A variety theorem without complementation

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1 Introduction.

The most important tool for classifying recognizable languages is Eilenberg's variety theorem [1], which gives a one-to-one correspondence between (pseudo)-varieties of finite semigroups and varieties of recognizable languages. Varieties of recognizable languages are classes of recognizable languages closed under union, intersection, complement, left and right quotients and inverse morphisms. Recall that one passes from a language to a finite semigroup by computing its syntactic semigroup.

However, certain interesting families of recognizable languages, which are not varieties of languages, also admit a syntactic characterization. The aim of this paper is to show that such results are not isolated, but are instances of a result as general as Eilenberg's theorem. On the language side, we consider *positive* varieties of languages, which have the same properties as varieties of languages except they are not supposed to be closed under complement. On the algebraic side, varieties of finite semigroups are replaced by varieties of finite ordered semigroups. Our main result states there is a one-to-one correspondence between positive varieties of languages and varieties of finite ordered semigroups. Due to the lack of space, we shall just give a few examples of this correspondence and defer to future papers the detailed study of our new types of varieties. For instance, P. Weil and the author have shown that the theorems of Birkhoff and Reiterman can be extended to ordered semigroups by replacing equations by inequations.

The proof of the main result is of course inspired by the proof of Eilenberg's theorem, although there are some subtle adjustments to do. The basic definitions and properties of ordered semigroups are presented in section 2. Recognizable sets are introduced in section 3, but our definition extends the standard one since we are dealing with ordered semigroups. The notion of syntactic ordered semigroup is defined in section 4. The main result is

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stated and proved in section 5 and a few examples are presented in section 6.

2 Ordered semigroups

An *ordered semigroup* $(S, \leq)$ is a semigroup S equipped with an order relation $\leq$ on S such that, for every $u, v, x \in S$, $u \leq v$ implies $ux \leq vx$ and $xu \leq xv$. An *order ideal* of $(S, \leq)$ is a subset I of S such that, if $x \leq y$ and $y \in I$, then $x \in I$. The order ideal *generated by an element* x is the set $[x]$ of all $y \in S$ such that $y \leq x$.

A *morphism of ordered semigroups* $\varphi : (S, \leq) \rightarrow (T, \leq)$ is a semigroup morphism from S into T such that, for every $x, y \in S$, $x \leq y$ implies $\varphi(x) \leq \varphi(y)$. If $\varphi : (R, \leq) \rightarrow (S, \leq)$ and $\psi : (S, \leq) \rightarrow (T, \leq)$ are morphisms of ordered semigroups, then $\psi \circ \varphi : (R, \leq) \rightarrow (T, \leq)$ is a morphism of ordered semigroups. A morphism of ordered semigroups $\varphi : (S, \leq) \rightarrow (T, \leq)$ is an *isomorphism* if there exists a morphism of ordered semigroups $\psi : (T, \leq) \rightarrow (S, \leq)$ such that $\varphi \circ \psi = Id_T$ and $\psi \circ \varphi = Id_S$. Therefore, one has the following characterization of isomorphisms.

Proposition 2.1 *A morphism of ordered semigroups $\varphi : (S, \leq) \rightarrow (T, \leq)$ is an isomorphism if and only if φ is a bijective semigroup morphism and, for every $x, y \in S$, $x \leq y$ is equivalent with $\varphi(x) \leq \varphi(y)$.*

Proof. If φ is an isomorphism then there exists a morphism of ordered semigroups $\psi : (T, \leq) \rightarrow (S, \leq)$ such that $\varphi \circ \psi = Id_S$ and $\psi \circ \varphi = Id_T$. In particular φ is a bijection. Furthermore $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ and $\varphi(x) \leq \varphi(y)$ implies $\psi(\varphi(x)) \leq \psi(\varphi(y))$, that is, $x \leq y$.

Conversely, suppose that φ is a bijection such that, for every $x, y \in S$, $x \leq y$ is equivalent with $\varphi(x) \leq \varphi(y)$. Let $\psi : (S, \leq) \rightarrow (T, \leq)$ be the inverse of φ. Then ψ is a semigroup morphism. It is also a morphism of ordered semigroups. Indeed, for every $x, y \in T$ such that $x \leq y$, one has $\psi(x) \leq \psi(y)$ since $x = \varphi(\psi(x))$ and $y = \varphi(\psi(y))$. Thus φ is an isomorphism. $\square$

The following example shows that a bijective morphism of ordered semigroups is not necessarily an isomorphism.

Example 2.1 Let $U_1 = \{0, 1\}$ be the two element semigroup defined by $1.1 = 1$ and $0.0 = 0.1 = 1.0 = 0$. Let $\leq$ the order on U_1 defined by $0 \leq 1$. Then the identity on U_1 defines a bijective morphism of ordered semigroups from $(U_1, =)$ into $(U_1, \leq)$. However, this is not an isomorphism.

The next propositions show that order ideals are preserved under union, intersection, inverse morphisms and residual.

