On uniformly continuous functions for some profinite topologies*

Dedicated to Antonio Restivo for his 70th birthday

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June 28, 2016

Abstract

Given a variety of finite monoids **V**, a subset of a monoid is a **V**-subset if its syntactic monoid belongs to **V**. A function between two monoids is **V**-preserving if it preserves **V**-subsets under preimages and it is hereditary **V**-preserving if it is **W**-preserving for every subvariety **W** of **V**. The aim of this paper is to study hereditary **V**-preserving functions when **V** is one of the following varieties of finite monoids: groups, p-groups, aperiodic monoids, commutative monoids and all monoids.

1 Introduction

This article is a follow-up of [12], where the authors started the study of **V**-preserving functions. Let us first remind the definition. Let M be a monoid and let **V** be a variety of finite monoids. A recognizable subset S of M is said to be a **V**-*subset* if its syntactic monoid belongs to **V**. A function $f : M \to N$ is called **V**-*preserving* if, for each **V**-subset of N, $f^{-1}(L)$ is a **V**-subset of M. A function is *hereditary* **V**-*preserving* if it is **W**-preserving for every subvariety **W** of **V**.

Let us first consider the case where f is a function from A^* to B^*, where A and B are finite alphabets. If **V** is the variety **M** of all finite monoids, a **V**-preserving function is also called *regularity-preserving*, according to the terminology used in [5, 16, 18]. The characterization of regularity-preserving functions is a long-term objective, but in spite of intensive research (see [10] for a detailed bibliography), it it is still out of reach. For the variety $\mathbf{G}_p$ of finite p-groups, the situation is more advanced. Indeed, the authors gave in [13] a characterization of $\mathbf{G}_p$-preserving functions when B is a one-letter alphabet

[1]The first author received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 670624). The second author was partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

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and a preliminary step towards a general solution can be found in [10]. For the variety $\mathbf{G}$ of finite groups and for the variety $\mathbf{A}$ of finite aperiodic monoids, the only known contribution to the study of $\mathbf{V}$-preserving functions seems to be the article of Reutenauer and Schützenberger on rational functions [14].

This paper focuses on hereditary $\mathbf{V}$-preserving functions when $\mathbf{V}$ is one of the varieties $\mathbf{M}$, $\mathbf{G}$, $\mathbf{G}_p$ and $\mathbf{A}$. We consider functions from a free monoid or a free commutative monoid to $\mathbb{N}$ and, in the case of the varieties $\mathbf{G}$ and $\mathbf{G}_p$, we also study functions from A^* to $\mathbb{Z}$ or from $\mathbb{Z}^k$ to $\mathbb{Z}$. The case of a one-letter alphabet was also discussed in [3]. Our results are summarized in the table below.

$\mathbf{V}$	$\mathbf{G}_p$	$\mathbf{G}$	$\mathbf{A}$	$\mathbf{M}$
$A^* \to \mathbb{Z}$	Theorem 3.19	Theorem 4.3	Open	Open
$\mathbb{Z}^k \to \mathbb{Z}$	Theorem 3.5	Theorem 4.1	Irrelevant	Corollary 7.5
$\mathbb{N}^k \to \mathbb{Z}$	Theorem 3.12	Theorem 4.2	Irrelevant	Open
$\mathbb{N}^k \to \mathbb{N}$	Theorem 3.12	Theorem 4.2	Theorem 5.4	Theorem 7.2

Characterization of hereditary V-preserving functions.

2 Preliminaries

In this section, we review the basic notions used in this paper.

2.1 Varieties

A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products. In the sequel, we shall use freely the term *variety* instead of *variety of finite monoids*.

We denote by $\mathbf{M}$ (respectively $\mathbf{Com}$, $\mathbf{G}$, $\mathbf{Ab}$, $\mathbf{A}$) the variety of all finite monoids (respectively finite commutative monoids, finite groups, finite abelian groups, finite aperiodic monoids). Given a prime number p, we denote by $\mathbf{G}_p$ the variety of all finite p-groups and by $\mathbf{Ab}_p$ the variety of all finite abelian p-groups. Each finite monoid M generates a variety, denoted by (M). The join of a family of varieties $(\mathbf{V}_i)_{i\in I}$ is the least variety containing all the varieties $\mathbf{V}_i$, for $i \in I$.

For $n > 0$, C_n denotes the cyclic group of order n. Throughout the paper, we shall use the well-known structure theorem for finite abelian groups [15], which shows that $\mathbf{Ab}$ is the variety generated by the finite cyclic groups.

Proposition 2.1 *Every finite abelian group is isomorphic to a direct product of finite cyclic groups.*

2.2 Ultrametrics and pseudo-ultrametrics

A *pseudo-ultrametric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties, for all $x, y, z \in X$:

(P_1) $d(x, y) \geqslant 0$,

(P_2) $d(x, x) = 0$,
(P_3) $d(x, y) = d(y, x)$,
(P_4) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$.

An *ultrametric* satisfies a stronger version of (P_2):

(P_5) $d(x, y) = 0$ if and only if $x = y$.

2.3 Uniformly continuous functions

Given two pseudometric spaces (X_1, d_1) and (X_2, d_2), a function $f : X_1 \to X_2$ is *uniformly continuous* if, for every positive real number ε there exists a positive real number $\delta > 0$ such that for all $(x, y) \in X^2$,

$$d_1(x, y) < \delta \text{ implies } d_2(f(x), f(y)) < \varepsilon. \tag{2.1}$$

It follows in particular that if $d_1(x, y) = 0$, then $d_2(f(x), f(y)) = 0$. Moreover this condition is sufficient if 0 is an isolated point in the range of d_1 and d_2. We shall only need a weaker version of this result.

Proposition 2.2 *If d_1 and d_2 have finite range, a function $f : (X_1, d_1) \to (X_2, d_2)$ is uniformly continuous if and only if*

$$d_1(x, y) = 0 \textit{ implies } d_2(f(x), f(y)) = 0. \tag{2.2}$$

Proof. Since d_2 has finite range, there exists a positive real number ε such that $d_2(u, v) < \varepsilon$ implies $d_2(u, v) = 0$. If f is uniformly continuous, there exists δ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. By the choice of ε, this actually implies $d_2(f(x), f(y)) = 0$ and thus (2.2) holds.

Since d_1 has finite range, there exists a positive real number δ such that $d_1(u, v) < \delta$ implies $d_1(u, v) = 0$. Suppose that (2.2) holds and let ε be a positive integer. If $d_1(u, v) < \delta$ then $d_1(u, v) = 0$ and by (2.2), $d_2(f(x), f(y)) = 0$. It follows in particular that $d_2(f(x), f(y)) < \varepsilon$ and thus f is uniformly continuous. $\square$

Nonexpansive functions form an interesting subclass of the class of uniformly continuous functions. A function $f : (X_1, d_1) \to (X_2, d_2)$ is *nonexpansive* if, for all $(x, y) \in X_1 \times X_1$,

$$d_2(f(x), f(y)) \leqslant d_1(x, y)$$

We shall use nonexpansive functions in Section 3.

2.4 Pro-V metrics

For the remainder of this section, let $\mathbf{V}$ denote a variety of finite monoids. Let M be a monoid and let $u, v \in M$. We say that a monoid N *separates* u and v if there exists a monoid morphism $\varphi : M \to N$ such that $\varphi(u) \neq \varphi(v)$. A monoid M is *residually* $\mathbf{V}$ if any two distinct elements of M can be separated by a monoid in $\mathbf{V}$.

We shall use the conventions $\min \emptyset = \infty$ and $2^{-\infty} = 0$. For all $u, v \in M$, let

$$r_{\mathbf{V}}(u, v) = \min \left\{ |N| \;\middle|\; N \text{ is in } \mathbf{V} \text{ and separates } u \text{ and } v \right\}$$

and $d_{\mathbf{V}}(u,v) = 2^{-r_{\mathbf{V}}(u,v)}$. Then $d_{\mathbf{V}}$ is a pseudo-ultrametric, called the *pro-*$\mathbf{V}$ metric on M (see [12]). If the monoid is residually $\mathbf{V}$, then $d_{\mathbf{V}}$ is an ultrametric.

In this paper, we consider free monoids, free commutative monoids and free abelian groups of finite rank: they are all finitely generated and residually $\mathbf{V}$ for the main varieties considered in this paper: monoids, (abelian) groups, abelian p-groups, (commutative) aperiodic monoids.

2.5 V-uniform continuity and V-hereditary continuity

Let M and N be monoids. A function $f : M \to N$ is said to be $\mathbf{V}$*-uniformly continuous* if it is uniformly continuous for the pro-$\mathbf{V}$ pseudometric on M and N. The following result was proved in [12, Theorem 4.1].

Proposition 2.3 *A function $f : M \to N$ is $\mathbf{V}$-preserving if and only if it is $\mathbf{V}$-uniformly continuous.*

We say that f is $\mathbf{V}$*-hereditarily continuous* if it is $\mathbf{W}$-uniformly continuous for each subvariety $\mathbf{W}$ of $\mathbf{V}$. Closure properties of this notion under various operators are analysed in [12, Subsection 4.3].

