On formations of monoids [1]

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Abstract

A formation of monoids is a class of finite monoids closed under taking quotients and subdirect products. Formations of monoids were first studied in connection with formal language theory, but in this paper, we come back to an algebraic point of view. We give two natural constructions of formations based on constraints on the minimal ideal and on the maximal subgroups of a monoid. Next we describe two sublattices of the lattice of all formations, and give, for each of them, an isomorphism with a known lattice of varieties of monoids. Finally, we study formations and varieties containing only Clifford monoids, completely describe such varieties and discuss the case of formations.

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In this paper, all semigroups, monoids and groups are considered to be finite, unless otherwise stated. The word *morphism* stands for *monoid morphism* with a few exceptions that will be clear from the context. Furthermore, we consider monoids up to isomorphism and thus the term *class of monoids* should be understood as *class of isomorphic types of monoids*.

1 Introduction

A formation of groups (or group formation) is a class of groups closed under taking quotients and subdirect products. Formations of groups were first introduced by Gaschütz [11] in his pioneering work on solvable groups. Although the notion of a formation was later extended to finite algebras in [19, 21, 20], formations of monoids were first studied in the papers [5, 6] in connection with formal languages.

The purpose of this paper is to study the algebraic foundations of this theory, without any reference to formal languages. Formations of monoids can be splitted into two categories: group formations and non-group formations. Since formations of groups are already extensively studied in the literature [4, 8, 15], we focus on non-group formations of monoids. These formations contain the two-element idempotent monoid U_1 and hence the variety of idempotent and commutative monoids.

In the first part of the paper, we compare the formation and the variety generated by a class of monoids. Surprisingly enough, non-group formations sometimes behave more smoothly than group formations. For instance, to obtain the variety generated by a non-group formation, it suffices to take the submonoids of the formation, see Corollary 3.12. This is no longer true for a group formation, as shown in Proposition 3.13. Non-group formations are also closed under taking submonoids with an added zero (see Proposition 3.10 for a precise statement) but this is also no longer true for group formations.

The second part of the paper (Section 4) is devoted to the description of formations on which various constraints are imposed. We study constraints on the minimal ideal in Section 4.1 and constraints on maximal subgroups in Section 4.2. This allows us to obtain new examples of formations.

Our main results are presented in Section 5. We identify two sublattices of the lattice of all formations, pictured in Figures 5.1 and 5.2, respectively, and, for each of them, we give a lattice isomorphism with a lattice of varieties of monoids. The second of these results states that the lattice of non-trivial formations of monoids with zero is isomorphic to the lattice of non-group varieties. The first one is more technical and involves a formation of groups.

Section 6 is dedicated to Clifford monoids and to formations containing only Clifford monoids, called *Clifford formations*. We first characterize the subdirectly irreducible Clifford monoids. Next we show that the minimal non-group formation above a group formation is a Clifford formation, and provide an in-depth description of this formation (Theorem 6.5). We also completely describe Clifford varieties, but the description of Clifford formations seems to be a much more difficult problem, as shown by the negative results presented in Proposition 6.10. To illustrate the difference between formations and varieties, we invite the reader to compare the following statements:

> *The variety generated by U_1 together with a variety of groups* $\mathbf{H}$ *is the class of Clifford monoids with groups in the group variety* $\mathbf{H}$.

> *The formation generated by U_1 together with a formation of groups* $\mathbf{H}$ *is the class of Clifford monoids with groups in the variety generated by* $\mathbf{H}$ *and whose minimal ideal is a group of* $\mathbf{H}$.

Much more pleasant results hold for the lattice of formations of Clifford monoids with zero, for which we get another lattice isomorphism, pictured in Figure 6.2.

Apart from Proposition 3.13 which requires some knowledge in group theory, the proofs make use of standard arguments of finite semigroup theory.

2 Background

The reader is referred to [12, 14, 16], [7, 22] and [13, 18] for the basics of semigroup theory, of universal algebra and of group theory, respectively. The books [4, Section 2.2], [8, Chapter 4] and [18, Section 9.5] contain a section dedicated to formations of groups.

The idempotent monoid U_1 with underlying set $\{0, 1\}$ will play an important role in what follows.

