MPRI, Fondations mathématiques de la théorie des automates

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Avertissement : On attachera une grande importance à la clarté, à la précision et à la concision de la rédaction.

Partie 1. Monoïde syntactique de $(abab)^*$.

Question 1. Soit $A = \{a, b\}$. On considère le langage $L = (abab)^*$. Calculer son automate minimal.

Question 2. Calculer le monoïde syntactique M de L. On donnera la liste des éléments et des relations permettant de définir M (vous devriez trouver 10 éléments, en comptant l'élément neutre).

Question 3. Donner la liste des idempotents de M.

Question 4. Déterminer la structure en $\mathcal{D}$-classes de M et dessiner les diagrammes boîtes à œufs.

Question 5. Le langage L est-il sans-étoile? Justifier votre réponse.

Question 6. Montrer que L vérifie l'équation profinie $x^\omega y^\omega = y^\omega x^\omega$ pour tout $x, y \in A^*$.

Partie 2. Ensembles préfixes maximaux finis.

On rappelle qu'un préfixe p d'un mot u est *propre* si $p \neq u$. Un ensemble X de mots non vides de A^+ est dit *préfixe* si aucun mot de X n'est préfixe propre d'un autre mot de X (autrement dit, si $u, uv \in X$ entraine $v = 1$). Un ensemble préfixe X est dit *maximal* si, pour tout mot $u \in A^+$, l'ensemble $X \cup \{u\}$ n'est pas préfixe.

Le but de l'exercice est d'étudier certaines propriétés des ensembles préfixes maximaux finis.

Question 7. Soit X un ensemble préfixe et soit $u \in X^*$. Montrer que $v \in X^*$ si et seulement si $uv \in X^*$.

Question 8. Montrer que l'ensemble

$$Z = \{aa, ab, baa, bab, bb\}$$

est un ensemble préfixe maximal.

Soit X un ensemble préfixe maximal fini et soit P l'ensemble des préfixes propres de ses mots. Par exemple, si $X = Z$, on a $P = \{1, a, b, ba\}$.

Question 9. En utilisant la maximalité de X, montrer que $A^* = P \cup XA^*$ et que cette union est disjointe.

Question 10. Déduire de la question précédente que $A^* = X^*P$ et que tout mot u de A^* admet une factorisation unique de la forme $u = x_1 \cdots x_n p$ avec $n \geqslant 0$, $x_1, \ldots, x_n \in X$ et $p \in P$.

Question 11. Soit $\mathcal{A}$ l'automate minimal déterministe de X^*. Montrer que l'état initial q_0 de $\mathcal{A}$ est aussi le seul état final.

On suppose maintenant que X est un ensemble préfixe maximal fini qui est en plus *suffixe* (aucun mot de X n'est suffixe propre d'un autre mot de X). Soit n la longueur maximale des mots de X.

Question 12. Soit q un état de $\mathcal{A}$ et u un mot. Montrer que si $q_0 \cdot u = q \cdot u$, alors $q = q_0$.

Question 13. Soit u un mot de longueur $\geqslant n$ et soient q_1, q_2 deux états de $\mathcal{A}$. Montrer que s'il existe un mot v tel que $q_1 \cdot uv = q_2 \cdot uv$, alors $q_1 \cdot u = q_2 \cdot u$.

Question 14. Soit $\eta : A^* \to M$ le monoïde syntactique de X^*. Montrer que si u est un mot de longueur $\geqslant n$, alors $\eta(u)$ appartient à l'idéal minimal de M.

Question 15. Que peut-on en déduire sur la structure de M?

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Warning : Clearness, accuracy and concision of the writing will be rewarded.

Part 1. Syntactic monoid of $(abab)^*$.

Question 1. Let $A = \{a, b\}$ and let $L = (abab)^*$. Compute the minimal automaton of L.

Question 2. Compute the syntactic monoid M of L. Give the list of its elements (you should find 10 elements, including the identity) and of its defining relations.

Question 3. Give the set of idempotents of M.

Question 4. Compute the $\mathcal{D}$-class structure of M and draw the egg-box pictures.

