.*

Proposition 10.8 *For every alphabet A, $\mathcal{L}\mathcal{I}(A^+)$ is the set of languages of the form $XA^*Y \cup Z$, where X, Y and Z are finite subsets of A^+.*

Note that the previous characterizations do not make use of the complement, although the sets $\mathcal{N}\text{il}(A^+)$, $\mathcal{L}^r\mathcal{I}(A^+)$, $\mathcal{L}^\ell\mathcal{I}(A^+)$ and $\mathcal{L}\mathcal{I}(A^+)$ are closed under complement. Actually, the following characterizations hold.

Proposition 10.9 *For every alphabet A,*

(1) *$\mathcal{L}^r\mathcal{I}(A^+)$ is the boolean algebra generated by the languages of the form A^*u, where $u \in A^+$,*
(2) *$\mathcal{L}^\ell\mathcal{I}(A^+)$ is the boolean algebra generated by the languages of the form uA^*, where $u \in A^+$,*
(3) *$\mathcal{L}\mathcal{I}(A^+)$ is the boolean algebra generated by the languages of the form uA^* or A^*u, where $u \in A^+$.*

It would be to long to state in full detail all known results on varieties of languages. Let us just mention that the languages corresponding to the following varieties of finite semigroups or monoids are known: all varieties of bands ([45] for the lower levels and [56] for the general case), the varieties of $\mathcal{R}$-trivial (resp. $\mathcal{L}$-trivial) monoids [17, 7, 37], the varieties of p-groups (resp. nilpotent groups) [17], the varieties of solvable groups [60], the varieties of monoids whose groups are commutative [54, 26], nilpotent [17], solvable [60], the variety of monoids with commuting idempotents [27], the variety of $\mathcal{J}$-trivial monoids with commuting idempotents [3], the variety of monoids whose regular $\mathcal{J}$-classes are rectangular bands [55], the variety of block groups (see the author's article "$\mathbf{BG} = \mathbf{PG}$, a success story" in this volume) and many others which follow in particular from the general results given in section 12.

11 Back to the early attempts

As the variety approach proved to be successful in many different situations, it was expected to shed some new light on the difficult problems mentioned in section 5. The reality is more contrasted. In brief, varieties do not seem to be helpful for the star height, it is so far the most successful approach for the dot-depth and the concatenation levels and, with regard to the extended star height, it seems to be a useful tool, but probably nothing more. Let us comment on this judgment in more details.

Varieties do not seem to be helpful for the star height, simply because the languages of a given star height are not closed under inverse morphisms between free monoids and thus, do not form a variety of languages. However, the notion of syntactic semigroup arises in the proof of Hashiguchi's theorems.

Schützenberger's theorem shows that the languages of extended star height 0 form a variety. However, it seems unlikely that a similar result holds for the languages of extended star height 1. Indeed, one can show [33] that every finite monoid divides the syntactic monoid of a language of the form L^*, where L is finite. It follows that if the languages of extended star height 1 form a variety of languages, then this variety is the variety of all rational languages. In particular, this would imply that every recognizable language is of extended star height 0 or 1.

Varieties are much more useful in the study of the concatenation product. We have already seen the syntactic characterization of the languages of concatenation level 1. There is a similar result for the languages of dot-depth one. It is easy to see from the general definition that a language of A^+ is of "dot-depth one" if it is a boolean combination of languages of the form

$$u_0 A^* u_1 A^* u_2 \cdots A^* u_{k-1} A^* u_k$$

where $k \geqslant 0$ and $u_i \in A^*$. The syntactic characterization of these languages was settled by Knast [24, 25].

Theorem 11.1 *A language of A^+ is of dot-depth one if and only if the graph of its syntactic semigroup satisfies the following condition : if e and f are two vertices, p and r edges from e to f, and q and s edges from f to e, then $(pq)^\omega ps(rs)^\omega = (pq)^\omega (rs)^\omega$.*

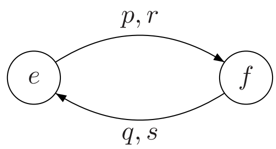

Figure 11.1:

More generally, one can show that the languages of dot-depth n form a +-variety of languages. The corresponding variety of finite semigroups is usually denoted by $\mathbf{B}_n$. Similarly, the languages of concatenation level n form a $*$-variety of languages and the corresponding variety of finite monoids is denoted $\mathbf{V}_n$. The two hierarchies are strict [8].