2.1 Basic constructions on monoids

Let us first recall a classical result of semigroup theory (see, for instance, [17, Lemma 4.6.10]).

Proposition 2.1. *Let $\varphi\colon S \to T$ be a surjective morphism of semigroups. Then φ maps the minimal ideal of S onto the minimal ideal of T. Moreover, for each $\mathcal{J}$-class K of T, there exists a minimal $\mathcal{J}$-class J of S such that $\varphi(J) \subseteq K$. Furthermore, for any such $\mathcal{J}$-class,*

(1) *$\varphi(J) = K$,*

(2) *φ maps $\mathcal{H}$-classes of S contained in J onto $\mathcal{H}$-classes of T contained in K.*

Moreover, if K is regular, then

(3) *J is regular as well, and it is uniquely determined,*

(4) *φ maps the maximal subgroups of S contained in J onto maximal subgroups of T contained in K, and conversely, any maximal subgroup of T contained in K is the image by φ of some maximal subgroup of S contained in J.*

Let S be a semigroup. As usual, let S^1 denote the monoid equal to S if S has an identity, and to $S \cup \{1\}$ if S is not a monoid. In the latter case, the operation of S is completed by the rules $1s = s1 = s$ for each $s \in S^1$. Let also $S^{\mathtt{I}}$ denote the monoid obtained from S by adjoining a new identity $\mathtt{I}$, even if S has an identity.

Lemma 2.2. *Let M be a monoid. Then M is a quotient of $M^{\mathtt{I}}$ and $M^{\mathtt{I}}$ is a submonoid of $M \times U_1$.*

Proof. For the first part, notice that the map $\alpha\colon M^{\mathtt{I}} \to M$ defined by $\alpha(\mathtt{I}) = 1$ and $\alpha(m) = m$ for all $m \in M$ is a surjective morphism. For the second part, observe that the map $\varphi\colon M^{\mathtt{I}} \to M \times U_1$ defined by $\varphi(\mathtt{I}) = (1, 1)$ and $\varphi(m) = (m, 0)$ if $m \in M$ is an injective morphism. ∎

Similarly, let S^0 be the semigroup equal to S if S has a zero, and to $S \cup \{0\}$ otherwise. In the latter case, the operation of S is completed by the rules $0s = s0 = 0$ for each $s \in S^0$. Note that G^0 is equal to G when G is a trivial group. For a semigroup S with or without a zero, let $S^{\square}$ denote the semigroup obtained from S by adjoining a new element and imposing to it the role of zero.

Lemma 2.3. *Let M be a monoid. Then M^0 is a quotient of $M^{\square}$ and $M^{\square}$ is the quotient of $M \times U_1$ by the ideal $M \times \{0\}$.*

Proof. If the monoid M does not have a zero, then $M^0 = M^{\square}$. Assume that M has a zero z_1. Then $M^0 = M$. Let z_2 be the zero of $M^{\square}$ and let $I = \{z_1, z_2\}$. Then I is an ideal of $M^{\square}$, and the Rees quotient semigroup $M^{\square}/I$ is isomorphic to M (even if M is trivial). Thus M^0 is a quotient of $M^{\square}$. The second part of the lemma is clear. ∎

2.2 Products and subdirect products

Given a (direct) product $\prod_{i \in I} M_i$ of monoids, let π_i denote the natural projection from $\prod_{i \in I} M_i$ onto M_i.

A monoid M is a *subdirect product* of a family of monoids $(M_i)_{i \in I}$ if M is isomorphic to a submonoid M' of the product $\prod_{i \in I} M_i$ such that each induced projection from M' to M_i is surjective. Such a monoid M' is called a *subdirect representation* of M by the family $(M_i)_{i \in I}$.

A monoid M is *subdirectly irreducible* if for every subdirect representation of M, at least one of the projections is an isomorphism. The complete classification of subdirectly indecomposable semigroups is still an open problem, discussed in [17, Section 4.7].

The following two results are instances of well-known results of universal algebra, see [7, Chap. II, Th. 8.4 and Cor. 8.7].