Proposition 2.2 *Let $(S, \leq)$ be an ordered semigroup and let $(P_i)_{i \in I}$ be a family of order ideals of S. Then $\bigcap_{i \in I} P_i$ and $\bigcup_{i \in I} P_i$ are order ideals of S.*

Proof. Let $y \in \bigcap_{i \in I} P_i$ and let $x \leq y$. Then for all $i \in I$, $y \in P_i$ and thus $x \in P_i$. It follows that $x \in \bigcap_{i \in I} P_i$ and thus $\bigcap_{i \in I} P_i$ is an order ideal of S. Similarly, if $y \in \bigcup_{i \in I} P_i$, there exists $i \in I$ such that $y \in P_i$ and thus $x \in P_i$. It follows that $x \in \bigcup_{i \in I} P_i$ and thus $\bigcup_{i \in I} P_i$ is an order ideal of S. $\square$

Proposition 2.3 *Let $\varphi : (S, \leq) \to (T, \leq)$ be a morphism of ordered semigroups. If I is an order ideal of T, then $\varphi^{-1}(I)$ is an order ideal of S.*

Proof. Let $y \in \varphi^{-1}(I)$ and let $x \leq y$. Then $\varphi(y) \in I$ and $\varphi(x) \leq \varphi(y)$. It follows $\varphi(x) \in I$ since I is an order ideal and thus $x \in \varphi^{-1}(I)$. $\square$

Proposition 2.4 *Let I be an order ideal of an ordered semigroup $(S, \leq)$. Then for every $s \in S^1$, $s^{-1}I$ and Is^{-1} are order ideals of S. More generally, if K is a subset of S^1, $K^{-1}I$ and IK^{-1} are order ideals of S.*

Proof. If $y \leq x$ and $x \in K^{-1}I$ then $sx \in I$ for some $x \in K$ and since $sy \leq sx$, it follows $sy \in I$, whence $y \in K^{-1}I$. Thus $K^{-1}I$ is an order ideal. The proof for IK^{-1} is dual. $\square$

We say that $(S, \leq)$ is an *ordered subsemigroup* of $(T, \leq)$ if S is a subsemigroup of T and the order on S is the restriction to S of the order on T.

We say that $(T, \leq)$ is an *ordered quotient* of $(S, \leq)$ if there exists a surjective morphism of ordered semigroups $\varphi : (S, \leq) \to (T, \leq)$. For instance, any ordered semigroup $(S, \leq)$ is a quotient of $(S, =)$. An ordered semigroup $(S, \leq)$ *divides* an ordered semigroup $(T, \leq)$ if $(S, \leq)$ is an ordered quotient of an ordered subsemigroup of $(T, \leq)$.

Given a family $(S_i, \leq)_{i \in I}$ of ordered semigroups, the product $\prod_{i \in I}(S_i, \leq)$ is the ordered semigroup defined on the set $\prod_{i \in I} S_i$ by the law

$$(s_i)_{i \in I}(s'_i)_{i \in I} = (s_i s'_i)_{i \in I}$$

and the order given by

$$(s_i)_{i \in I} \leq (s'_i)_{i \in I} \text{ if and only if, for all } i \in I,\ s_i \leq s'_i.$$

Let A be a set and let A^+ be the free semigroup on A. Then $(A^+, =)$ is an ordered semigroup. We show that $(A^+, =)$ is in fact the free ordered semigroup on A.

Proposition 2.5 *If $\varphi : A \to S$ is a function from A into an ordered semigroup $(S, \leq)$, there exists a unique morphism of ordered semigroups $\bar{\varphi} : (A^+, =) \to (S, \leq)$ such that $\varphi(a) = \bar{\varphi}(a)$ for every $a \in A$. Moreover $\bar{\varphi}$ is surjective if and only if $\varphi(A)$ is a generator set of S.*

Proof. Since A^+ is the free semigroup on A, there exists a unique semigroup morphism $\bar{\varphi} : A^+ \to S$ such that $\varphi(a) = \bar{\varphi}(a)$ for every $a \in A$. Now $u = v$ implies $\varphi(u) = \varphi(v)$ and thus also $\varphi(u) \leq \varphi(v)$. Thus φ is a morphism of ordered semigroups. □

Corollary 2.6 *Let $\eta : (A^+, =) \to (S, \leq)$ be a morphism of ordered semigroups and let $\beta : (T, \leq) \to (S, \leq)$ be a surjective morphism of ordered semigroups. Then there exists a morphism of ordered semigroups $\varphi : (A^+, =) \to (T, \leq)$ such that $\eta = \beta \circ \varphi$.*

Proof. Let us associate with each letter $a \in A$ an element $\varphi(a)$ of $\beta^{-1}(\eta(a))$. We thus define a function $\varphi : A \to T$ which can be extended to a semigroup morphism $\varphi : A^+ \to T$ such that $\eta = \beta \circ \varphi$.

$$\begin{tikzcd}
& (A^+, =) \arrow[dl, "\varphi"'] \arrow[dr, "\eta"] & \\
(T, \leq) \arrow[rr, "\beta"] & & (S, \leq)
\end{tikzcd}$$

But since the order on A^+ is the equality, φ is in fact a morphism of ordered semigroups. □

3 Recognizable sets

Let $(S, \leq)$ be an ordered semigroup and let $\eta : (S, \leq) \to (T, \leq)$ be a surjective morphism of ordered semigroups. An order ideal Q of S is said to be *recognized* by η if there exists an order ideal P of T such that $Q = \eta^{-1}(P)$. Notice that this condition implies $\eta(Q) = \eta\eta^{-1}(P) = P$. By extension, an order ideal Q of S is said to be *recognized* by $(T, \leq)$ if there exists a surjective morphism of ordered semigroups from $(S, \leq)$ onto $(T, \leq)$ that recognizes Q. The next propositions show how the main operations on order ideals are captured by this definition.