A monoid N is called $\mathbf{V}$*-projective* if the following property holds: if $\alpha : N \to R$ is a morphism and if $\beta : T \to R$ is a surjective morphism, where T (and hence R) is a monoid of $\mathbf{V}$, then there exists a morphism $\gamma : N \to T$ such that $\alpha = \beta \circ \gamma$.

$$\begin{tikzcd} & T \arrow[d, "\beta"] \\ N \arrow[ur, dashed, "\gamma"] \arrow[r, "\alpha"'] & R \end{tikzcd}$$

For example, any free monoid (in particular $\mathbb{N}$) is $\mathbf{V}$-projective for every variety of finite monoids. Similarly, any free group (in particular $\mathbb{Z}$) is $\mathbf{V}$-projective for every variety of finite groups. Note that a $\mathbf{V}$-projective monoid is $\mathbf{W}$-projective for every subvariety $\mathbf{W}$ of $\mathbf{V}$.

The following results were proved in [12]:

Proposition 2.4 [12, Proposition 5.7] *Let $\mathbf{V}$ be the join of a family $(\mathbf{V}_i)_{i\in I}$ of varieties of finite commutative monoids. A function from a monoid to a $\mathbf{V}$-projective monoid is $\mathbf{V}$-hereditarily continuous if and only if it is $\mathbf{V}_i$-hereditarily continuous for all $i \in I$.*

Proposition 2.5 [12, Proposition 5.4] *A function from a monoid to a commutative monoid is $\mathbf{V}$-hereditarily continuous if and only if it is $(\mathbf{V} \cap \mathbf{Com})$-hereditarily continuous.*

In constrast, note that a $\mathbf{V}$-uniformly continuous function from a monoid to a commutative monoid is not necessarily $(\mathbf{V} \cap \mathbf{Com})$-hereditarily continuous. For instance, the function f from $\{a,b\}^*$ to $\mathbb{N}$ defined by $f(ab) = 1$ and $f(u) = 0$ if $u \neq ab$ is $\mathbf{M}$-uniformly continuous but is not $\mathbf{Com}$-uniformly continuous.

2.6 p-adic valuations

Let p be a prime number. If n is a non-zero integer, the *p-adic valuation* of n is the integer

$$v_p(n) = \max\left\{k \in \mathbb{N} \mid p^k \text{ divides } n\right\}$$

By convention, $v_p(0) = +\infty$. Note that the equality $v_p(nm) = v_p(n) + v_p(m)$ holds for all integers n, m.

The *p-adic norm* of n is the real number

$$|n|_p = p^{-v_p(n)}.$$

The p-adic norm satisfies the following properties, for all $n, m \in \mathbb{Z}$:

(N_1) $|n|_p \geqslant 0$,
(N_2) $|n|_p = 0$ if and only if $n = 0$,
(N_3) $|mn|_p = |m|_p|n|_p$,
(N_4) $|m+n|_p \leqslant \max\{|m|_p, |n|_p\}$.

The p-adic valuation and the p-adic norm can be extended to $\mathbb{Z}^k$ as follows. Given $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, we set

$$v_p(n) = \min_{1 \leqslant j \leqslant k} \{v_p(n_j)\} \quad \text{and} \quad |n|_p = p^{-v_p(n)} = \max_{1 \leqslant j \leqslant k} \{|n_j|_p\}.$$

The p-adic norm on $\mathbb{Z}^k$ still satisfies (N_1), (N_2) and (N_4), as well as the following weaker version of (N_3):

(N_5) for all $n, m \in \mathbb{Z}^k$, $|mn|_p \leqslant |m|_p|n|_p$.

The p-adic norm on $\mathbb{Z}^k$ induces the *p-adic ultrametric* d_p on $\mathbb{Z}^k$, defined by $d_p(u,v) = |u - v|_p$. Note that the pro-$\mathbf{Ab}_p$ metric $d_{\mathbf{Ab}_p}$ and d_p are strongly equivalent metrics.

2.7 Binomial coefficients

Let A be a finite alphabet. We denote by A^* the free monoid on A. Note that if $|A| = 1$, then A^* is isomorphic to the additive monoid $\mathbb{N}$.

Let u and v be two words of A^*. Let $u = a_1 \cdots a_n$, with $a_1, \ldots, a_n \in A$. Then u is a *subword* of v if there exist $v_0, \ldots, v_n \in A^*$ such that $v = v_0 a_1 v_1 \cdots a_n v_n$. Set

$$\tbinom{v}{u} = |\{(v_0, \ldots, v_n) \mid v = v_0 a_1 v_1 \cdots a_n v_n\}|.$$

Note that if $A = \{a\}$, $u = a^n$ and $v = a^m$, then $\binom{v}{u} = \binom{m}{n}$ and hence these numbers constitute a generalization of the classical binomial coefficients. See [7, Chapter 6] for more details. Sometimes, it will be useful to use the convention $\binom{m}{n} = 0$ for $m \geqslant 0$ and $n \in \mathbb{Z} \setminus \{0, \ldots, m\}$, which is compatible with the usual properties of binomial coefficients.

2.8 Mahler expansions

For a fixed $v \in A^*$, we can view the generalized binomial coefficient $\binom{-}{v}$ as a function from A^* to $\mathbb{N}$. The functions $\left\{\binom{-}{v} \mid v \in A^*\right\}$ constitute a *locally finite* family of functions in the sense that, for each $u \in A^*$, the image of u is 0 for all but finitely many elements of the family.

It is clear that the sum of a locally finite family of functions is well defined. In particular, if $(g_v)_{v\in A^*}$ is a family of elements of an abelian group G, then there is a well-defined function f from A^* into G defined by the formula (in additive notation)

$$f(u) = \sum_{v\in A^*} g_v \binom{u}{v}$$

The generalized binomial coefficients provide a unique decomposition of the functions from A^* into G, which will be referred as *Mahler expansion*:

Proposition 2.6 (Lothaire [7]) *Let G be an abelian group and let $f : A^* \to G$ be an arbitrary function. Then there exists a unique family $\langle f, v\rangle_{v\in A^*}$ of elements of G such that, for all $u \in A^*$, $f(u) = \sum_{v\in A^*} \langle f, v\rangle \binom{u}{v}$. This family is given by the inversion formula*

$$\langle f, v\rangle = \sum_{w\in A^*} (-1)^{|v|+|w|} \binom{v}{w} f(w) \tag{2.3}$$

A similar result holds for functions from $\mathbb{N}^k$ to an abelian group G. If r is an element of $\mathbb{N}^k$ (or more generally of $\mathbb{Z}^k$), we denote by r_i its i-th component, so that $r = (r_1, \ldots, r_k)$. First observe that the family

$$\left\{ \binom{-}{r_1} \cdots \binom{-}{r_k} \mid r \in \mathbb{N}^k \right\}$$

is a locally finite family of functions from $\mathbb{N}^k$ into $\mathbb{N}$. Thus, given a family $(g_r)_{r\in\mathbb{N}^k}$, the formula

$$f(n) = \sum_{r\in\mathbb{N}^k} g_r \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

defines a function $f : \mathbb{N}^k \to G$. Conversely, each function from $\mathbb{N}^k$ to G admits a unique *Mahler expansion*, a result proved in a more general setting in [2, 1].

Proposition 2.7 *Let G be an abelian group and let $f : \mathbb{N}^k \to G$ be an arbitrary function. Then there exists a unique family $\langle f, r\rangle_{r\in\mathbb{N}^k}$ of elements of G such that, for all $n \in \mathbb{N}^k$,*

$$f(n) = \sum_{r\in\mathbb{N}^k} \langle f, r\rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}.$$

The coefficients $\langle f, r\rangle$ are given by

$$\langle f, r\rangle = \sum_{i_1=0}^{r_1} \ldots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \binom{r_1}{i_1} \cdots \binom{r_k}{i_k} f(i).$$

3 $\mathbf{G}_p$-hereditary continuity

Let p be a prime number. We proved in [11, 13] that $\mathbf{G}_p$-uniformly continuous functions from A^* to $\mathbb{Z}$ can be characterized by properties of their Mahler expansions. The case where A is a one-letter alphabet corresponds to the classical Mahler's Theorem from p-adic number theory [8, 9].

Theorem 3.1 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v\in A^*} \langle f, v\rangle \binom{u}{v}$ be its Mahler expansion. Then the following conditions are equivalent:*

(1) f *is* $\mathbf{G}_p$*-uniformly continuous;*

(2) $\lim_{|v|\to\infty}|\langle f, v\rangle|_p = 0$.

A similar result (Amice, [2]) holds when A^* is replaced by $\mathbb{Z}^k$ (see also [13, Corollary 6.3] for an alternative proof). In this section, we obtain analogous results for $\mathbf{G}_p$-hereditary continuity. A first step is to reduce $\mathbf{G}_p$-hereditary continuity to a simpler property.

Lemma 3.2 *A function from a monoid to a* $\mathbf{G}_p$*-projective commutative monoid is* $\mathbf{G}_p$*-hereditarily continuous if and only if it is* (C_{p^n})*-uniformly continuous for all* $n > 0$.

Proof. By Proposition 2.5, f is $\mathbf{G}_p$-hereditarily continuous if and only if it is $(\mathbf{G}_p \cap \mathbf{Com})$-hereditarily continuous. Since

$$\mathbf{G}_p \cap \mathbf{Com} = \mathbf{G}_p \cap \mathbf{Ab} = \mathbf{Ab}_p = \bigvee_{n>0} (C_{p^n})$$

by Proposition 2.1, Proposition 2.4 implies that f is $\mathbf{G}_p$-hereditarily continuous if and only if f is (C_{p^n})-hereditarily continuous for every $n \in \mathbb{N}$. Since the only subvarieties of (C_{p^n}) are those of the form (C_{p^i}) with $i \leqslant n$, the lemma follows. $\square$

Let $\mathbf{V}$ be a variety of groups. Since any morphism from $\mathbb{N}^k$ to a finite group extends uniquely to a morphism from $\mathbb{Z}^k$ to that same group, the pro-$\mathbf{V}$ pseudo-metric on $\mathbb{N}^k$ is the restriction of the pro-$\mathbf{V}$ pseudo-metric on $\mathbb{Z}^k$. Therefore the forthcoming results hold for $\mathbb{N}^k$ even though they are stated and proved for $\mathbb{Z}^k$.