Proposition 2.4. *A non-trivial monoid is subdirectly irreducible if and only if it has a smallest non-trivial congruence.*

Proposition 2.5. *Let **C** be a class of monoids closed under taking quotients. Any monoid of **C** is a subdirect product of a finite family of subdirectly irreducible monoids of **C**.*

Minimal ideals are preserved under subdirect products. More precisely, the following result holds:

Proposition 2.6. *Let M be a subdirect product of a finite family $(M_j)_{j \in J}$ of monoids. Then the minimal ideal of M is a subdirect product of the minimal ideals of the monoids M_j.*

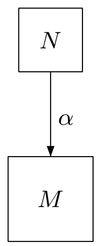
Proof. By definition, there exists an embedding $\varphi\colon M \to \prod_{j \in J} M_j$ such that each induced projection $\varphi_j\colon M \to M_j$ is surjective. Let I be the minimal ideal of M. Since φ_j is surjective, Proposition 2.1 shows that the minimal ideal of M_j is $\varphi_j(I)$. It follows that I is a subdirect product of the family $\big(\varphi_j(I)\big)_{j \in J}$. ∎

2.3 A semilattice of monoids

We now present a construction that is a particular case of a semilattice of monoids (see [14]) and that plays an important role in the sequel. Let N and M be monoids and let $\alpha\colon N \to M$ be a morphism. On the disjoint union T of M and N, define a product $\cdot$ that extends both products on M and on N by setting, for all $m \in M$ and $n \in N$,

$$m \cdot n = m\alpha(n) \qquad \text{and} \qquad n \cdot m = \alpha(n)m.$$

Then T is a monoid, whose identity is the identity of N. This monoid, denoted $\mathcal{U}_1(N, M, \alpha)$, is a semilattice over U_1 of M and N.



Proposition 2.7. *The monoid $\mathcal{U}_1(N, M, \alpha)$ is a subdirect product of M and $N^\square$. Moreover, if α is injective, then $\mathcal{U}_1(N, M, \alpha)$ is a subdirect product of M and U_1.*

Proof. Let $T = \mathcal{U}_1(N, M, \alpha)$ and let $\varphi_1\colon T \to M$ and $\varphi_2\colon T \to N^\square$ be the surjective morphisms defined respectively by

$$\varphi_1(x) = \begin{cases} \alpha(x) & \text{if } x \in N \\ x & \text{if } x \in M \end{cases} \qquad\qquad \varphi_2(x) = \begin{cases} x & \text{if } x \in N \\ 0 & \text{if } x \in M \end{cases}$$

Since the morphism $\varphi_1 \times \varphi_2\colon T \to M \times N^\square$ is injective, T is a subdirect product of M and $N^\square$.

Suppose now that α is injective and let $\varphi_3\colon T \to U_1$ be the surjective morphism defined by

$$\varphi_3(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \in M \end{cases}$$

Since the morphism $\varphi_1 \times \varphi_3\colon T \to M \times U_1$ is injective, T is a subdirect product of M and U_1. ∎

If N is a submonoid of M and if α is the embedding morphism, we simply denote $\mathcal{U}_1(N, M, \alpha)$ by $\mathcal{U}_1(N, M)$.

3 Formations and varieties

3.1 Definitions and examples

A *variety of monoids* is a class of monoids closed under taking submonoids, quotients and finite direct products. A *formation of monoids* is a class of monoids closed under taking quotients and subdirect products of finite families of monoids. *Varieties of semigroups*, *varieties of groups*, *formations of semigroups* and *formations of groups* are defined in a similar way. Varieties are obviously formations.

In a category with limits, the product of a family of objects indexed by the empty set is the terminal object. In particular, the product of the empty family of semigroups is the trivial semigroup $\{1\}$. It follows that a class $\mathbf{C}$ of semigroups is closed under finite direct products if and only if it satisfies the following conditions:

(1) $\{1\} \in \mathbf{C}$,
(2) if $S_0, S_1 \in \mathbf{C}$, then $S_0 \times S_1 \in \mathbf{C}$.

Similarly, a class $\mathbf{C}$ of semigroups is closed under finite subdirect products if and only if it satisfies the following conditions:

(1) $\{1\} \in \mathbf{C}$,
(2) if $S_0, S_1 \in \mathbf{C}$, then all subdirect products of S_0 and S_1 belongs to $\mathbf{C}$.

In particular, every formation of semigroups contains the trivial semigroup $\{1\}$.