Theorem 11.2 *For every $n \geqslant 0$, there exists a language of dot-depth $n+1$ which is not of dot-depth n and a language of concatenation level $n+1$ which is not of concatenation level n.*

An important connection between the two hierarchies was found by Straubing [67]. Given a variety of finite monoids $\mathbf{V}$ and a variety of finite semigroups $\mathbf{W}$, denote by $\mathbf{V} * \mathbf{W}$ the variety of finite semigroups generated by the semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$ such that the action of T on S is right unitary.

Theorem 11.3 *For every $n > 0$, one has $\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI}$ and $\mathbf{V}_n = \mathbf{B}_n \cap \mathbf{M}$.*

In particular $\mathbf{B}_1 = \mathbf{J} * \mathbf{LI}$. It follows also, thanks to e deep result of Straubing [67] that $\mathbf{B}_n$ is decidable if and only if $\mathbf{V}_n$ is decidable. However, it is still an open problem to know whether the varieties $\mathbf{V}_n$ are decidable for $n \geqslant 2$. The case $n = 2$ is especially frustrating, but although several partial results have been obtained [44, 68, 72, 70, 74, 13], the general case remains open.

12 Recent developments

We shall not discuss in detail the numerous developments of the theory since Eilenberg's variety theorem, but we shall indicate the main trends. A quick glance at the known examples shows that the combinatorial description of a variety of languages follow most often the following pattern: the variety is described as the smallest variety closed under a given class of operations, such as boolean operations, product, etc. Varieties of semigroups are also often defined with the help of operators: join, semidirect products, Malcev products, etc. In view of Eilenberg's theorem, one may expect some relationship between the operators on languages (of combinatorial nature) and the operators on semigroups (of algebraic nature).

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{\text{Operations on semigroups}} & \mathbf{W} \\ \downarrow & & \downarrow \\ \mathcal{V} & \xrightarrow{\text{Operations on languages}} & \mathcal{W} \end{array}$$

In the late seventies, several results of this type were established, in particular by H. Straubing. We first consider the marked product. One of the most useful tools for studying this product is the *Schützenberger product* of n monoids, which was originally defined by Schützenberger for two monoids [53], and extended by Straubing [64] for any number of monoids.

Given a monoid M, the set of subsets of M, denoted $\mathcal{P}(M)$, is a semiring under union as addition and the product of subsets as multiplication, defined, for all $X, Y \subset M$ by $XY = \{xy \mid x \in X \text{ and } y \in Y\}$.

Let $M_1, \ldots, M_n$ be monoids. We denote by M the product monoid $M_1 \times \cdots \times M_n$, k the semiring $\mathcal{P}(M)$ and by $M_n(k)$ the semiring of square matrices of size n with entries in k. The Schützenberger product of $M_1, \ldots, M_n$, denoted $\diamondsuit_n(M_1, \ldots, M_n)$ is the submonoid of the multiplicative monoid $M_n(k)$ composed of all the matrices P satisfying the three following conditions:

(1) If $i > j$, $P_{i,j} = 0$

(2) If $1 \leqslant i \leqslant n$, $P_{i,i} = \{(1, \ldots, 1, s_i, 1, \ldots, 1)\}$ for some $s_i \in S_i$

(3) If $1 \leqslant i \leqslant j \leqslant n$, $P_{i,j} \subset 1 \times \cdots \times 1 \times M_i \times \cdots \times M_j \times 1 \cdots \times 1$.

Condition (1) indicates that the matrices of the Schützenberger product are upper triangular, condition (2) enables to identify the diagonal coefficient $P_{i,i}$ with an element s_i of M_i and condition (3) shows that if $i < j$, $P_{i,j}$ can be identified with a subset of $M_i \times \cdots \times M_j$. With this convention, a matrix of $\diamondsuit_3(M_1, M_2, M_3)$ will have the form

$$\begin{pmatrix} s_1 & P_{1,2} & P_{1,3} \\ 0 & s_2 & P_{2,3} \\ 0 & 0 & s_3 \end{pmatrix}$$

with $s_i \in M_i$, $P_{1,2} \subset M_1 \times M_2$, $P_{1,3} \subset M_1 \times M_2 \times M_3$ and $P_{2,3} \subset M_2 \times M_3$.

Notice that the Schützenberger product is not associative, in the sense that in general the monoids $\diamondsuit_2(M_1, \diamondsuit_2(M_2, M_3))$, $\diamondsuit_2(\diamondsuit_2(M_1, M_2), M_3)$ and $\diamondsuit_3(M_1, M_2, M_3)$ are pairwise distinct.