Proposition 3.1 *Let I be a set and, for each $i \in I$, let $\eta_i : (S, \leq) \to (S_i, \leq)$ be a surjective morphism of ordered semigroups. If each η_i recognizes an order ideal Q_i of S, then the order ideals $\cap_{i \in I} Q_i$ and $\cup_{i \in I} Q_i$ are recognized by an ordered subsemigroup of the product $\prod_{i \in I}(S_i, \leq)$.*

Proof. For each $i \in I$, let P_i be an order ideal of S_i such that $Q_i = \eta_i^{-1}(P_i)$. Let $\eta : S \to \prod_{i \in I}(S_i, \leq)$ be the morphism defined by $\eta(u) = (\eta_i(u))_{i \in I}$. Put $S' = \eta(S)$, $P = \prod_{i \in I} P_i$ and

$$P' = \bigcup_{j \in I} \prod_{i \in I} P'_{i,j} \quad \text{where } P'_{i,j} = \begin{cases} P_i & \text{if } i = j \\ S_i & \text{if } i \neq j \end{cases}$$

Then $(S', \leq)$ is an ordered subsemigroup of $\prod_{i \in I}(S_i, \leq)$. Furthermore $P \cap S'$ and $P' \cap S'$ are order ideals of $(S', \leq)$. Indeed, if $y \leq x$ and $x \in P$ (resp. $x \in P'$), then for all (resp. for at least one) $i \in I$, $x_i \in P_i$. Thus $y_i \in P_i$ and thus $y \in P$ (resp. $y \in P'$). It follows that $P \cap S'$ and $P' \cap S'$ are order ideals of $(S', \leq)$. Finally, $\cap_{i \in I} Q_i = \eta^{-1}(P \cap S')$ and $\cup_{i \in I} Q_i = \eta^{-1}(P' \cap S')$. Thus $(S', \leq)$ recognizes $\cap_{i \in I} Q_i$ and $\cup_{i \in I} Q_i$. □

Proposition 3.2 *Let $\varphi : (R, \leq) \to (S, \leq)$ and $\eta : (S, \leq) \to (T, \leq)$ be two surjective morphisms of ordered semigroups. If η recognizes an order ideal Q of S, then $\eta \circ \varphi$ recognizes $\varphi^{-1}(Q)$.*

Proof. By definition, there exists an order ideal P of T such that $Q = \eta^{-1}(P)$. It follows that $\varphi^{-1}(Q) = \varphi^{-1}(\eta^{-1}(P)) = (\eta \circ \varphi)^{-1}(P)$. □

Proposition 3.3 *Let $\eta : (S, \leq) \to (T, \leq)$ be a surjective morphism of ordered semigroups. If η recognizes an order ideal Q of S, it also recognizes $K^{-1}Q$ and QK^{-1} for every subset K of S^1.*

Proof. Indeed, if $Q = \eta^{-1}(P)$ and $R = \eta(K)$, then $K^{-1}Q = \eta(R^{-1}P)$ and $QK^{-1} = \eta(PR^{-1})$. Now, by Proposition 2.4, $R^{-1}P$ and PR^{-1} are order ideals of $(T, \leq)$. □

4 Syntactic ordered semigroup

Let $(T, \leq)$ be an ordered semigroup and let P be an order ideal of T. We define on T two relations $\preceq_P$ and $\sim_P$ by setting

$u \preceq_P v$ if and only if, for every $x, y \in T^1$, $xvy \in P$ implies $xuy \in P$

$u \sim_P v$ if and only if $u \preceq_P v$ and $v \preceq_P u$

Proposition 4.1 *The relation $\preceq_P$ is a stable quasiorder on T and the relation $\sim_P$ is a semigroup congruence.*

Proof. The relation $\preceq_P$ is clearly reflexive. Suppose that $u \preceq_P v$ and $v \preceq_P w$. Then, for every $x, y \in T^1$, $xwy \in P$ implies $xvy \in P$ and $xvy \in P$ implies $xuy \in P$. Therefore $xwy \in P$ implies $xuy \in P$ and thus $u \preceq_P w$. It follows that $\preceq_P$ is transitive. Let $x, y \in T^1$. If $u \preceq_P v$, then $xuy \preceq_P xvy$ since if $sxvyt \in P$ then $sxuyt \in P$. Thus $\preceq_P$ is stable. It follows that $\sim_P$ is stable and thus, is a congruence. □

The congruence $\sim_P$ is called the *syntactic congruence* of P in T. The quasiorder $\preceq_P$ on T induces a stable order $\leq_P$ on $S(P) = T/{\sim_P}$. The ordered semigroup $(S(P), \leq_P)$ is called the *syntactic ordered semigroup* of P, the relation $\leq_P$ is called the *syntactic order* of P and the canonical morphism η_P from T onto $S(P)$ is called the *syntactic morphism* of P.

Proposition 4.2 *The map η_P defines a surjective morphism of ordered semigroups from $(T, \leq)$ onto $(S(P), \leq_P)$ which recognizes P.*

Proof. Since $\sim_P$ is a semigroup congruence, η_P is a semigroup morphism. Let u and v be two elements of T such that $u \leq v$. Suppose that $xvy \in P$. Then $xuy \leq xvy$ since $u \leq v$ and thus $xuy \in P$ since P is an order ideal. It follows that $u \preceq_P v$, whence $\eta_P(u) \leq_P \eta_P(v)$. Thus η_P defines a surjective morphism of ordered semigroups from $(T, \leq)$ onto $(S(P), \leq_P)$. We claim that η_P recognizes P. If $u \in P$ and $u \sim_P v$, then, for all $s, t \in T^1$, $sut \in P$ if and only if $svt \in P$. In particular, for $s = t = 1$, $u \in P$ implies $v \in P$. Therefore P is saturated by $\sim_P$, which proves the claim. □