As the intersection of any nonempty family of formations is a formation, we may define the *formation generated by a class* $\mathbf{C}$ of monoids as the smallest formation of monoids containing $\mathbf{C}$. It is denoted by $\mathrm{Form}(\mathbf{C})$. Similarly, we let $\mathrm{Var}(\mathbf{C})$ denote the *variety generated by* $\mathbf{C}$.

In this paper, we consider the following classes of semigroups:

- $\mathbf{M}$ – all monoids,
- $\mathbf{J}_1$ – all idempotent and commutative monoids,
- $\mathbf{G}$ – all groups,
- $\mathbf{ZE}$ – all monoids in which idempotents are in the center,
- $\mathbf{C\ell}$ – all Clifford monoids, that is, regular monoids of $\mathbf{ZE}$,
- $\mathbf{CS}$ – all completely simple semigroups,
- $\mathbf{Zr}$ – all monoids with zero.

Note that $\mathbf{CS}$ is a variety of semigroups, $\mathbf{M}$, $\mathbf{J}_1$, $\mathbf{G}$, $\mathbf{ZE}$, $\mathbf{C\ell}$ are varieties of monoids and $\mathbf{Zr}$ is a formation of monoids.

In what follows we mainly deal with monoids, and so, varieties and formations refer to monoids unless otherwise stated.

Remark 3.1. Let $\mathbf{V}$ be a variety of monoids. We claim that if $M \in \mathbf{V}$ and S is a subsemigroup of M which is a monoid, then $S \in \mathbf{V}$. In particular, if $M \in \mathbf{V}$ and G is a subgroup of M, then $G \in \mathbf{V}$. Indeed, let e be the identity of S. If e is also the identity of M, then S is a submonoid of M and thus $S \in \mathbf{V}$. Otherwise, since $S^{\mathtt{I}}$ is obtained by adding to S the identity of M, $S^{\mathtt{I}}$ is a submonoid of M. Now, the map $\pi : S^{\mathtt{I}} \to S$ defined by $\pi(\mathtt{I}) = e$ and $\pi(s) = s$ if $s \in S$ is a surjective morphism. It follows that S is a quotient of $S^{\mathtt{I}}$, whence $S \in \mathbf{V}$.

3.2 Operators on classes of monoids

We already defined the operators Form and Var on a class $\mathbf{C}$ of monoids, which are related with the following other operators:

- $\mathrm{S}(\mathbf{C})$ – submonoids of monoids of $\mathbf{C}$;
- $\mathrm{H}(\mathbf{C})$ – homomorphic images (quotients) of monoids of $\mathbf{C}$;
- $\mathrm{P}(\mathbf{C})$ – direct products of finite families of monoids of $\mathbf{C}$;
- $\mathrm{P_s}(\mathbf{C})$ – subdirect products of finite families of monoids of $\mathbf{C}$.

The next proposition gathers properties of these operators. Recall that an operator O is a *closure operator* if it is *extensive*, *increasing* and *idempotent*, meaning respectively $\mathbf{C} \subseteq O(\mathbf{C})$, $\mathbf{C}_1 \subseteq \mathbf{C}_2$ implies $O(\mathbf{C}_1) \subseteq O(\mathbf{C}_2)$ and $O(\mathbf{C}) = O(O(\mathbf{C}))$ for any classes C, C_1 and C_2. Given two operators O_1 and O_2, we write $O_1 \leqslant O_2$ if, for any class $\mathbf{C}$ of monoids, $O_1(\mathbf{C}) \subseteq O_2(\mathbf{C})$.

Proposition 3.1. *The operators* $\mathrm{S}, \mathrm{H}, \mathrm{P}$*,* $\mathrm{P_s}$*,* Form *and* Var *are closure operators. Moreover,*

(1) $\mathrm{SH} \leqslant \mathrm{HS}$*,*
(2) $\mathrm{PH} \leqslant \mathrm{HP}$*,*

(3) $\mathrm{PS} \leqslant \mathrm{SP}$,
(4) $\mathrm{P_sH} \leqslant \mathrm{HP_s}$,
(5) $\mathrm{P} \leqslant \mathrm{P_s} \leqslant \mathrm{SP}$,
(6) $\mathrm{PP_s} = \mathrm{P_s} = \mathrm{P_sP}$,
(7) $\mathrm{SP} = \mathrm{SP_s} = \mathrm{P_sS}$,
(8) $\mathrm{Form} = \mathrm{HP_s}$,
(9) $\mathrm{Var} = \mathrm{HSP}$,
(10) $\mathrm{Var}\,\mathrm{Form} = \mathrm{Var}$.