The following result shows that the Schützenberger product is the algebraic operation on monoids that corresponds to the marked product.

Proposition 12.1 *Let L_0, L_1, ..., L_n be languages of A^* recognized by monoids M_0, M_1, ..., M_n and let a_1, ..., a_n be letters of A. Then the marked product $L_0a_1L_1 \cdots a_nL_n$ is recognized by the monoid $\diamondsuit_{n+1}(M_0, M_1, \ldots, M_n)$.*

This result was extended to varieties by Reutenauer [50] for $n = 1$ and by the author [36] in the general case (see also [73] for a simpler proof). Let $\mathbf{V}_0$, ..., $\mathbf{V}_n$ be varieties of finite monoids and let $\diamondsuit_{n+1}(\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_n)$ be the variety of finite monoids generated by the Schützenberger products of the form $\diamondsuit_{n+1}(M_0, M_1, \ldots, M_n)$ with $M_0 \in \mathbf{V}_0$, $M_1 \in \mathbf{V}_1$, ..., $M_n \in \mathbf{V}_n$.

Theorem 12.2 *Let $\mathcal{V}$ be the $*$-variety corresponding to the variety of finite monoids $\diamondsuit_{n+1}(\mathbf{V}_0, \mathbf{V}_1, \ldots, \mathbf{V}_n)$. Then, for all alphabet A, $\mathcal{V}(A^*)$ is the boolean algebra generated by all the marked products of the form $L_0a_1L_1 \cdots a_nL_n$ where $L_0 \in \mathcal{V}_0(A^*)$,..., $L_n \in \mathcal{V}_n(A^*)$ and $a_1, \ldots, a_n \in A$.*

If $\mathbf{V}_0 = \mathbf{V}_1 = \ldots = \mathbf{V}_n = \mathbf{V}$, the variety $\diamondsuit_{n+1}(\mathbf{V}, \mathbf{V}, \ldots, \mathbf{V})$ is denoted $\diamondsuit_{n+1}\mathbf{V}$ and $\diamondsuit\mathbf{V} = \bigcup_{n>0} \diamondsuit_n\mathbf{V}$ denotes the union of all $\diamondsuit_n\mathbf{V}$. Given a $*$-variety of languages $\mathcal{V}$, the *extension of $\mathcal{V}$ under marked product* is the $*$-variety $\mathcal{V}'$ such that, for every alphabet A, $\mathcal{V}'(A^*)$ is the boolean algebra generated by the marked products of the form $L_0a_1L_1 \cdots a_nL_n$ where $L_0, L_1, \ldots, L_n \in \mathcal{V}(A^*)$ and $a_1, \ldots, a_n \in A$. The *closure of $\mathcal{V}$ under marked product* is the smallest $*$-variety $\bar{\mathcal{V}}$ such that, for every alphabet A, $\bar{\mathcal{V}}(A^*)$ contains $\mathcal{V}(A^*)$ and all the marked products of the form $L_0a_1L_1 \cdots a_nL_n$ where $L_0, L_1, \ldots, L_n \in \bar{\mathcal{V}}(A^*)$ and $a_1, \ldots, a_n \in A$. The $*$-variety corresponding to $\diamondsuit\mathbf{V}$ is described in the following theorem.

Theorem 12.3 *Let $\mathbf{V}$ be a monoid variety and let $\mathcal{V}$ be the corresponding $*$-variety. Then the $*$-variety corresponding to $\diamondsuit\mathbf{V}$ is the extension of $\mathcal{V}$ under marked product.*

Corollary 12.4 *A $*$-variety is closed under marked product if and only if the corresponding variety of monoids $\mathbf{V}$ satisfies $\mathbf{V} = \diamondsuit\mathbf{V}$.*

The Schützenberger product has a remarkable algebraic property [64, 39]. Let M_1, ..., M_n be monoids and let $\pi : \diamondsuit_n(M_1, \ldots, M_n) \to M_1 \times \cdots \times M_n$ be the monoid morphism that maps a matrix onto its diagonal.

Theorem 12.5 *For every idempotent e of $M_1 \times \cdots \times M_n$, the semigroup $\pi^{-1}(e)$ is in the variety $\mathbf{B}_1$.*

Given a variety of finite semigroups $\mathbf{V}$, a finite monoid M is called a $\mathbf{V}$-*extension of a finite monoid N* if there exists a surjective morphism $\varphi : M \to N$ such that, for every idempotent e of N, $\varphi^{-1}(e) \in \mathbf{V}$. Theorem 12.5 shows that the Schützenberger product of n finite monoids is a $\mathbf{B}_1$-extension of their product. Given a variety of finite monoids $\mathbf{W}$, the Malcev product $\mathbf{V} \textcircled{\scriptsize M} \mathbf{W}$ is the variety of finite monoids generated by all the $\mathbf{V}$-extensions of monoids of $\mathbf{W}$. This gives the following relation between the $\mathbf{V}_n$.