Proposition 4.3 *Let $\varphi : (R, \leq) \to (S, \leq)$ be a surjective morphism of ordered semigroups and let P be an order ideal of $(R, \leq)$. The following properties hold:*

(1) *The morphism φ recognizes P if and only if η_P factorizes through it.*

(2) *Let $\pi : (S, \leq) \to (T, \leq)$ be a surjective morphism of ordered semigroups. If $\pi \circ \varphi$ recognizes P, then φ recognizes P.*

Proof. (1) First, by Proposition 4.2, $(S(P), \leq_P)$ recognizes P and $Q = \eta_P(P)$ is an order ideal of $(S(P), \leq_P)$ such that $P = \eta_P^{-1}(Q)$. If η_P factorizes through φ, there exists a morphism of ordered semigroups $\psi : (S, \leq) \to (S(P), \leq_P)$ such that $\eta_P = \psi \circ \varphi$.

$$\begin{tikzcd} & (T, \leq) \arrow[dl, "\varphi"'] \arrow[dr, "\eta_P"] & \\ (S, \leq) \arrow[rr, "\psi"] & & (S(P), \leq_P) \end{tikzcd}$$

Then ψ is onto and, by Proposition 2.3, $K = \psi^{-1}(Q)$ is an order ideal of S. Furthermore, $\varphi^{-1}(K) = \varphi^{-1}(\psi^{-1}(Q)) = \eta_P^{-1}(Q) = P$. Thus φ recognizes P.

Conversely, if φ recognizes P, there exists an order ideal Q of S such that $P = \varphi^{-1}(Q)$. Let $u, v \in R$ be such that $\varphi(u) \leq \varphi(v)$. We claim that $u \preceq_P v$. If $x, y \in R^1$ and $xvy \in P$, then $\varphi(xvy) \in \varphi(P) = Q$. Now since $\varphi(xuy) = \varphi(x)\varphi(u)\varphi(y) \leq \varphi(x)\varphi(v)\varphi(y) = \varphi(xvy)$, it follows $xuy \in \varphi^{-1}(Q) = P$, proving the claim. Therefore $\varphi(u) = \varphi(v)$ implies $u \sim_P v$ and thus, by [4], Proposition I.1.4, there exists a surjective semigroup morphism $\psi : S \to S(P)$ such that $\eta_P = \psi \circ \varphi$. In fact ψ is a morphism of ordered semigroups. Indeed, let u' and v' be two elements of S such that $u' \leq v'$. Since φ is onto, there exist two elements u and v of R such that $u' = \varphi(u)$ and $v' = \varphi(v)$. By the claim, $u \preceq v$ and thus $\eta_P(u) \leq_P \eta_P(v)$, that is, $\psi(u') \leq \psi(v')$. Thus η_P factorizes through φ.

(2) If $\pi \circ \varphi$ recognizes P then by (1), η_P factorizes through $\pi \circ \varphi$. But then η_P factorizes through φ and thus by (1), φ recognizes P. $\square$

The previous definitions and results apply in particular for free semigroups. Indeed, if A is a finite alphabet, then $(A^+, =)$ is an ordered semigroup and every subset of A^+ is an order ideal. Furthermore, if $(S, \leq)$ is an ordered semigroup, every surjective semigroup morphism $\eta : A^+ \to S$ induces a surjective morphism of ordered semigroups from $(A^+, =)$ onto $(S, \leq)$. Therefore, a language $L \subset A^+$ is said to be *recognized* by a semigroup morphism $\eta : A^+ \to (S, \leq)$ if there exists an order ideal P of S such that $L = \eta^{-1}(P)$. By extension, given an ordered semigroup $(S, \leq)$ and an order ideal P of S, we say that (S, P) recognizes $L \subset A^+$ if there exists a surjective semigroup morphism $\eta : A^+ \to S$ such that $L = \eta^{-1}(P)$.

In particular, we shall denote by $(S(L), \leq_L)$ (or simply $S(L)$) the syntactic ordered semigroup of a language L.

Corollary 4.4 *Let L a language of A^+ and let $(S, \leq)$ be an ordered semigroup. Then*

(1) *$(S, \leq)$ recognizes L if and only if $(S(L), \leq_L)$ is a quotient of $(S, \leq)$.*

(2) *If $(T, \leq)$ recognizes L and $(T, \leq)$ is a quotient of $(S, \leq)$, then $(S, \leq)$ recognizes L.*

Proof. (1) One applies Proposition 4.3 with $(R, \leq) = (A^+, =)$ and $P = L$. Thus if $\varphi : (A^+, =) \to (S, \leq)$ recognizes L, then η_L factorizes through φ and therefore $(S(L), \leq_L)$ is a quotient of $(S, \leq)$. Conversely, suppose there exists a surjective morphism of ordered semigroups $\beta : (S, \leq) \to (S(L), \leq_L)$. Then, by Corollary 2.6, there exists a morphism of ordered semigroups $\varphi : (A^+, =) \to (S, \leq)$ such that $\eta_L = \beta \circ \varphi$. Therefore, η_L factorizes through φ and by Proposition 4.3, φ recognizes L.

(2) If $(T, \leq)$ recognizes L, then by (1), $(S(L), \leq_L)$ is a quotient of $(T, \leq)$. Therefore if $(T, \leq)$ is a quotient of $(S, \leq)$, then $(S(L), \leq_L)$ is a quotient of $(S, \leq)$ and thus by (1), $(S, \leq)$ recognizes L. $\square$

5 Varieties of ordered semigroups

A *variety of finite ordered semigroups* is a class of finite ordered semigroups closed under the taking of ordered subsemigroups, ordered quotients and finite products.