We denote by $e_1, \ldots, e_k$ the canonical generators of both $\mathbb{N}^k$ and $\mathbb{Z}^k$. Thus $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in position j.

Lemma 3.3 *Let* $n \in \mathbb{N}$ *and let* d *be the pro-*(C_{p^n}) *pseudo-metric on* $\mathbb{Z}^k$*. For* $r, s \in \mathbb{Z}^k$*, one has* $d(r, s) = 2^{-p^m}$ *where*

$$m = \min\left\{ i \leqslant n \mid \text{there exists } j \in \{1, \ldots, k\} \text{ such that } r_j \not\equiv s_j \bmod p^i \right\}.$$

Proof. Suppose that $r_j \not\equiv s_j \pmod{p^i}$ for some $i \leqslant n$ and $j \in \{1, \ldots, k\}$. Let $f : \mathbb{Z}^k \to C_{p^i}$ be defined by $f(n) = n_j$. Clearly, $C_{p^i} \in (C_{p^n})$ and f separates r and s, hence $d(r, s) \geqslant 2^{-p^i}$ and so $d(r, s) \geqslant 2^{-p^m}$. Note that this last inequality holds trivially if $m = \infty$.

If $d(r, s) = 0$, equality follows. Otherwise, we may assume that $f : \mathbb{Z}^k \to G \in (C_{p^n})$ is a morphism that separates r and s with $|G|$ minimum. By Proposition 2.1, G is a direct product of cyclic groups. Since their order must divide $|G|$ which is a power of p, each one of these factor groups is of the form C_{p^i}. Since any group in (C_{p^n}) must satisfy the identity $x^{p^n} = 1$, we conclude that $i \leqslant n$ in each case. If G were a nontrivial direct product, we could decompose f into its components and contradict the minimality of G, thus $G = C_{p^i}$ with $i \leqslant n$.

Suppose that $r_j \equiv s_j \pmod{p^i}$ for every $j \in \{1, \ldots, k\}$. Then $r_j = s_j$ in C_{p^i} for every j and so

$$f(r) = \sum_{j=1}^{k} r_j f(e_j) = \sum_{j=1}^{k} s_j f(e_j) = f(s),$$

a contradiction. Thus $r_j \not\equiv s_j \pmod{p^i}$ for some $j \in \{1, \dots, k\}$ and so $i \geqslant m$. It follows that $d(r, s) = 2^{-p^i} \leqslant 2^{-p^m}$ and so $d(r, s) = 2^{-p^m}$ as required. □

The next corollary shows how the pro-(C_{p^n}) pseudo-metric relates to the p-adic norm:

Corollary 3.4 *Let $n \in \mathbb{N}$ and let d denote the pro-(C_{p^n}) pseudo-metric on $\mathbb{Z}^k$. For all $r, s \in \mathbb{Z}^k$, we have*

$$d(r,s) = \begin{cases} 2^{-\frac{p}{|r-s|_p}} & \text{if } |r-s|_p > p^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let

$$m = \min \left\{ i \leqslant n \mid \text{ exists } j \in \{1, \dots, k\} \text{ such that } r_j \not\equiv s_j \bmod p^i \right\}.$$

It is easy to check that

$$m = \begin{cases} v_p(r-s)+1 & \text{if } v_p(r-s) < n \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $v_p(r-s) < n$ if and only if $|r-s|_p > p^{-n}$. In this case,

$$p^m = p^{v_p(r-s)+1} = \frac{p}{|r-s|_p}$$

and the claim follows from Lemma 3.3. □

We arrive to our characterization of $\mathbf{G}_p$-hereditarily continuous functions.

Theorem 3.5 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}_p$-hereditarily continuous if and only if it is nonexpansive for the p-adic norm.*

Proof. Let d_n denote the pro-(C_{p^n}) pseudo-metric. By Lemma 3.2, f is hereditarily $\mathbf{G}_p$-uniformly continuous if and only if, for all $n > 0$, it is uniformly continuous for d_n. By Proposition 2.2, this holds if and only if, for all $r, s \in \mathbb{Z}^k$,

$$d_n(r,s) = 0 \text{ implies } d_n(f(r), f(s)) = 0. \tag{3.4}$$

By Corollary 3.4, $d_n(r, s) = 0$ if and only if $|r-s|_p \leqslant p^{-n}$, thus (3.4) is equivalent to stating that for all $r, s \in \mathbb{Z}^k$,

$$|r-s|_p \leqslant p^{-n} \text{ implies } |f(r) - f(s)|_p \leqslant p^{-n}. \tag{3.5}$$

Clearly, (3.5) holds for every n if and only if $|f(r) - f(s)|_p \leqslant |r-s|_p$, which proves the result. □

It follows easily from Theorem 3.5 that all polynomial functions from $\mathbb{Z}^k$ to $\mathbb{Z}$ are $\mathbf{G}_p$-hereditarily continuous. We shall use the Mahler expansion of functions given by Proposition 2.7 to characterize all the $\mathbf{G}_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$. Polynomial functions will appear then as the finitely generated case. We shall need a few lemmas:

Lemma 3.6 *The sum of a locally finite family of $\mathbf{G}_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$ is $\mathbf{G}_p$-hereditarily continuous.*

Proof. Let $\{f_i : \mathbb{N}^k \to \mathbb{Z} \mid i \in I\}$ be a locally finite family of $\mathbf{G}_p$-hereditarily continuous functions and let $f = \sum_{i\in I} f_i$. By Theorem 3.5, each f_i is nonexpansive for the p-adic norm, and since the p-adic norm satisfies (N$_4$), f is also nonexpansive. $\square$

The following result is due to Kummer [6]. See also [17, 4].

Proposition 3.7 *Let $n, r \in \mathbb{N}$ with $0 \leqslant r \leqslant n$. Then $v_p\big(\binom{n}{r}\big)$ is equal to the number of carries it takes to add r and $n - r$ in base p.*

Taking $n = p^s$ yields the following corollary

Lemma 3.8 *Let $r, s \in \mathbb{N}$ with $0 < r \leqslant p^s$. Then $v_p\Big(\binom{p^s}{r}\Big) = s - v_p(r)$.*

We also need a result stated in [3, Lemma 2.8], for which we give a shorter proof.

Lemma 3.9 *Let $n, r, s \in \mathbb{N}$. Then*

$$p^s \text{ divides } \big(\operatorname*{lcm}_{1\leqslant j\leqslant r} j\big)\big(\tbinom{n+p^s}{r} - \tbinom{n}{r}\big) \tag{3.6}$$

or equivalently,

$$s \leqslant \max_{1\leqslant j\leqslant r} v_p(j) + \nu_p\big(\tbinom{n+p^s}{r} - \tbinom{n}{r}\big). \tag{3.7}$$

Proof. Since $\binom{n+p^s}{r} - \binom{n}{r} = \sum_{j=1}^{r} \binom{p^s}{j}\binom{n}{r-j}$, one gets by Lemma 3.8 the relation

$$\left|\binom{n+p^s}{r} - \binom{n}{r}\right|_p \leqslant \max_{1\leqslant j\leqslant r} \left|\binom{p^s}{j}\right|_p \left|\binom{n}{r-j}\right|_p \leqslant \max_{1\leqslant j\leqslant r} \left|\binom{p^s}{j}\right|_p = \max_{1\leqslant j\leqslant r} p^{v_p(j)-s}$$

or equivalently,

$$v_p\left(\binom{n+p^s}{r} - \binom{n}{r}\right) \geqslant \min_{1\leqslant j\leqslant r} (s - v_p(j)) = s - \max_{1\leqslant j\leqslant r} v_p(j)$$

which gives (3.7). $\square$

We shall need two elementary results on nonexpansive functions.