Theorem 12.6 *For every $n \geqslant 0$, $\mathbf{V}_{n+1}$ is contained in $\mathbf{B}_1 \textcircled{\scriptsize M} \mathbf{V}_n$.*

It is conjectured that $\mathbf{V}_{n+1} = \mathbf{B}_1 ⓜ \mathbf{V}_n$ for every n. If this conjecture were true, it would reduce the decidability of the dot-depth to a problem on the Malcev products of the form $\mathbf{B}_1 ⓜ \mathbf{V}$.

Malcev products actually play an important role in the study of the marked product. For instance, Straubing has established the important following result, which gives support to the previous conjecture.

Theorem 12.7 *Let* $\mathbf{V}$ *be a monoid variety and let* $\mathcal{V}$ *be the corresponding* $*$*-variety. Then the* $*$*-variety corresponding to* $\mathbf{A} ⓜ \mathbf{V}$ *is the closure of* $\mathcal{V}$ *under marked product.*

Example 12.1 Let $\mathbf{H}$ be a variety of finite groups (for instance, the variety of all finite commutative groups, nilpotent groups, solvable groups, etc.). Denote by $\bar{\mathbf{H}}$ the variety of all monoids whose subgroups (that is, $\mathcal{H}$-classes containing an idempotent) belong to $\mathbf{H}$. One can show that $\mathbf{A} ⓜ \bar{\mathbf{H}} = \bar{\mathbf{H}}$. Therefore, the corresponding $*$-variety is closed under marked product.

The marked product $L = L_0 a_1 L_1 \cdots a_n L_n$ of n languages $L_0, L_1, \ldots, L_n$ is *unambiguous* if every word u of L admits a unique factorization of the form $u_0 a_1 u_1 \cdots a_n u_n$ with $u_0 \in L_0$, $u_1 \in L_1, \ldots, u_n \in L_n$. The following result was established in [35, 46] as a generalization of a former result of Schützenberger [55].

Theorem 12.8 *Let* $\mathbf{V}$ *be a monoid variety and let* $\mathcal{V}$ *be the corresponding* $*$*-variety. Then the* $*$*-variety corresponding to* $\mathbf{LI} ⓜ \mathbf{V}$ *is the closure of* $\mathcal{V}$ *under unambiguous marked product.*

The extension of a given $*$-variety is also characterized in [46]. Other variations of the marked product have been considered [55, 35, 49]. They lead to some interesting algebraic constructions.

Another operation on semigroups has a natural counterpart in terms of languages. Given a variety of finite monoids $\mathbf{V}$, denote by $\mathbf{PV}$ the variety of finite monoids generated by all the monoids of the form $\mathcal{P}(M)$, for $M \in \mathbf{V}$. A monoid morphism $\varphi : B^* \to A^*$ is *length preserving* if it maps a letter of B onto a letter of A. Given a $*$-variety of languages $\mathcal{V}$, the *extension of* $\mathcal{V}$ *under length preserving morphisms* is the smallest $*$-variety $\mathcal{V}'$ such that, for every alphabet A, $\mathcal{V}'(A^*)$ contains the languages of the form $\varphi(L)$ where $L \in \mathcal{V}(B^*)$ and $\varphi : B^* \to A^*$ is a length preserving morphism. The *closure of* $\mathcal{V}$ *under length preserving morphisms* is the smallest $*$-variety $\bar{\mathcal{V}}$ containing $\mathcal{V}$ such that, for every length preserving morphism $\varphi : B^* \to A^*$, $L \in \bar{\mathcal{V}}(A^*)$ implies $\varphi^{-1}(L) \in \bar{\mathcal{V}}(B^*)$. We can now state the result found independently by Reutenauer [50] and Straubing [62].

Theorem 12.9 *Let* $\mathbf{V}$ *be a monoid variety and let* $\mathcal{V}$ *be the corresponding* $*$*-variety. Then the* $*$*-variety corresponding to* $\mathbf{PV}$ *is the extension of* $\mathcal{V}$ *under length preserving morphisms.*

Corollary 12.10 *A* $*$*-variety is closed under length preserving morphisms if and only if the corresponding variety of monoids* $\mathbf{V}$ *satisfies* $\mathbf{V} = \mathbf{PV}$*.*

These results motivated the systematic study of the varieties of the form **PV**, which is not yet achieved. See the survey article [38] for the known results prior to 1986 and the book of J. Almeida [1] for the more recent results.