Recall that a *class of recognizable languages* is a correspondence $\mathcal{C}$ which associates with each finite alphabet A a set $\mathcal{C}(A^+)$ of recognizable languages of A^+.

If $\mathbf{V}$ is a variety of finite ordered semigroups, we denote by $\mathcal{V}(A^+)$ the set of recognizable languages of A^+ whose ordered syntactic semigroup belongs to $\mathbf{V}$. The following is an equivalent definition:

Proposition 5.1 *$\mathcal{V}(A^+)$ is the set of languages of A^+ recognized by an ordered semigroup of $\mathbf{V}$.*

Proof. If $L \in \mathcal{V}(A^+)$, then the ordered syntactic semigroup of L, which recognizes L, belongs to $\mathbf{V}$. Conversely, if L is recognized by an ordered semigroup $(S, \leq)$ of $\mathbf{V}$, then by Corollary 4.4, the ordered syntactic semigroup of L is a quotient of $(S, \leq)$ and thus belongs also to $\mathbf{V}$. $\square$

The correspondence $\mathbf{V} \to \mathcal{V}$ associates with each variety of finite ordered semigroups a class of recognizable languages. The next proposition shows that this correspondence is one to one.

Theorem 5.2 *Let $\mathbf{V}$ and $\mathbf{W}$ be two varieties of finite ordered semigroups. Suppose that $\mathbf{V} \to \mathcal{V}$ and $\mathbf{W} \to \mathcal{W}$. Then $\mathbf{V} \subset \mathbf{W}$ if and only if, for every finite alphabet A, $\mathcal{V}(A^+) \subset \mathcal{W}(A^+)$. In particular, $\mathbf{V} = \mathbf{W}$ if and only if $\mathcal{V} = \mathcal{W}$.*

Proof. If $\mathbf{V} \subset \mathbf{W}$, it follows immediately from the definitions that $\mathcal{V}(A^+) \subset \mathcal{W}(A^+)$. The converse is based on the following proposition.

Proposition 5.3 *Let $\mathbf{V}$ be a variety of ordered semigroups and let $(S, \leq) \in \mathbf{V}$. Then there exist a finite alphabet A and languages $L_1, \ldots, L_k \in \mathcal{V}(A^+)$ such that $(S, \leq)$ is isomorphic with a subsemigroup of $S(L_1) \times \cdots \times S(L_k)$.*

Proof. Since S is finite, there exists a finite alphabet A and a surjective semigroup morphism $\varphi : A^+ \to S$. For each $s \in S$, put $L_s = \varphi^{-1}([s])$. Since $[s]$ is an order ideal, L_s is recognized by $(S, \leq)$ and thus $L_s \in \mathcal{V}(A^+)$. Let $(S_s, \leq_s)$ be the ordered syntactic semigroup of L_s. Since $(S, \leq)$ recognizes

L_s, Corollary 4.4 shows that $(S_s, \leq_s)$ is a quotient of $(S, \leq)$. We denote by $\pi_s : (S, \leq) \rightarrow (S_s, \leq_s)$ the projection and by $\pi : S \rightarrow \prod_{s \in S}(S_s, \leq_s)$ the morphism of ordered semigroups defined by $\pi(x) = (\pi_s(x))_{s \in S}$. We claim that, for each $x, y \in S$, $x \leq y$ if and only if $\pi(x) \leq \pi(y)$. Since π is a morphism of ordered semigroups, it is clear that $x \leq y$ implies $\pi(x) \leq \pi(y)$. Conversely, if $\pi(x) \leq \pi(y)$, then in particular, $\pi_y(x) \leq_y \pi_y(y)$. This means that, for every $s, t \in S^1$, $syt \leq y$ implies $sxt \leq y$ and for $s = t = 1$, one gets $x \leq y$. This proves the claim. It follows in particular that π is one to one and by Proposition 2.1 that S is isomorphic with a subsemigroup of $\prod_{s \in S}(S_s, \leq_s)$. $\square$

We can now complete the proof of theorem 5.2. Suppose that $\mathcal{V}(A^+) \subset \mathcal{W}(A^+)$ for every finite alphabet A and let $S \in \mathbf{V}$. Then by Proposition 5.3, S is isomorphic with a subsemigroup of the form $S(L_1) \times \cdots \times S(L_k)$, where $L_1, \ldots, L_k \in \mathcal{V}(A^+)$. It follows that $L_1, \ldots, L_k \in \mathcal{W}(A^+)$, i.e. $S(L_1), \ldots, S(L_k) \in \mathbf{W}$. Therefore $S \in \mathbf{W}$. $\square$

We now characterize the classes of languages which can be associated with a variety of ordered semigroups.
A *positive variety of languages* is a class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $\mathcal{V}(A^+)$ is closed under finite union and finite intersection,
(2) if $\varphi : A^+ \rightarrow B^+$ is a semigroup morphism, $L \in \mathcal{V}(B^+)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^+)$,
(3) if $L \in \mathcal{V}(A^+)$ and if $a \in A$, then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^+)$.

In particular, $\mathcal{V}(A^+)$ always contain the empty set and the set A^+ since $\emptyset = \bigcup_{i \in \emptyset} L_i$ and $A^+ = \bigcap_{i \in \emptyset} L_i$.