Proof. The six operators are extensive and increasing by construction. The fact that the operators Form and Var are idempotent is also an immediate consequence of their definition. It is shown in [2, Lemma 1.3.3] that the operators S, H and P are idempotent. Formulas (1), (2) and (3) are also stated in the same lemma. The inequality (4) is proved in [5, Prop. 1.3]. Formula (5) is trivial since a direct product is a special case of a subdirect product and a subdirect product is by definition a submonoid of a direct product.

Let us show that $\mathrm{P_s}$ is idempotent. Clearly, $\mathrm{P_s} \leqslant \mathrm{P_sP_s}$. Now, let N be a subdirect product of a family $(M_i)_{i\in I}$ of monoids and suppose that each M_i is a subdirect product of a family $(M_{i_j})_{i_j\in J_i}$ of monoids. Then N is a submonoid of the direct product $\prod_{i\in I}\prod_{i_j\in J_i} M_{i_j}$. Furthermore, since the projections of N on each M_i are surjective, and the projections of each M_i on each M_{i_j} are surjective, then the projections of N on each M_{i_j} are also surjective. Thus N is a subdirect product of the monoids M_{i_j}, which proves that $\mathrm{P_sP_s} \leqslant \mathrm{P_s}$. Hence $\mathrm{P_sP_s} = \mathrm{P_s}$.

(6) Since P is extensive, one gets $\mathrm{P_s} \leqslant \mathrm{PP_s}$. Furthermore, since $\mathrm{P} \leqslant \mathrm{P_s}$ by (5), one gets $\mathrm{PP_s} \leqslant \mathrm{P_sP_s} = \mathrm{P_s}$, whence $\mathrm{P_s} = \mathrm{PP_s}$ and similarly $\mathrm{P_s} \leqslant \mathrm{P_sP} \leqslant \mathrm{P_sP_s} = \mathrm{P_s}$, whence $\mathrm{P_s} = \mathrm{P_sP}$.

(7) The relation $\mathrm{SP} \leqslant \mathrm{SP_s}$ follows from $\mathrm{P} \leqslant \mathrm{P_s}$. Moreover, since $\mathrm{P_s} \leqslant \mathrm{SP}$, one gets $\mathrm{SP_s} \leqslant \mathrm{SSP} = \mathrm{SP}$. Thus $\mathrm{SP} = \mathrm{SP_s}$. By (5) and (3), it follows that

$$\mathrm{P_sS} \leqslant \mathrm{SPS} \leqslant \mathrm{SSP} = \mathrm{SP} = \mathrm{SP_s}\,.$$

Finally, we claim that $\mathrm{SP} \leqslant \mathrm{P_sS}$. Let N be a submonoid of a direct product $\prod_{i\in I} M_i$ and for each $i \in I$, let N_i be the projection of N on M_i. Then N is by construction a subdirect product of the N_i's, which proves the claim and concludes the proof of (7).

(8) This was proved in [5, Prop. 1.4]. We reproduce the proof for the convenience of the reader. By definition of a formation, $\mathrm{HP_s}(\mathbf{C})$ is contained in $\mathrm{Form}(\mathbf{C})$. Moreover by (4) we get that $\mathrm{P_sHP_s}(\mathbf{C}) \subseteq \mathrm{HP_s}(\mathbf{C})$ and since H is idempotent, one also gets $\mathrm{HHP_s}(\mathbf{C}) = \mathrm{HP_s}(\mathbf{C})$. Thus $\mathrm{HP_s}(\mathbf{C})$ is a formation containing $\mathbf{C}$, and hence $\mathrm{Form}(\mathbf{C}) = \mathrm{HP_s}(\mathbf{C})$.

(9) This is also well known and is proved in the same way as (8).

(10) This follows from the idempotency of Var, since $\mathrm{Var} \leqslant \mathrm{Var}\,\mathrm{Form} \leqslant \mathrm{Var}\,\mathrm{Var} = \mathrm{Var}$. ∎

When applied to a formation, the equality given by (9) of Proposition 3.1 can be improved as follows.