The Schützenberger product and the power monoid are actually particular cases of a general construction which gives the monoid counterpart of a given operation on languages [42, 43, 40]. This general construction works for most operations on languages, with the notable exception of the star operation, but its presentation would take us to far afield. We conclude this section by a few results on the semidirect product of two varieties.

We have already defined the semidirect product $\mathbf{V} * \mathbf{W}$ of a variety of finite monoids $\mathbf{V}$ and a variety of finite semigroups $\mathbf{W}$. One can define similarly the semidirect product of two varieties of finite monoids or of two varieties of finite semigroups. For instance, if $\mathbf{V}$ and $\mathbf{W}$ are two varieties of finite monoids, $\mathbf{V} * \mathbf{W}$ is the variety of finite monoids generated by the semidirect products $M * N$ with $M \in \mathbf{V}$ and $N \in \mathbf{W}$ such that the action of N on M is unitary. This variety is also generated by the wreath products $M \circ N$ with $M \in \mathbf{V}$ and $N \in \mathbf{W}$.

Straubing has given a general construction to describe the languages recognized by the wreath product of two finite monoids. Let $M \in \mathbf{V}$ and $N \in \mathbf{W}$ be two finite monoids and let $\eta : A^* \to M \circ N$ be a monoid morphism. We denote by $\pi : M \circ N \to N$ the monoid morphism defined by $\pi(f, n) = n$ and we put $\varphi = \pi \circ \eta$. Thus φ is a monoid morphism from A^* into N. Let $B = N \times A$ and $\sigma : A^* \to B^*$ be the map (which is not a morphism!) defined by

$$\sigma(a_1 a_2 \cdots a_n) = (1, a_1)(\varphi(a_1), a_2) \cdots (\varphi(a_1 a_2 \cdots a_{n_1}), a_n)$$

Then Straubing's "wreath product principle" can be stated as follows.

Theorem 12.11 *If a language L is recognized by $\eta : A^* \to M \circ N$, then L is a finite boolean combination of languages of the form $X \cap \sigma^{-1}(Y)$, where $Y \subset B^*$ is recognized by M and where $X \subset A^*$ is recognized by N.*

Conversely, the finite boolean combinations of languages of the form $X \cap \sigma^{-1}(Y)$ are not necessarily recognized by $M \circ N$, but are certainly recognized by a monoid of the variety $\mathbf{V} * \mathbf{W}$. Therefore, a careful study of the languages of the form $\sigma^{-1}(Y)$ usually suffices to give a combinatorial description of the languages corresponding to $\mathbf{V} * \mathbf{W}$. A similar wreath product principle holds when $\mathbf{V}$ or $\mathbf{W}$ are varieties of finite semigroups. Examples of application of this technique include Proposition 8.2 and the proof of Schützenberger's theorem based on the fact that every finite aperiodic monoid divides a wreath product of copies of U_2. Straubing also has successfully used this principle to describe the variety of languages corresponding to solvable groups (solvable groups are wreath products of commutative groups) and in his proof of the equality $\mathbf{B}_n = \mathbf{V}_n * \mathbf{LI}$.

13 Conclusion

We have centered our presentation around the notion of variety and voluntarily left out several aspects of the theory which are developed extensively in other articles of this volume: H. Straubing, D. Thérien and W. Thomas survey the connections with formal logic and boolean circuits, J. Almeida and P. Weil present the implicit operations, D. Perrin and the author treat the theory of automata on infinite words, J. Rhodes states a general conjecture on Malcev products, the topological aspects are mentioned in the author's account of the success story $\mathbf{BG} = \mathbf{PG}$, S.W. Margolis and J. Meakin cover the extensions of automata theory to inverse monoids, M. Sapir demarcates the border between decidable and undecidable and H. Short shows that automata are also useful in group theory. Some other extensions are not covered at all in this volume, in particular the connections with the variable length codes, the rational and recognizable sets on arbitrary monoids and the extension of the theory to power series and algebras.

We hope that the reading of the articles of this volume will convince the reader that the algebraic theory of automata is a recent but flourishing subject. It is intimately related to the theory of finite semigroups and certainly one of the most convincing applications of this theory.

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