Question 5. Is the language L star-free? Justify your answer.

Question 6. Show that L satisfies the profinite equations $x^\omega y^\omega = y^\omega x^\omega$ for all $x, y \in A^*$.

Part 2. Finite maximal prefix sets.

Recall that a prefix p of a word u is *proper* if $p \neq u$. A set X of nonempty words of A^+ is *prefix* if no word of X is a proper prefix of another word of X (in other words, if $u, uv \in X$ implies $v = 1$). A prefix set X is *maximal* if, for all $u \in A^+$, the set $X \cup \{u\}$ is not prefix.

The aim of this exercise is to study some properties of finite maximal prefix sets.

Question 7. Let X be a prefix set and let $u \in X^*$. Show that $v \in X^*$ if and only if $uv \in X^*$.

Question 8. Show that the set

$$Z = \{aa, ab, baa, bab, bb\}$$

is a maximal prefix set.

Let X be a finite maximal prefix set and let P be the set of all proper prefixes of its words. For instance, if $X = Z$, then $P = \{1, a, b, ba\}$.

Question 9. Using the maximality of X, show that $A^* = P \cup XA^*$ and that this union is a disjoint one.

Question 10. Deduce from the previous question that $A^* = X^*P$ and that any word $u \in A^*$ admits a unique factorization of the form $u = x_1 \cdots x_n p$ with $n \geqslant 0$, $x_1, \ldots, x_n \in X$ and $p \in P$.

Question 11. Let $\mathcal{A}$ be the deterministic minimal automaton of X^*. Show that the initial state q_0 of $\mathcal{A}$ is also its unique final state.

Suppose now that X is a finite maximal prefix set which is also *suffix* (no word of X is a proper suffix of another word of X). Let n be the maximal length of all words in X

Question 12. Let q be a state of $\mathcal{A}$ and let u be a word. Show that if $q_0 \cdot u = q \cdot u$, then $q = q_0$.

Question 13. Let u be a word of length $\geqslant n$ and let q_1, q_2 be two states of $\mathcal{A}$. Prove that if there exists a word v such that $q_1 \cdot uv = q_2 \cdot uv$, then $q_1 \cdot u = q_2 \cdot u$.

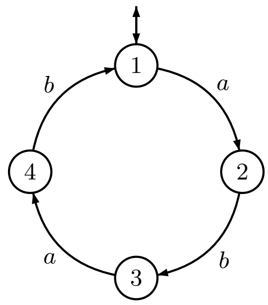
Question 14. Let $\eta : A^* \to M$ be the syntactic monoid of X^*. Show that if u is a word of length $\geqslant n$, then $\eta(u)$ belongs to the minimal ideal of M.

Question 15. What can be said about the structure of M?

Corrigé

Part 1. Syntactic monoid of $(abab)^*$.

Question 1. The minimal automaton of L is represented below



Question 2. The elements of M are given in the table below

	1	2	3	4
$*\ 1$	1	2	3	4
a	2	0	4	0
b	0	3	0	1
$*\ a^2$	0	0	0	0
ab	3	0	1	0
ba	0	4	0	2
aba	4	0	2	0
bab	0	1	0	3
$*\ abab$	1	0	3	0
$*\ baba$	0	2	0	4

This monoid has 10 elements. It has a zero since $a^2 = 0$ and is defined by the relations $a^2 = 0$, $b^2 = 0$, $ababa = a$ and $babab = b$

Question 3. Idempotents:

$$E(S) = \{1, a^2, abab, baba\}$$

Question 4. $\mathcal{D}$-class structure:

^{*1}

$^*baba \quad ba$	$bab \quad b$
$a \quad aba$	$^*abab \quad ab$

$^*a^2$

Question 5. The monoid M is not aperiodic, since the identity $x^\omega = x^{\omega+1}$ is not satisfied for $x = ab$. Therefore L is not star-free.

Question 6. The idempotents of M commute and thus L satisfies the profinite identity $x^\omega y^\omega = y^\omega x^\omega$.

Part 2. Finite maximal prefix sets.