Proposition 5.4 *Let* $\mathbf{V}$ *be a variety of finite ordered semigroups. If* $\mathbf{V} \rightarrow \mathcal{V}$*, then* $\mathcal{V}$ *is a positive variety of languages.*

Proof. Let $L, L_1, L_2 \in \mathcal{V}(A^+)$ and let $a \in A$. Then by definition $S(L)$, $S(L_1)$, $S(L_2)$ are in $\mathbf{V}$. By Proposition 3.1, the languages $L_1 \cup L_2$ and $L_1 \cap L_2$ are recognized by an ordered subsemigroup T of $S(L_1) \times S(L_2)$. Now since $\mathbf{V}$ is a variety of ordered semigroups, $T \in \mathbf{V}$ and thus $L_1 \cup L_2$ and $L_1 \cap L_2$ belong to $\mathcal{V}(A^+)$. Since $\emptyset$ and A^+ are recognized by the trivial ordered semigroup $(\{1\}, =)$, which is certainly in $\mathbf{V}$, condition (1) is satisfied. Similarly, Proposition 3.3 shows that the languages $a^{-1}L$ and La^{-1} are recognized by $S(L)$ and Proposition 3.2 shows that, if $\varphi : B^+ \rightarrow A^+$ is a semigroup morphism, then $\varphi^{-1}(L)$ is recognized by $S(L)$. Thus conditions (2) and (3) are satisfied. $\square$

Theorem 5.5 *For every positive variety of languages $\mathcal{V}$, there exists a variety of finite ordered semigroups* $\mathbf{V}$ *such that* $\mathbf{V} \to \mathcal{V}$.

Proof. Let $\mathbf{V}$ be the variety of ordered semigroups generated by the ordered semigroups of the form $S(L)$ where $L \in \mathcal{V}(A^+)$ for a certain alphabet A. Suppose that $\mathbf{V} \to \mathcal{W}$; we shall in fact show that $\mathcal{V} = \mathcal{W}$. First, if $L \in \mathcal{V}(A^+)$, we have $S(L) \in \mathbf{V}$ by definition and therefore $L \in \mathcal{W}(A^+)$, still by definition. Therefore, for every alphabet A, $\mathcal{V}(A^+) \subset \mathcal{W}(A^+)$.

The inclusion $\mathcal{W}(A^+) \subset \mathcal{V}(A^+)$ is more difficult to prove. Let $L \in \mathcal{W}(A^+)$. Then $S(L) \in \mathbf{V}$ and since $\mathbf{V}$ is the variety generated by the ordered semigroups of the form $S(L)$ where L is a language of $\mathcal{V}$, there exist an integer $n > 0$ and, for $1 \leq i \leq n$, alphabets A_i and languages $L_i \in \mathcal{V}_i(A_i^+)$ such that $S(L)$ divides $S(L_1) \times \cdots \times S(L_k) = (S, \leq)$. Since $S(L)$ divides $(S, \leq)$, $S(L)$ is quotient of an ordered subsemigroup $(T, \leq)$ of $(S, \leq)$. By Corollary 4.4, $(T, \leq)$ recognizes L. Therefore there exists a surjective morphism of ordered semigroups $\varphi : (A^+, =) \to (T, \leq)$ and an order ideal P of T such that $L = \varphi^{-1}(P)$. Let $\iota : (T, \leq) \to (S, \leq)$ be the identity on T and let $\pi_i : (S, \leq) \to S(L_i)$ be the i-th projection defined by $\pi_i(s_1, \ldots, s_n) = s_i$. Put $\varphi' = \iota \circ \varphi$, $\varphi_i = \pi_i \circ \varphi'$ and let $\eta_i : A^+ \to S(L_i)$ be the syntactic morphism of L_i. Since η_i is onto, there exists by Corollary 2.6 a morphism of ordered semigroups $\psi_i : A^+ \to A_i^+$ such that $\varphi_i = \eta_i \circ \psi_i$. We can summarize the situation by a diagram:

$$\begin{tikzcd}
A^+ \arrow[r, "\psi_i"] \arrow[d, "\varphi'"'] \arrow[dr, "\varphi_i"] & A_i^+ \arrow[d, "\eta_i"] \\
S \arrow[r, "\pi_i"] & S(L_i)
\end{tikzcd}$$

We recall that we are seeking to prove that $L \in \mathcal{V}(A^+)$, which is finally obtained by a succession of reductions of the problem.

(1) We have
$$L = \varphi^{-1}(P) = \bigcup_{s \in P} \varphi^{-1}([s])$$
Since $\mathcal{V}(A^+)$ is closed under union, it suffices to establish that for every $s \in T$ we have $\varphi^{-1}([s]) \in \mathcal{V}(A^+)$.

(2) Put $\iota(s) = (s_1, \ldots, s_n)$. We claim that
$$\varphi^{-1}([s]) = \bigcap_{1 \leq i \leq n} \varphi_i^{-1}([s_i])$$

Indeed, we have the following sequence of equivalences

$$\begin{aligned}
u \in \varphi^{-1}([s]) &\iff s \geq \varphi(u) \\
&\iff \iota(s) \geq \iota(\varphi(u)) = \varphi'(u) \\
&\iff \text{for } 1 \leq i \leq n,\ \pi_i(\iota(s)) \geq \pi_i(\varphi'(u)) \\
&\iff \text{for } 1 \leq i \leq n,\ s_i \geq \varphi_i(u) \\
&\iff \text{for } 1 \leq i \leq n,\ \varphi_i(u) \in [s_i] \\
&\iff u \in \bigcap_{1 \leq i \leq n} \varphi_i^{-1}([s_i])
\end{aligned}$$

As $\mathcal{V}(A^+)$ is closed under intersection, it suffices to establish that, for $1 \leq i \leq n$, $\varphi_i^{-1}([s_i]) \in \mathcal{V}(A^+)$.