Lemma 3.10 *Let $f : \mathbb{N} \to \mathbb{Z}$ be a nonexpansive function for the p-adic norm and let $s \in \mathbb{N}$. Then for $0 \leqslant i \leqslant p^s$, p^s divides $\binom{p^s}{i}(f(i) - f(0))$, or equivalently, $s \leqslant v_p\Big(\binom{p^s}{i}\Big) + v_p(f(i) - f(0))$.*

Proof. Since f is nonexpansive, one has $|f(i) - f(0)|_p \leqslant |i - 0|_p$ and thus $v_p(f(i) - f(0)) \geqslant v_p(i)$. Since $v_p\Big(\binom{p^s}{i}\Big) = s - v_p(i)$ by Lemma 3.8, the relation $s \leqslant v_p\Big(\binom{p^s}{i}\Big) + v_p(f(i) - f(0))$ follows immediately. $\square$

Corollary 3.11 *Let $f : \mathbb{N} \to \mathbb{Z}$ be a nonexpansive function for the p-adic norm and let $s \in \mathbb{N}$. Then p^s divides $\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i)$.*

Proof. Newton's binomial formula yields

$$0 = (1-1)^{p^s} = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i},$$

hence

$$\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i) = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} (f(i) - f(0)).$$

The result now follows from Lemma 3.10. □

Theorem 3.12 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r\rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \to \mathbb{Z}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{G}_p$-hereditarily continuous,*

(2) *$v_p(j) \leqslant v_p(\langle f, r\rangle)$ holds for all j, r such that $1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}$.*

Proof. (1) $\Rightarrow$ (2). For all $r, t \in \mathbb{N}^k$, let us set

$$m_r(t) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \cdots \tbinom{r_k}{i_k} f(i+t).$$

By Proposition 2.7, we have $m_r(0, \ldots, 0) = \langle f, r\rangle$. We next show that

$$\min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\} \leqslant \min_{t \in \mathbb{N}^k} \{v_p(m_{r+s}(t))\} \tag{3.8}$$

for all $r, s \in \mathbb{N}^k$. By transitivity, we may assume that $s_1 + \ldots + s_k = 1$. By symmetry, we may assume that $s = (1, 0, \ldots, 0)$. Let $\ell = \min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\}$. For all $t \in \mathbb{N}^k$, we have

$$\begin{aligned}
m_{r+s}(t) &= \sum_{i_1=0}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{1+r_1}{i_1} \tbinom{r_2}{i_2} \cdots \tbinom{r_k}{i_k} f(i+t) \\
&= \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \cdots \tbinom{r_k}{i_k} f(i+t) \\
&+ \sum_{i_1=1}^{r_1+1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1-1} \tbinom{r_2}{i_2} \cdots \tbinom{r_k}{i_k} f(i+t) \\
&= -\sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \cdots \tbinom{r_k}{i_k} f(i+t) \\
&+ \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \cdots \tbinom{r_k}{i_k} f(i+t+s) \\
&= -m_r(t) + m_r(t+s).
\end{aligned}$$

Since $p^\ell \mid m_r(t)$ and $p^\ell \mid m_r(t+s)$, it follows that $p^\ell \mid m_{r+s}(t)$ and so (3.8) holds.

Now we show that

$$s \leqslant v_p\big(m_{p^s e_j}(t)\big) \tag{3.9}$$

for all $s \in \mathbb{N}$, $t \in \mathbb{N}^k$ and $j = 1, \ldots, k$.

By symmetry, we may assume that $j = 1$, so that (3.9) becomes

$$p^s \mid \sum_{i=0}^{p^s} (-1)^{p^s+i} \tbinom{p^s}{i} f(i + t_1, t_2, \ldots, t_k). \tag{3.10}$$

Fix $t \in \mathbb{N}^k$ and let $g : \mathbb{N} \to \mathbb{Z}$ be the function defined by

$$g(n) = f(n + t_1, t_2, \ldots, t_k).$$

By Theorem 3.5, g is $\mathbf{G}_p$-hereditarily continuous and thus (3.10) follows from Corollary 3.11. Therefore (3.10) holds and so does (3.9).

We now show that

$$1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\} \Rightarrow v_p(j) \leqslant v_p(m_r(t)) \tag{3.11}$$

holds for all $j \in \mathbb{N}$ and $r, t \in \mathbb{N}^k$.

We use induction on $q = r_1 + \ldots + r_k$. The claim holds trivially for $q = 0$, hence we assume that $q > 0$ and (3.11) holds for smaller values of q. By symmetry, we may assume that $r_1 > 0$.

Assume first that $1 \leqslant j \leqslant \max\{r_1 - 1, \ldots, r_k\}$. By the induction hypothesis on q, we have $v_p(j) \leqslant v_p(m_{r_1-1,r_2,\ldots,r_k}(t))$ for all $t \in \mathbb{N}^k$. Thus $v_p(j) \leqslant v_p(m_r(t))$ by (3.8).

The remaining case corresponds to $j = r_1 > \max\{r_1 - 1, \ldots, r_k\}$. If j is not a power of p, then we may write $j = j_1 j_2$ with $j_1 < j$ and $v_p(j_1) = v_p(j)$, falling into the previous case. Thus we may assume that $j = p^i$ for some $i \in \mathbb{N}$. By (3.9), we have $i \leqslant v_p\big(m_{p^i,0,\ldots,0}(t)\big)$ for all $t \in \mathbb{N}^k$. Since $r_1 = j = p^i$, it follows from (3.8) that $v_p(j) = i \leqslant v_p(m_r(t))$ and (3.11) holds.

Considering now the particular case $t = 0$, we obtain Condition (2).

(2) $\Rightarrow$ (1). By Lemma 3.6, it is enough to show that the function

$$g(n) = \langle f, r\rangle \tbinom{n_1}{r_1} \cdots \tbinom{n_k}{r_k}$$

is $\mathbf{G}_p$-hereditarily continuous for a fixed $r \in \mathbb{N}^k$. Write $m = \langle f, r\rangle$. Let $x, y \in \mathbb{N}^k$ and assume that $p^s \mid x - y$. By Theorem 3.5, it suffices to show that

$$p^s \mid m\Big(\tbinom{x_1}{r_1} \cdots \tbinom{x_k}{r_k} - \tbinom{y_1}{r_1} \cdots \tbinom{y_k}{r_k}\Big). \tag{3.12}$$

We have $p^s \mid x - y$ if and only $y = x + p^s z$ for some $z \in \mathbb{Z}^k$. Clearly, we can obtain y from x by successively adding or subtracting $p^s e_i$ $(i = 1, \ldots, k)$. Since $p^s \mid \ell$ and $p^s \mid \ell'$ together imply $p^s \mid \ell - \ell'$, we may assume without loss of generality that $x = y + p^s e_i$. By symmetry, we may also assume that $i = 1$. Therefore (3.12) will follow from

$$p^s \mid m\Big(\tbinom{y_1 + p^s}{r_1} - \tbinom{y_1}{r_1}\Big). \tag{3.13}$$

By condition (2), we have $v_p(j) \leqslant v_p(m)$ if $1 \leqslant j \leqslant r_1$, hence Lemma 3.9 yields

$$s \leqslant \max_{1 \leqslant j \leqslant r_1} v_p(j) + \nu_p\Big(\tbinom{y_1+p^s}{r_1} - \tbinom{y_1}{r_1}\Big) \leqslant v_p(m) + \nu_p\Big(\tbinom{y_1+p^s}{r_1} - \tbinom{y_1}{r_1}\Big)$$

and (3.13) holds as required. $\square$

It followed from Theorem 3.5 that all polynomial functions $f : \mathbb{N}^k \to \mathbb{Z}$ with integer coefficients are $\mathbf{G}_p$-hereditarily continuous. There are of course only countably many such functions. Theorem 3.12 implies the existence of uncountably many $\mathbf{G}_p$-hereditarily continuous functions:

Corollary 3.13 *There are uncountably many $\mathbf{G}_p$-hereditarily continuous functions $f : \mathbb{N}^k \to \mathbb{Z}$.*

Proof. For every $r \in \mathbb{N}^k$, let

$$\ell_r = \max\{v_p(j) \mid 1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}\}.$$

By Theorem 3.12 and Proposition 2.7, the map

$$(n_r)_{r\in\mathbb{N}^k} \mapsto \sum_{r\in\mathbb{N}^k} p^{\ell_r} n_r \binom{-}{r_1} \cdots \binom{-}{r_k}$$

is a bijection between $\mathbb{Z}^{(\mathbb{N}^k)}$ and the set of all $\mathbf{G}_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$. $\square$

We now consider functions from a free monoid A^* to $\mathbb{Z}$. Let $h : A^* \to \mathbb{N}^A$ be the canonical morphism defined by $h(u) = (|u|_a)_{a\in A}$, where $|u|_a$ denotes as usual the number of occurrences of the letter a in u. Let $\sim$ be the *commutative equivalence*, formally defined by $u \sim v$ if and only if $h(u) = h(v)$.

Lemma 3.14 *Let $f : A^* \to \mathbb{Z}$ be a $\mathbf{G}_p$-hereditarily continuous function and let $u, v \in A^*$ be commutatively equivalent. Then $f(u) = f(v)$.*

Proof. Let us choose s such that $p^s > |f(u)-f(v)|$ and let d (respectively d') be the pro-(C_{p^s}) pseudo-metric on A^* (respectively $\mathbb{Z}$). Since f is hereditarily $\mathbf{G}_p$-uniformly continuous, it is in particular (C_{p^s})-uniformly continuous. Now, if u and v are commutatively equivalent, then $d(x, y) = 0$ and hence $d'(f(x), f(y)) = 0$, which means that $f(x) \equiv f(y) \bmod p^s$. Since $p^s > |f(u) - f(v)|$, this finally implies that $f(u) = f(v)$. $\square$

Lemma 3.15 *Let $u \in A^*$ and $r = (r_a)_{a\in A} \in \mathbb{N}^A$. Then $\sum_{v\in h^{-1}(r)} \binom{u}{v} = \prod_{a\in A} \binom{|u|_a}{r_a}$.*