Question 7. If $v \in X^*$, then clearly $uv \in X^*$. Since $u \in X^*$, one gets $u = x_1 \cdots x_n$ for some $x_1, \ldots, x_n \in X$. Suppose that $uv \in X^*$. Then $uv = z_1 \cdots z_m$ for some $z_1, \ldots, z_m \in Z$. Now, the words x_1 and z_1 are comparable in the prefix order. But they are both in X, which is a prefix set. Therefore $x_1 = z_1$. A similar argument would show that $x_2 = z_2$, ..., $x_n = z_n$. Therefore $v = z_{n+1} \cdots z_m \in X^*$.

Question 8. Clearly, no word of Z is a proper prefix of another word of Z. Therefore Z is prefix. Suppose that $Z \cup \{u\}$ is a prefix set. First suppose that $|u| \geqslant 3$. Since $aa, ab, bb \in Z$, the prefix of length 2 of u has to be ba. But then one of the word baa or bab is a prefix of u, a contradiction. Thus $|u| \leqslant 2$, but this is also impossible since all the words of length $\leqslant 2$ are prefixes of some word of Z. Thus Z is maximal.

Question 9. Let $u \in A^*$. Since $X \cup \{u\}$ is not a prefix set, either u is a prefix of some word of X or some word of X is a prefix of u. This gives the equality $A^* = P \cup XA^*$. The union is disjoint since if $u \in P \cap XA^*$, then $u = xv$ for some $x \in X$ and $v \in A^*$, which implies that x is a prefix of some word of X, a contradiction.

Question 10. Let us prove the result by induction on the length of u. The result is trivial if $u \in P$ and in particular for $u = 1$. Suppose that $u \notin P$. Then since $A^* = P \cup XA^*$, one gets $u \in XA^*$ and there exist $x \in X$ and $v \in A^*$ such that $u = xv$. Note that the pair (x, v) is unique since X is prefix. It suffices now to apply the induction hypothesis to v to conclude.

Question 11. Since $1 \in X^*$, the initial state is also a final state. Let q be a final state. Since q is accessible, there is a word u such that $q_0 \cdot u = q$ and since q is final, one has $u \in X^*$. Now, by Question 7, the conditions $v \in X^*$ and $uv \in X^*$ are equivalent. It follows that the conditions $q_0 \cdot v \in F$ and $q \cdot v \in F$ are equivalent and thus the states q_0 and q are equivalent. Since $\mathcal{A}$ is minimal, they are equal. Therefore q_0 is the unique final state.

Question 12. Since X is suffix, the dual version of Question 7 shows that the conditions $u, vu \in X^*$ imply $v \in X^*$. Let q be a state. Since q is accessible and coaccesible, there are two words v and w such that $q_0 \cdot v = q$ and $q \cdot w = q_0$. Now, if $q_0 \cdot u = q \cdot u$, then $q_0 \cdot uw = q_0 \cdot vuw = q_0$. It follows that $uw, vuw \in X^*$ and thus $v \in X^*$. Therefore $q_0 \cdot v = q_0$ and thus $q = q_0$.

Question 13. Suppose that $q_1 \cdot uv = q_2 \cdot uv$, where u is a word of length $\geqslant n$. Let w be a word such that $q_0 \cdot w = q_1$. Since $A^* = X^*P$ by Question 10, $wu = wrp$ for some words r and p such that $rp = u$, $wr \in X^*$ and $p \in P$. It follows that $q_0 = q_0 \cdot wr = q_1 \cdot r$. Since $q_1 \cdot rpv = q_2 \cdot rpv$, one gets $q_0 \cdot pv = (q_2 \cdot r) \cdot pv$. It follows by Question 12 that $q_0 = q_2 \cdot r$, whence $q_1 \cdot r = q_2 \cdot r$ and finally $q_1 \cdot u = q_2 \cdot u$.

Question 14. It follows from Question 13 that all words of length $\leqslant n$ have minimal rank in $\mathcal{A}$. Therefore their syntactic image belong to the minimal ideal of M.

Question 15. Consequently, M consists of the identity 1, sone nonregular elements and a unique regular $\mathcal{D}$-class, its minimal ideal.