(3) Since $\varphi_i = \eta_i \circ \psi_i$, one has $\varphi_i^{-1}([s_i]) = \psi^{-1}(\eta_i^{-1}([s_i])$. Now since $\mathcal{V}$ is a positive variety of languages, it suffices to prove that $\eta_i^{-1}([s_i]) \in \mathcal{V}(A_i^+)$, which results from the following lemma.

Lemma 5.6 *Let $\mathcal{V}$ be a positive variety of languages, A a finite alphabet and let $L \in \mathcal{V}(A^+)$. Let $\eta : A^+ \to (S(L), \leq_L)$ be the syntactic morphism of L. Then for every $x \in S(L)$, $\eta^{-1}([x]) \in \mathcal{V}(A^+)$.*

Proof. Let $P = \eta(L)$. Then P is an order ideal of $(S(L), \leq_L)$ such that $L = \eta^{-1}(P)$. We claim that

$$[x] = \bigcap_{(s,t) \in E} s^{-1}Pt^{-1} \quad \text{where } E = \{(s,t) \mid sxt \in P\}$$

Indeed, let $x \in s^{-1}Pt^{-1}$ and let $u \in [x]$, that is, $u \leq_L x$. Since $s^{-1}Pt^{-1}$ is an order ideal by Proposition 2.4, it follows $u \in s^{-1}Pt^{-1}$. This proves that $[x]$ is contained in $\bigcap_{(s,t) \in E} s^{-1}Pt^{-1}$. Conversely, let $u \in \bigcap_{(s,t) \in E} s^{-1}Pt^{-1}$. If $sxt \in P$, then $(s,t) \in E$ and thus $u \in s^{-1}Pt^{-1}$, that is, $sut \in P$. It follows that $u \leq_L x$ and thus $u \in [x]$, which proves the claim. It follows in particular

$$\eta^{-1}([x]) = \bigcap_{(s,t) \in E} \big(\eta^{-1}(s)\big)^{-1} L \big(\eta^{-1}(t)\big)^{-1}$$

Now $L \in \mathcal{V}(A^+)$ by hypothesis and $\mathcal{V}(A^+)$ is closed under finite intersection and left and right quotients. Therefore $\eta^{-1}([x]) \in \mathcal{V}(A^+)$. □

In conclusion, we have proved the following theorem.

Theorem 5.7 *The correspondence $\mathbf{V} \to \mathcal{V}$ defines a one to one correspondence between the varieties of finite ordered semigroups and the positive varieties of languages.*

As for Eilenberg's variety theorem, there is an analogous theorem for varieties of ordered monoids. In this case, the definitions of a class of languages and of a positive variety have to be modified as follows.

A *class of recognizable languages* is a correspondence $\mathcal{C}$ which associates with each finite alphabet A a set $\mathcal{C}(A^*)$ of recognizable languages of A^*. A *positive variety of languages* is a class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $\mathcal{V}(A^*)$ is closed under finite union and finite intersection,

(2) if $\varphi : A^* \to B^*$ is a monoid morphism, $L \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^*)$,

(3) if $L \in \mathcal{V}(A^*)$ and if $a \in A$, then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^*)$.

The monoid version of our result is proved exactly in the same way.

Theorem 5.8 *The correspondence* $\mathbf{V} \to \mathcal{V}$ *defines a one to one correspondence between the varieties of finite ordered monoids and the positive varieties of languages.*

6 Some examples

Since equality is an order relation, any finite semigroup S can be considered as a an ordered semigroup $(S, =)$ and any variety of finite semigroups $\mathbf{V}$ can be regarded as a variety of finite ordered semigroups. Note that, since any ordered semigroup $(S, \leq)$ is a quotient of $(S, =)$, this variety contains all ordered semigroups $(S, \leq)$ such that $S \in \mathbf{V}$. In particular, Eilenberg's varieties theorem is a special instance of our result corresponding to varieties of ordered monoids in which the order is the equality relation. Equivalently, varieties of languages are also positive varieties of languages.

We just present three other instances of the correspondence between positive varieties of languages and varieties of ordered semigroups. Other examples will be given in subsequent papers. Recall that a language is *cofinite* if its complement is finite.

Theorem 6.1 *A language is empty or cofinite if and only if it is recognized by a finite ordered nilpotent semigroup S in which $0 \leq s$ for all $s \in S$.*

Proof. If L is empty, its ordered syntactic semigroup is trivial and the result is clear. If L is cofinite, its ordered syntactic semigroup is nilpotent [1, 4]. Let $\eta : A^+ \to S$ be the syntactic morphism and let P be the image of L in S. Since L is cofinite, it contains arbitrary long words and thus P contains 0. Therefore, for every $x, y \in S^1$, $xsy \in P$ implies $x0y = 0 \in P$. Thus $0 \leq s$ in S.

Conversely, let $(S, \leq)$ be a finite ordered nilpotent semigroup in which $0 \leq s$ for all $s \in S$ and let $\varphi : A^+ \to (S, \leq)$ be a morphism of ordered semigroups. Finally, let I be an order ideal of S. If I is empty, $\varphi^{-1}(I)$ is the empty language. If I is non-empty, it certainly contains 0, since 0 is the minimal element of S. It follows that $\varphi^{-1}(I)$ is cofinite. □

Since the finite ordered nilpotent semigroups S in which 0 is the smallest element form a variety of finite ordered semigroups, one obtains the following corollary.