Proof. Let $\mathbb{Z}\langle A\rangle$ be the ring of polynomials in noncommutative variables in A with integer coefficients. The monoid morphism μ from A^* to the multiplicative monoid $\mathbb{Z}\langle A\rangle$ defined, for each letter $a \in A$, by $\mu(a) = 1 + a$, is called the *Magnus transformation*. By [7, Proposition 6.3.6], the following formula holds for all $u \in A^*$:

$$\mu(u) = \sum_{v\in A^*} \binom{u}{v} v \tag{3.14}$$

Let $\mathbb{Z}[A]$ be the ring of polynomials in commutative variables in A with integer coefficients. The commutative version of the Magnus transformation is the monoid morphism $\underline{\mu}$ from A^* to the multiplicative monoid $\mathbb{Z}[A]$ defined, for each letter $a \in A$, by $\underline{\mu}(a) = 1 + a$. Thus by definition, one has, for each word $v \in A^*$,

$$\underline{\mu}(u) = \prod_{a \in A} (1+a)^{|u|_a} = \prod_{a \in A} \left(\sum_{0 \leqslant r_a \leqslant |u|_a} \binom{|u|_a}{r_a} a^{r_a} \right) = \sum_{0 \leqslant r_a \leqslant |u|_a} \left(\prod_{a \in A} \binom{|u|_a}{r_a} \right) \prod_{a \in A} a^{r_a} \tag{3.15}$$

and on the other hand, (3.14) shows that

$$\underline{\mu}(u) = \sum_{v \in A^*} \binom{u}{v} \prod_{a \in A} a^{|v|_a} = \sum_{r \in \mathbb{N}^A} \left(\sum_{v \in h^{-1}(r)} \binom{u}{v} \right) \prod_{a \in A} a^{r_a} \tag{3.16}$$

Comparing (3.15) and (3.16) now gives the formula $\sum_{v \in h^{-1}(r)} \binom{u}{v} = \prod_{a \in A} \binom{|u|_a}{r_a}$. $\square$

Lemma 3.16 *Let $f : A^* \to G$ be a function from A^* to some abelian group with Mahler expansion $f(\_) = \sum_{w \in A^*} \langle f, w \rangle \binom{\_}{w}$. Then the following conditions are equivalent:*

(1) *for any two commutatively equivalent words u and v, $\langle f, u \rangle = \langle f, v \rangle$,*

(2) *for any two commutatively equivalent words u and v, $f(u) = f(v)$.*

Proof. (1) implies (2). Suppose that (2) holds. For each $r \in \mathbb{N}^k$, let $\langle k, r \rangle$ be the common value of $\langle f, v \rangle$ for all $v \in h^{-1}(r)$. With the help of Lemma 3.15, we now obtain

$$f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \sum_{v \in h^{-1}(r)} \langle k, r \rangle \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \langle k, r \rangle \sum_{v \in h^{-1}(r)} \binom{u}{v} = \sum_{r \in \mathbb{N}^k} \langle k, r \rangle \prod_{a \in A} \binom{|u|_a}{r_a}$$

It follows immediately that if u and v are commutatively equivalent, then $f(u) = f(v)$.

(2) implies (1). Let $g : A^* \to G$ be the function defined by $g(u) = (-1)^{|u|} \langle f, u \rangle$. It follows from the inversion formula (2.3) that $\langle g, x \rangle = (-1)^{|x|} f(x)$. Thus if (2) holds, then for any two commutatively equivalent words u and v, $\langle g, u \rangle = \langle g, v \rangle$. By the first part of the proof applied to g, it follows that $g(u) = g(v)$ and thus $\langle f, u \rangle = \langle f, v \rangle$. $\square$

Lemma 3.17 *Let $g : \mathbb{N}^k \to \mathbb{Z}$ be a function and let $\mathbf{V}$ be a variety of finite groups. Then g is $\mathbf{V}$-hereditarily continuous if and only if $g \circ h$ is $\mathbf{V}$-hereditarily continuous.*

Proof. By Proposition 2.5, g or $g \circ h$ are $\mathbf{V}$-hereditarily continuous if and only if they are $(\mathbf{V} \cap \mathbf{Ab})$-hereditarily continuous. Let $\mathbf{W}$ be a subvariety of $\mathbf{V} \cap \mathbf{Ab}$ and let d denote the pro-$\mathbf{W}$ pseudo-metric. Since h is surjective, every element of $\mathbb{N}^k$ can be written in the form $h(u)$ for some $u \in A^*$. Therefore g is $\mathbf{W}$-uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $u, v \in A^*$,

$$d\,(h(u), h(v)) < \delta \text{ implies } d\,(g \circ h(u), g \circ h(v)) < \varepsilon \tag{3.17}$$

Since any morphism from A^* to an abelian group factors through $\mathbb{N}^k$, one has $d(u,v) = d(h(u), h(v))$ for all $u, v \in A^*$. Therefore (3.18) can be rewritten as

$$d(u,v) < \delta \text{ implies } d\left(g \circ h(u), g \circ h(v)\right) < \varepsilon \tag{3.18}$$

and thus g is **W**-uniformly continuous if and only if $g \circ h$ is **W**-uniformly continuous. $\square$

Lemma 3.18 *Let $g : \mathbb{N}^k \to \mathbb{Z}$ be a function and let*

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \text{ and } g \circ h(u) = \sum_{v \in A^*} \langle g \circ h, v \rangle \binom{u}{v}$$

be the Mahler expansions of g and $g \circ h$. Then $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$.

Proof. We have

$$\begin{aligned} g(n) &= g \circ h(a_1^{n_1} \cdots a_k^{n_k}) = \sum_{v \in A^*} \langle g \circ h, v \rangle \binom{a_1^{n_1} \cdots a_k^{n_k}}{v} \\ &= \sum_{v \in a_1^* \cdots a_k^*} \langle g \circ h, v \rangle \binom{a_1^{n_1} \cdots a_k^{n_k}}{v} = \sum_{r_1=0}^{n_1} \cdots \sum_{r_k=0}^{n_k} \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}. \end{aligned}$$

By the uniqueness of the Mahler expansion in Proposition 2.7, we conclude that $\langle g, r \rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k} \rangle$ for every $r \in \mathbb{N}^k$. $\square$

Theorem 3.19 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then f is $\mathbf{G}_p$-hereditarily continuous if and only if it satisfies the following conditions:*

1. *for any two commutatively equivalent words u and v, $\langle f, u \rangle = \langle f, v \rangle$,*
2. *$v_p(j) \leqslant v_p(\langle f, v \rangle)$ holds for all $v \in A^*$ and $1 \leqslant j \leqslant \max_{a \in A} |v|_a$.*

Proof. Assume that f is $\mathbf{G}_p$-hereditarily continuous. By Lemmas 3.14 and 3.16, condition (1) holds. Moreover, by Lemma 3.14, we may write $f = g \circ h$, where $h : A^* \to \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \to \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \cdots a_k^{n_k}).$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$.

Assume that $v \in A^*$ and $j \in \mathbb{N}$ are such that $1 \leqslant j \leqslant |v|_{a_i}$ for every $i \in \{1, \ldots, k\}$. Let $r = (|v|_{a_1}, \ldots, |v|_{a_k})$. By Lemma 3.17, g is $\mathbf{G}_p$-hereditarily continuous and so we get $v_p(j) \leqslant v_p(\langle g, r \rangle) = v_p(\langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle)$ by Theorem 3.12. Since $v \sim a_1^{r_1} \cdots a_k^{r_k}$, we get $\langle f, v \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$ by Lemma 3.16 and so $v_p(j) \leqslant v_p(\langle f, v \rangle)$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \to \mathbb{Z}$ such that $f = g \circ h$. By Lemma 3.17, it suffices to show that g is $\mathbf{G}_p$-hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}$. By Theorem 3.12, we only need to show that

$$v_p(j) \leqslant v_p(\langle g, r \rangle). \tag{3.19}$$

By Lemma 3.18, we have $\langle g, r \rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle$. Since

$$1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\} = \max\{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \ldots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $v_p(j) \leqslant v_p(\langle f, a_1^{r_1} \cdots a_k^{r_k} \rangle)$ and so (3.19) holds as required. □

4 G-hereditary continuity

Let $\mathbb{P}$ denote the set of all positive primes.

Theorem 4.1 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}$-hereditarily continuous if and only if, for each prime p, it is nonexpansive for the p-adic norm.*

Proof. Since $\mathbf{G} \cap \mathbf{Com} = \bigvee_{p \in \mathbb{P}} (\mathbf{G}_p \cap \mathbf{Com})$, it follows from Propositions 2.4 and 2.5 that a function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}$-hereditarily continuous if and only if it is $\mathbf{G}_p$-hereditarily continuous for every $p \in \mathbb{P}$. It now remains to apply Theorem 3.5 to conclude. □

Theorem 3.12 yields:

Theorem 4.2 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \to \mathbb{Z}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{G}$-hereditarily continuous;*

(2) *j divides $\langle f, r \rangle$ for all $j \in \mathbb{N}$ and $r \in \mathbb{N}^k$ such that $1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}$.*

We present now the analogue of Theorem 3.19 through an adaptation of its proof. We keep the notation introduced in Section 3.

Theorem 4.3 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then f is $\mathbf{G}$-hereditarily continuous if and only if it satisfies the following conditions:*

(1) *if u and v are commutatively equivalent, then $\langle f, u \rangle = \langle f, v \rangle$,*

(2) *j divides $\langle f, v \rangle$ for all $v \in A^*$ and $1 \leqslant j \leqslant \max_{a \in A} |v|_a$.*

Proof. Assume that f is $\mathbf{G}$-hereditarily continuous. Since $\mathbf{G}$-hereditarily continuous implies $\mathbf{G}_p$-hereditarily continuous, Lemma 3.14 remains valid for $\mathbf{G}$. Together with Lemma 3.16, this yields condition (1). Moreover, by Lemma 3.14, we may write $f = gh$, where $h : A^* \to \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \to \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \cdots a_k^{n_k})$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r\rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r\rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k}\rangle$.

Assume that $1 \leqslant j \leqslant |v|_{a_i}$ for some $v \in A^*$ and $i \in \{1, \dots, k\}$. Let $r = (|v|_{a_1}, \dots, |v|_{a_k})$. By Lemma 3.17, g is $\mathbf{G}$-hereditarily continuous and so we get $j \mid \langle g, r\rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k}\rangle$ by Theorem 4.2. Since $v \sim a_1^{r_1} \cdots a_k^{r_k}$, we get $\langle f, v\rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k}\rangle$ by Lemma 3.16 and so $j \mid \langle f, v\rangle$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \to \mathbb{Z}$ such that $f = gh$. By Lemma 3.17, it suffices to show that g is $\mathbf{G}$-hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r\rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leqslant j \leqslant \max\{r_1, \dots, r_k\}$. By Theorem 4.2, we only need to show that

$$j \mid \langle g, r\rangle. \tag{4.20}$$

By Lemma 3.18, we have $\langle g, r\rangle = \langle f, a_1^{r_1} \cdots a_k^{r_k}\rangle$. Since

$$1 \leqslant j \leqslant \max\{r_1, \dots, r_k\} = \max\{|a_1^{r_1} \cdots a_k^{r_k}|_{a_1}, \dots, |a_1^{r_1} \cdots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $j \mid \langle f, a_1^{r_1} \cdots a_k^{r_k}\rangle$ and so (4.20) holds as required. $\square$

5 A-uniform continuity

Given a variety $\mathbf{V}$, let $\mathbf{CV} = \mathbf{Com} \cap \mathbf{V}$. In particular $\mathbf{CA}$ is the variety of commutative and aperiodic monoids. For each $t \in \mathbb{N}$, let $\mathbf{A}_t = [\![x^{t+1} = x^t]\!]$ and $\mathbf{CA}_t = \mathbf{Com} \cap \mathbf{A}_t$ be the variety of commutative aperiodic monoids of exponent t.

Let also N_t denote the monogenic monoid presented by $\langle x \mid x^t = x^{t+1}\rangle$. We usually view N_t as a quotient of $\mathbb{N}$ in order to represent its elements by natural numbers. The following results are folklore.

Proposition 5.1 *Every variety of commutative monoids is generated by its monogenic monoids. In particular* $\mathbf{CA}_t = (N_t)$ *for every* $t \in \mathbb{N}$*. Moreover, if* $\mathbf{V} \subseteq \mathbf{CA}$*, then* $\mathbf{V} = \mathbf{CA}_t$ *for some* $t \in \mathbb{N}$*.*

Given $m, n \in \mathbb{N}$, let us set

$$(m \wedge n) = \begin{cases} \min\{m, n\} & \text{if } m \neq n \\ \infty & \text{if } m = n \end{cases}$$

More generally, for $u, v \in \mathbb{N}^k$, we set write

$$(u \wedge v) = \min\{u_1 \wedge v_1, \ldots, u_k \wedge v_k\}.$$

Lemma 5.2 *Let $u, v \in \mathbb{N}^k$ and $t \in \mathbb{N}$. Then:*

(1) $r_{\mathbf{A}}(u, v) = r_{\mathbf{CA}}(u, v) = (u \wedge v) + 2$;

(2) $r_{\mathbf{A}_t}(u, v) = r_{\mathbf{CA}_t}(u, v) = \begin{cases} (u \wedge v) + 2 & \text{if } (u \wedge v) < t \\ \infty & \text{otherwise} \end{cases}$

Proof. We may assume that $u \neq v$. Let $\mathbf{V} \subseteq \mathbf{A}$. Since $\mathbf{CV} \subseteq \mathbf{V}$ and every quotient of $\mathbb{N}^k$ in $\mathbf{V}$ is necessarily in $\mathbf{CV}$, we have $r_{\mathbf{V}}(u, v) = r_{\mathbf{CV}}(u, v)$. We show next that

$$r_{\mathbf{CV}}(u, v) = \min\{|N_t| \mid N_t \in \mathbf{CV} \text{ and separates } u \text{ and } v\}. \tag{5.21}$$

Indeed, if $M \in \mathbf{CV}$ separates u and v through $\psi : \mathbb{N}^k \to M$, it follows from the proof of Proposition 5.1 that there exists an onto homomorphism $\varphi : N_{t_1} \times \cdots \times N_{t_n} \to M$, where each N_{t_i} may be assumed to be a submonoid of M. Since $\mathbb{N}^k$ is a free commutative monoid, we may factor ψ through θ:

$$\begin{tikzcd} \mathbb{N}^k \arrow[rr, "\theta"] \arrow[dr, dashed, "\psi"'] & & N_{t_1} \times \cdots \times N_{t_n} \arrow[dl, "\varphi"] \\ & M & \end{tikzcd}$$

Since $\psi(u) \neq \psi(v)$, one of the component morphisms $\theta_i : \mathbb{N}^k \to N_{t_i}$ must separate u and v. Therefore the smallest $M \in \mathbf{CV}$ separating u and v must be of the form N_t and so (5.21) holds.

(1) By (5.21), we have

$$r_{\mathbf{CA}}(u, v) = \min\{|N_t| \mid N_t \text{ separates } u \text{ and } v\}. \tag{5.22}$$

If $u \wedge v = u_i \wedge v_i$, it is immediate that the projection on the i-th component induces a morphism from $\mathbb{N}^k$ to $N_{(u \wedge v)+1}$ separating u and v.

Suppose now that $\eta : \mathbb{N}^k \to N_t$ separates u and v with $t \leqslant (u \wedge v)$. Since

$$\sum_{i=1}^{k} \eta(u_i e_i) = \eta(u) \neq \eta(v) = \sum_{i=1}^{k} \eta(v_i e_i),$$

we have $\eta(u_i e_i) \neq \eta(v_i e_i)$ for some $i \in \{1, \ldots, k\}$. Hence $\eta(e_i) \geqslant 1$. Since $u_i, v_i \geqslant t$, it follows that $\eta(u_i e_i) = t = \eta(v_i e_i)$, a contradiction.

Thus $N_{(u \wedge v)+1}$ is the smallest N_t separating u and v. In view of (5.21), it follows that

$$r_{\mathbf{CA}}(u, v) = |N_{(u \wedge v)+1}| = (u \wedge v) + 2.$$

(2) By (5.21) and Proposition 5.1, we have

$$r_{\mathbf{CA}_t}(u,v) = \min\{|N_s| \mid s \leqslant t \text{ and } N_s \text{ separates } u \text{ and } v\}.$$

In view of (5.22), it follows that

$$r_{\mathbf{CA}_t}(u,v) = \begin{cases} r_{\mathbf{CA}}(u,v) & \text{if } r_{\mathbf{CA}}(u,v) \leqslant t+1 = |N_t| \\ \infty & \text{otherwise} \end{cases}$$

By (1), $r_{\mathbf{CA}}(u,v) \leqslant t+1$ is equivalent to $(u \wedge v) < t$ and the claim follows. □

Theorem 5.3 *Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is* **A***-uniformly continuous,*

(2) *for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$, $u \wedge v \geqslant s$ implies $f(u) \wedge f(v) \geqslant n$,*

(3) *for every $n \in \mathbb{N}$, $f^{-1}(n)$ is either finite or cofinite.*

Proof. (1) ⇔ (2). It follows from the definition that f is **A**-uniformly continuous if and only if for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$,

$$r_{\mathbf{A}}(u,v) \geqslant s \text{ implies } r_{\mathbf{A}}(f(u), f(v)) \geqslant n,$$

that is equivalent to (2) by Lemma 5.2.

(2) ⇒ (3). Suppose that $f^{-1}(m)$ is neither finite nor cofinite. Let $s \in \mathbb{N}$ be arbitrary. Take $u_s \in f^{-1}(m)$ and $v_s \in \mathbb{N} \setminus f^{-1}(m)$ such that $u_s, v_s \geqslant s$. Thus the relation

$$u_s \wedge v_s \geqslant s \text{ and } f(u_s) \wedge f(v_s) \leqslant m$$

holds for all $s \in \mathbb{N}$, and so (2) fails.

(3) ⇒ (2). Let $n \in \mathbb{N}$. Suppose first that $f^{-1}(m)$ is cofinite for some $m \in \mathbb{N}$. Let $s = \max(\mathbb{N} \setminus f^{-1}(m))$. If $u \wedge v \geqslant s+1$, then $u \neq v$ implies $u, v \geqslant s+1$ and so $f(u) = m = f(v)$, hence we have $f(u) = f(v)$ in any case and $f(u) \wedge f(v) > n$ trivially.

Assume now that $f^{-1}(i)$ is finite for every $i \in \mathbb{N}$. Let $s = \max \cup_{i=0}^{n-1} f^{-1}(i)$. If $u \wedge v \geqslant s+1$ and $u \neq v$, then $u, v \geqslant s+1$ and so $u, v \notin \cup_{i=0}^{n} f^{-1}(i)$. Hence $f(u), f(v) \geqslant n$ and so $f(u) \wedge f(v) \geqslant n$. Therefore (2) holds. □

Similarly, we get

Theorem 5.4 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is* **A***-uniformly continuous;*

(2) *for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant s$ implies $f(u) \wedge f(v) \geqslant n$.*

However, there is no analogue of condition (3) of Theorem 5.3 in this case: if we define $f : \mathbb{N}^2 \to \mathbb{N}$ by $f(m,n) = m$, it is immediate that f is **A**-uniformly continuous and $f^{-1}(m)$ is infinite for every $m \in \mathbb{N}$.