Corollary 6.2 *Let, for each alphabet A, $\mathcal{V}(A^+)$ be the class of empty or cofinite languages of A^+. Then $\mathcal{V}$ is a positive variety of languages.*

Our second example deals with idempotent and commutative monoids (or semilattices). Let A be an alphabet and B be a subset of A. Denote by $L(B)$ the set of words containing at least one occurrence of every letter of B. Equivalently,

$$L(B) = \bigcap_{a \in B} A^* a A^*$$

Theorem 6.3 *Let A be an alphabet. A language of A^* is a finite union of languages of the form $L(B)$ for some subset B of A if and only if it is recognized by an ordered idempotent and commutative monoid M in which the identity is the greatest element.*

Proof. Let $L = L(B)$ for some subset B of A, let $\eta : A^* \to M$ be the syntactic morphism of L and let P be the image of L in M. It is well known [1, 4] that M is idempotent and commutative. Let $x, y, s \in A^*$. If $xy \in L$, then $xsy \in L$ by construction. It follows that, for every $x, y, s \in M$, $xy \in P$ implies $xsy \in P$. Therefore $s \leq 1$ for all $s \in M$.

Conversely, let M be an ordered idempotent and commutative monoid in which 1 is the greatest element and let $\varphi : A^* \to (M, \leq)$ be a morphism of ordered monoids. Let I be an order ideal and let $L = \varphi^{-1}(I)$. Let $u \in L$ and let $c(u)$ be the set of letters occurring in u. We claim that $L(c(u))$ is a subset of L. First, since M is idempotent and commutative, $\varphi(u) = \prod_{b \in c(u)} \varphi(b)$. On the other hand, if $v \in L(c(u))$, then $\varphi(v) \leq \prod_{b \in c(u)} \varphi(b)$ and thus $\varphi(v) \in I$, since I is an ideal. Thus v is in L. It follows that $L = \bigcup_{u \in L} L(c(u))$. But since each $c(u)$ is a subset of A, this apparently infinite union is in fact finite. □

For our last example, we need two definitions. Recall that a word $u = a_1 a_2 \cdots a_n$ (where the a_i's are letters) is a subword of a word v if $v = v_0 a_1 v_1 a_2 \cdots a_n v_n$ for some words v_0, v_1, ..., v_n. A language L is a *shuffle ideal* if any word which has a subword in L is also in L.

Theorem 6.4 *A language is a shuffle ideal if and only if it is recognized by a finite ordered monoid in which 1 is the greatest element.*

Proof. By a well-known theorem of Higman (cf [3], chapter 6), every shuffle ideal is a finite union of languages of the form $A^*a_1A^*a_2\cdots a_kA^*$, where the a_i's are letters. In particular, every shuffle ideal is recognizable.

Let L be a shuffle ideal and let $\eta : A^* \to (M, \leq)$ be its syntactic morphism. If $xy \in L$, then $xsy \in L$ for every $s \in A^*$. It follows that, for every $x, y, s \in M$, $xy \in P$ implies $xsy \in P$. Therefore $s \leq 1$ and 1 is the greatest element.

Conversely, let $(M, \leq)$ be a finite ordered monoid in which 1 is the greatest element and let $\varphi : A \to (M, \leq)$ be a morphism of ordered monoids. Let I be an order ideal of S and let $L = \varphi^{-1}(I)$. If $xy \in L$, then $\varphi(xy) = \varphi(x)\varphi(y) \in I$. Now since $\varphi(s) \leq 1$, $\varphi(xsy) = \varphi(x)\varphi(s)\varphi(y) \leq \varphi(x)\varphi(y)$ and thus $\varphi(xsy) \in I$. It follows that $xsy \in L$ and thus L is a shuffle ideal. $\square$

Notice that every finite ordered monoid in which 1 is the maximum is $\mathcal{J}$-trivial. This implies that the variety $\mathbf{V}$ of ordered monoids in which 1 is the maximum is a subvariety of the variety $\mathbf{J}$ of ordered $\mathcal{J}$-trivial monoids. But $\mathbf{V}$ is a proper subvariety of $\mathbf{J}$, since the languages corresponding to $\mathbf{J}$ are closed under complement. This seems to be in contradiction with the result of Straubing and Thérien [5] stating that every finite $\mathcal{J}$-trivial monoid is the quotient of an ordered monoid in which 1 is the maximum. However, this quotient is in the sense of monoids and not in the sense of ordered monoids.

Also note that our three examples have a dual version, obtained by reversing the order. For instance, the dual version of our first example can be stated as follows. Let, for each alphabet A, $\mathcal{V}(A^+)$ be the class formed by the finite languages and by A^+. Then $\mathcal{V}$ is a positive variety. The corresponding variety of ordered semigroup is the variety of ordered finite nilpotent semigroups in which 0 is the maximum.

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References

[1] Eilenberg, S., *Automata, Languages and Machines*, Vol. A, Academic Press, New York, 1974. Vol. B, Academic Press, New York, 1976.

[2] Lallement, G., *Semigroups and combinatorial applications*, Wiley, New York, 1979.

[3] M. Lothaire, *Combinatorics on Words*, Cambridge University Press, 1982.

[4] J.E. Pin, *Variétés de langages formels*, 160 p., Masson, Paris (1984). *Varieties of formal languages*, 138 p., North Oxford Academic (London), 1986 and Plenum (New York), 1986.

[5] H. Straubing and D. Thérien, Partially ordered finite monoids and a theorem of I. Simon, *J. of Algebra* **119**, (1985) 393–399.