Theorem 5.5 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping and $t \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{A}_t$-uniformly continuous,*

(2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant t$ implies $f(u) \wedge f(v) \geqslant t$.*

Proof. Since $\mathbb{N}^k$ and $\mathbb{N}$ are commutative, the pseudo-metrics $d_{\mathbf{A}_t}$ and $d_{\mathbf{CA}_t}$ coincide in both monoids. Hence f is $\mathbf{A}_t$-uniformly continuous if and only if it is $\mathbf{CA}_t$-uniformly continuous.

Since $\operatorname{Im} r_{\mathbf{CA}_t} = \{2, \ldots, t+1, \infty\}$ by Lemma 5.2 (2), it follows from Proposition 2.2 that f is $\mathbf{CA}_t$-uniformly continuous if and only if for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{CA}_t}(u, v) = \infty$ implies $r_{\mathbf{CA}_t}(f(u), f(v)) = \infty$. Now the claim follows from the same Lemma 5.2 (2). $\square$

6 A-hereditary continuity

Lemma 6.1 *A function from a monoid M to $\mathbb{N}$ is $\mathbf{A}$-hereditarily continuous if and only if it is $\mathbf{CA}_t$-uniformly continuous for every $t \in \mathbb{N}$.*

Proof. By Proposition 2.5, a function is $\mathbf{A}$-hereditarily continuous if and only if it is $\mathbf{CA}$-hereditarily continuous. The lemma now follows from [12, Proposition 5.9]. $\square$

Theorem 6.2 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is $\mathbf{A}$-hereditarily continuous,*

(2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \leqslant f(u) \wedge f(v)$,*

(3) *f is $\mathbf{A}$-nonexpansive.*

Proof. (1) is equivalent to (2). By Lemma 6.1, f is $\mathbf{A}$-hereditarily continuous if and only it is $\mathbf{CA}_t$-uniformly continuous for every $t \in \mathbb{N}$. In view of Lemma 5.2, this amounts to stating that, for all $t \in \mathbb{N}$ and for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant t$ implies $f(u) \wedge f(v) \geqslant t$.

(2) is equivalent to (3). By Lemma (5.2) (1), an equivalent formulation of (2) is that, for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{A}}(u, v) \leqslant r_{\mathbf{A}}(f(u), f(v))$, which is equivalent to (3). $\square$

We now look for a more explicit characterization of $\mathbf{A}$-hereditary continuity. Given a function $f : \mathbb{N}^k \to \mathbb{N}$, we say that $g : \mathbb{N} \to \mathbb{N}$ is a *slice function* of f if there exists some $j \in \{1, \ldots, k\}$ and $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \in \mathbb{N}$ such that $g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k)$ for every $x \in \mathbb{N}$.

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *extensive* if $x \leqslant f(x)$ for every $x \in \mathbb{N}$ and *truncated* if there exists some $m \in \mathbb{N}$ such that $x \leqslant f(x)$ for $x \leqslant m$ and $f(x) = m$ for $x > m$. Functions that are either extensive or truncated can be described by the following single property:

(C) if $b = \min\{x \in \mathbb{N} \mid f(x) < x\}$, then $f(x) = b - 1$ for every $x \geqslant b$.

Indeed, the case $b = \infty$ corresponds to extensive functions and the case b finite corresponds to truncated functions.

Lemma 6.3 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping satisfing condition (C) and assume that $f(a_1, \dots, a_k) < \min\{a_1, \dots, a_k\}$. Then:*

(1) *$f(x_1, \dots, x_k) = f(a_1, \dots, a_k)$ for all $x_1 \geqslant a_1, \dots, x_k \geqslant a_k$;*

(2) *there exists some $c \leqslant \min\{a_1, \dots, a_k\}$ such that $f(a_1, \dots, a_k) = f(c, \dots, c) = c - 1$.*

Proof. (1) We use induction on k. For $k = 1$, assume that $f(a) < a$ and $x \geqslant a$. Then there exists $b = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $x \geqslant a \geqslant b$ implies $f(x) = f(a) = b - 1$.

Assume now that $k > 1$ and (1) holds for smaller values of k. Let $x_1 \geqslant a_1$, $\dots$, $x_k \geqslant a_k$. By condition (C), we have $f(a_1, \dots, a_{k-1}, x_k) = f(a_1, \dots, a_k)$: indeed, if we take $b = \min\{x \in \mathbb{N} \mid f(a_1, \dots, a_{k-1}, x) < x\}$, then $b \leqslant a_k \leqslant x_k$ and so $f(a_1, \dots, a_{k-1}, x) = b - 1 = f(a_1, \dots, a_k)$.

Define now $g : \mathbb{N}^{k-1} \to \mathbb{N}$ by $g(y_1, \dots, y_{k-1}) = f(y_1, \dots, y_{k-1}, x_k)$. Since f satisfies (C), so does g. Moreover,

$$g(a_1, \dots, a_{k-1}) = f(a_1, \dots, a_{k-1}, x) = b-1 = f(a_1, \dots, a_k) < \min\{a_1, \dots, a_{k-1}\}.$$

By the induction hypothesis, we get $g(x_1, \dots, x_{k-1}) = g(a_1, \dots, a_{k-1})$ since $x_1 \geqslant a_1, \dots, x_{k-1} \geqslant a_{k-1}$. Thus

$$\begin{aligned} f(x_1, \dots, x_k) &= g(x_1, \dots, x_{k-1}) = g(a_1, \dots, a_{k-1}) \\ &= f(a_1, \dots, a_{k-1}, x_k) = f(a_1, \dots, a_k) \end{aligned}$$

as required.

(2) We use induction on k. For $k = 1$, assume that $f(a) < a$. Then there exists $c = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $a \geqslant c$ implies $f(a) = f(c) = c - 1$.

Assume now that $k > 1$ and (ii) holds for smaller values of k. Let

$$b = \min\{y \in \mathbb{N} \mid f(a_1, \dots, a_{k-1}, y) < y\}.$$

Then $b \leqslant a_k$. Define g as above. We have

$$g(a_1, \dots, a_{k-1}) = f(a_1, \dots, a_{k-1}, b) = b - 1 = f(a_1, \dots, a_k)$$

by condition (C). By the induction hypothesis, there exists $c \leqslant a_1, \dots, a_{k-1}$ such that $g(a_1, \dots, a_{k-1}) = g(c, \dots, c) = c - 1$. Thus $c - 1 = g(a_1, \dots, a_{k-1}) = b - 1$ and so $b = c$. Since $b \leqslant a_k$, we get $c \leqslant a_1, \dots, a_k$. Thus $f(a_1, \dots, a_k) = c - 1 = g(c, \dots, c) = f(c, \dots, c)$ as required. $\square$

Theorem 6.4 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is **A**-hereditarily continuous;*

(2) *every slice function of f is either extensive or truncated.*

Proof. (1) $\Rightarrow$ (2). Let g be the slice function of f defined by $g(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k)$ and let $a_j = \min\{x \in \mathbb{N} \mid g(x) < x\}$. If $x > a_j$, then one gets by Theorem 6.2

$$a_j = (a_1, \dots, a_k) \wedge (a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k) \leqslant g(a_j) \wedge g(x),$$

and since $g(a_j) < a_j$, it follows that $g(a_j) = g(x)$.

Let $z = g(a_j)$. It remains to prove that $z = a_j - 1$. Since $z < a_j$, we have $z + 1 \leqslant a_j$. Suppose that $z + 1 < a_j$. By Theorem 6.2, one has

$$z+1 = (a_1, \dots, a_k) \wedge (a_1, \dots, a_{j-1}, z+1, a_{j+1}, \dots, a_k) \leqslant g(a_j) \wedge g(z+1) = z \wedge g(z+1),$$

hence $g(z + 1) = z < z + 1$. Since $z + 1 < a_j$, this contradicts the minimality of a_j. Thus $z + 1 = a_j$ and (C) holds.

(2) $\Rightarrow$ (1). We use induction on k. For $k = 1$, assume that f satisfies condition (C) and let $u, v \in \mathbb{N}$ be distinct. By Theorem 6.2, we must prove that $u \wedge v \leqslant f(u) \wedge f(v)$. Let $Y = \{y \in \mathbb{N} \mid f(y) < y\}$. If $u, v \notin Y$, the claim follows. Hence we may assume that $Y \neq \emptyset$ and $b = \min Y$. If $u, v \in Y$, then $f(u) = b - 1 = f(v)$ by condition (C) and so $u \wedge v \leqslant \infty = f(u) \wedge f(v)$. Finally, assume that $u \in Y$ and $v \notin Y$. Then $f(u) = b - 1$. Without loss of generality, we may assume that $f(v) \neq b - 1$. Hence $v < b$ and so $v \leqslant (f(u) \wedge f(v))$. Thus $u \wedge v \leqslant f(u) \wedge f(v)$ and the result holds for $k = 1$.

Assume now that $k > 1$ and the theorem holds for smaller values of k. Assume that $f : \mathbb{N}^k \to \mathbb{N}$ satisfies condition (C) and let $u, v \in \mathbb{N}^k$ be distinct.

Assume first that $u_i = v_i$ for some $i \in \{1, \dots, k\}$. Without loss of generality, we may assume that $i = k$. Define $g : \mathbb{N}^{k-1} \to \mathbb{N}$ by $g(y_1, \dots, y_{k-1}) = f(y_1, \dots, y_{k-1}, u_k)$. Since f satisfies (C), so does g. By the induction hypothesis and Theorem 6.2, we get

$$\begin{aligned} u \wedge v &= (u_1, \dots, u_{k-1}) \wedge (v_1, \dots, v_{k-1}) \\ &\leqslant g(u_1, \dots, u_{k-1}) \wedge g(v_1, \dots, v_{k-1}) = f(u) \wedge f(v) \end{aligned}$$

as required.

Hence we may assume that $u_i \neq v_i$ for every $i \in \{1, \dots, k\}$. Without loss of generality, we may also assume that $u \wedge v = u_1$. Suppose first that $f(v) < u_1$. Since $u_1 \leqslant v_1, \dots, v_k$, we may apply Lemma 6.3(2) and get some $c \leqslant v_1, \dots, v_k$ such that $f(v) = f(c, \dots, c) = c - 1$. Thus $c - 1 < u_1$ and so $c < u_1 \leqslant u_2, \dots, u_k$. By Lemma 6.3(1), it follows that $f(u) = f(c, \dots, c) = f(v)$. Therefore $u \wedge v \leqslant f(u) \wedge f(v)$.

Next we assume that $f(u) < u_1$. Since $u_1 \leqslant u_2, \dots, u_k$, we may apply Lemma 6.3(2) and get some $c \leqslant u_1$ such that $f(u) = f(c, \dots, c) = c - 1$. Since $v_1, \dots, v_k \geqslant u_1 \geqslant c$, it follows from Lemma 6.3(1) that $f(v) = f(c, \dots, c) = f(u)$. Therefore $u \wedge v \leqslant f(u) \wedge f(v)$ also in this case.

The final case $f(u), f(v) \geqslant u_1 = u \wedge v$ is trivial. □

7 M-hereditary continuity

Proposition 7.1 *Let M be a monoid and $f : M \to \mathbb{N}$ a mapping. Then the following conditions are equivalent:*

(1) *f is **M**-hereditarily continuous;*

(2) *f is both **G**- and **A**-hereditarily continuous;*

(3) *f is both **Ab**- and **CA**-hereditarily continuous.*

Proof. The equivalence of (1) and (3) follows from [12, Proposition 5.8] and that of (2) and (3) from Proposition 2.5. □

Theorem 7.2 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then f is* **M***-hereditarily continuous if and only if:*

(1) $\gcd\{u_i - v_i \mid i = 1, \ldots, k\}$ *divides* $f(u) - f(v)$ *for all* $u, v \in \mathbb{N}^k$,

(2) *every slice function of f is either extensive or constant.*

Proof. By Proposition 7.1, f is **M**-hereditarily continuous if and only if it is both **G**- and **A**-hereditarily continuous. Now condition (1) is equivalent to **G**-hereditary continuity by Theorem 4.1. By Theorem 6.4, **A**-hereditary continuity is equivalent to every slice function of f being either extensive or truncated. Clearly, every constant function $f : \mathbb{N} \to \mathbb{N}$ is necessarily truncated. It remains to prove that every truncated slice function must be indeed constant in these circumstances.

Suppose that $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k)$ is truncated with $f(x) = m$ for every $x > m$. Let $M = \max(\operatorname{Im} f)$ and take $s < m$ arbitrary. We consider

$$u = (a_1, \ldots, a_{j-1}, s, a_{j+1}, \ldots, a_k), \quad v = (a_1, \ldots, a_{j-1}, M+s+1, a_{j+1}, \ldots, a_k).$$

Since

$$M + 1 = \gcd_{1 \leqslant i \leqslant k} (u_i - v_i) \mid f(u) - f(v)$$

and $|f(u) - f(v)| \leqslant M$, it follows that $f(u) = f(v)$, hence $g(s) = g(M+s+1) = m$. Therefore g is constant as claimed. □

Corollary 7.3 *Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping. Then f is* **M***-hereditarily continuous if and only if f is extensive or constant, and $u - v$ divides $f(u) - f(v)$ for all $u, v \in \mathbb{N}$.*

We can now adapt the proof of Corollary 3.13 to strengthen it:

Theorem 7.4 *There are uncountably many* **M***-hereditarily continuous functions from $\mathbb{N}$ to $\mathbb{N}$.*

Proof. By the uniqueness of Mahler expansions, the mapping

$$(n_r)_{r \in \mathbb{N}} \mapsto \sum_{r \in \mathbb{N}} n_r \operatorname{lcm}(1, \ldots, r) \binom{-}{r}$$

induces an injection θ from $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. Let $f \in \operatorname{Im} \theta$. By Theorem 4.2, f is **G**-hereditarily continuous. Since $n_1 \geqslant 1$, we have

$$\sum_{r \in \mathbb{N}} n_r \operatorname{lcm}(1, \ldots, r) \binom{x}{r} \geqslant n_1 \binom{x}{1} \geqslant x$$

for every $x \in \mathbb{N}$ and so f is extensive and thus **A**-hereditarily continuous by Theorem 6.4. Therefore f is **M**-hereditarily continuous by Proposition 7.1. Since $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ is uncountable and θ is one-to-one, $\operatorname{Im} \theta$ is an uncountable set of **M**-hereditarily continuous functions from $\mathbb{N}$ to $\mathbb{N}$. □

We can also settle the case of functions from $\mathbb{Z}^k$ to $\mathbb{Z}$.

Corollary 7.5 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{M}$-hereditarily continuous if and only if, for each prime p, it is nonexpansive for the p-adic norm.*

Proof. Let $f : \mathbb{Z}^k \to \mathbb{Z}$ be a function and let $\mathbf{V}$ denote a subvariety of $\mathbf{M}$. Since every quotient of $\mathbb{Z}^k$ is necessarily a group, the pseudo-metrics $d_{\mathbf{V}}$ and $d_{\mathbf{V}\cap\mathbf{G}}$ coincide in $\mathbb{Z}^k$ (and in particular in $\mathbb{Z}$). It follows that f is $\mathbf{V}$-uniformly continuous if and only if it is $\mathbf{V}\cap\mathbf{G}$-uniformly continuous. Since $\mathbf{V}\cap\mathbf{G}$ takes all possible values among the subvarieties of $\mathbf{G}$, it follows that f is $\mathbf{M}$-hereditarily continuous if and only if it is $\mathbf{G}$-hereditarily continuous. One can now apply Theorem 4.1 to conclude. $\square$

Acknowledgements

We would like to thank the anonymous referees for their helpful comments.

References

[1] R. Ahlswede and R. Bojanic, Approximation of continuous functions in p-adic analysis, *J. Approximation Theory* **15**,3 (1975), 190–205.

[2] Y. Amice, Interpolation p-adique, *Bull. Soc. Math. France* **92** (1964), 117–180.

[3] P. Cégielski, S. Grigorieff and I. Guessarian, Newton representation of functions over natural integers having integral difference ratios, *Int. J. Number Theory* **11**,4 (2015), 2109–2139.

[4] S. Frisch, Binomial coefficients generalized with respect to a discrete valuation, in *Applications of Fibonacci numbers, Vol. 7 (Graz, 1996)*, pp. 133–144, Kluwer Acad. Publ., Dordrecht, 1998.

[5] S. R. Kosaraju, Regularity preserving functions, *SIGACT News* **6 (2)** (1974), 16–17.

[6] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, *J. Reine Angew. Math.* **44** (1852), 93–146.

[7] M. Lothaire, *Combinatorics on words*, *Cambridge Mathematical Library*, Cambridge University Press, Cambridge, 1997.

[8] K. Mahler, An interpolation series for continuous functions of a p-adic variable., *J. Reine Angew. Math.* **199** (1958), 23–34. Correction **208** (1961), 70–72.

[9] K. Mahler, A correction to the paper "An interpolation series for continuous functions of a p-adic variable.", *J. Reine Angew. Math.* **208** (1961), 70–72.

[10] J.-É. Pin, Newton's forward difference equation for functions from words to words, in *Evolving Computability*, A. Beckmann, V. Mitrana and M. Soskova (eds.), pp. 71–82, *Lect. Notes Comp. Sci.* vol. 9136, Springer International Publishing, 2015.

[11] J.-É. Pin and P. V. Silva, A Mahler's theorem for functions from words to integers, in *25th International Symposium on Theoretical Aspects of Computer Science (STACS 2008)*, S. Albers and P. Weil (eds.), Dagstuhl, Germany, 2008, pp. 585–596, Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany.

[12] J.-É. Pin and P. V. Silva, On profinite uniform structures defined by varieties of finite monoids, *Internat. J. Algebra Comput.* **21** (2011), 295–314.

[13] J.-É. Pin and P. V. Silva, A noncommutative extension of Mahler's theorem on interpolation series, *European J. Combin.* **36** (2014), 564–578.

[14] C. Reutenauer and M.-P. Schützenberger, Variétés et fonctions rationnelles, *Theoret. Comput. Sci.* **145**,1-2 (1995), 229–240.

[15] J. J. Rotman, *The theory of groups. An introduction,* second ed., Allyn and Bacon Inc., Boston, Mass., 1973. Allyn and Bacon Series in Advanced Mathematics.

[16] J. I. Seiferas and R. McNaughton, Regularity-preserving relations, *Theoret. Comp. Sci.* **2** (1976), 147–154.

[17] D. Singmaster, Notes on binomial coefficients. I. A generalization of Lucas' congruence, *J. London Math. Soc. (2)* **8** (1974), 545–548.

[18] R. E. Stearns and J. Hartmanis, Regularity preserving modifications of regular expressions, *Information and Control* **6** (1963), 55–69.