Corollary 3.2 [5, Prop. 1.4]**.** *If* $\mathbf{F}$ *is a formation of monoids, then* $\mathrm{Var}(\mathbf{F}) = \mathrm{HS}(\mathbf{F})$*. Consequently, a formation of monoids* $\mathbf{F}$ *is a variety if and only if* $\mathrm{S}(\mathbf{F}) \subseteq \mathbf{F}$*.*

Proof. If $\mathbf{F}$ is a formation, then $\mathrm{P}(\mathbf{F}) = \mathbf{F}$. Thus by Proposition 3.1 (9), the variety generated by $\mathbf{F}$ is $\mathrm{HS}(\mathbf{F})$.

If $\mathbf{F}$ is a variety, then $\mathrm{S}(\mathbf{F}) \subseteq \mathbf{F}$. Conversely, if $\mathbf{F}$ is a formation such that $\mathrm{S}(\mathbf{F}) \subseteq \mathbf{F}$, then $\mathbf{F}$ is closed under taking submonoids, quotients and finite direct products, and hence it is a variety. ∎

For classes of monoids, a similar result holds.

Proposition 3.3. *Let* $\mathbf{C}$ *be a class of monoids. If* $\mathrm{S}(\mathbf{C}) \subseteq \mathrm{Form}(\mathbf{C})$*, then* $\mathrm{Form}(\mathbf{C}) = \mathrm{Var}(\mathbf{C})$*.*

Proof. By Corollary 3.2, we only need to prove that $\mathrm{S}\,\mathrm{Form}(\mathbf{C}) \subseteq \mathrm{Form}(\mathbf{C})$. By hypothesis and Proposition 3.1, we have $\mathrm{S}(\mathbf{C}) \subseteq \mathrm{Form}(\mathbf{C}) = \mathrm{HP_s}(\mathbf{C})$, and hence

$$\mathrm{HP_sS}(\mathbf{C}) \subseteq \mathrm{HP_sHP_s}(\mathbf{C})$$

Applying several times Proposition 3.1 together with the idempotency of H and $\mathrm{P_s}$, one gets

$$\begin{aligned}\mathrm{S}\,\mathrm{Form}(\mathbf{C}) = \mathrm{SHP_s}(\mathbf{C}) \subseteq \mathrm{HSP_s}(\mathbf{C}) \subseteq \mathrm{HP_sS}(\mathbf{C})\\ \subseteq \mathrm{HP_sHP_s}(\mathbf{C}) \subseteq \mathrm{HHP_sP_s}(\mathbf{C}) = \mathrm{HP_s}(\mathbf{C}) = \mathrm{Form}(\mathbf{C}),\end{aligned}$$

as desired. ∎

This last proposition provides a useful corollary, that guarantees that there is no ambiguity when we talk about joins of formations that are varieties.

Corollary 3.4. *For any varieties* $\mathbf{V}$ *and* $\mathbf{W}$*, the formation generated by* $\mathbf{V} \cup \mathbf{W}$ *is a variety. The lattice of varieties is a sublattice of the lattice of formations.*

In the next proposition, the same symbol $\vee$ is used for the join of two varieties or for the join of two formations, but the context avoids any ambiguity.

Proposition 3.5. *The formula* $\mathrm{Var}(\mathbf{F}_1 \vee \mathbf{F}_2) = \mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$ *holds for all formations* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *of monoids.*

Proof. First, one has $\mathbf{F}_1 \subseteq \mathrm{Var}(\mathbf{F}_1) \subseteq \mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$ and similarly, $\mathbf{F}_2 \subseteq \mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$. Since $\mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$ is a variety, and hence a formation, it follows that $\mathbf{F}_1 \vee \mathbf{F}_2 \subseteq \mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$ and finally $\mathrm{Var}(\mathbf{F}_1 \vee \mathbf{F}_2) \subseteq \mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2)$.

Conversely, since $\mathbf{F}_1, \mathbf{F}_2 \subseteq \mathbf{F}_1 \vee \mathbf{F}_2$, one gets $\mathrm{Var}(\mathbf{F}_1), \mathrm{Var}(\mathbf{F}_2) \subseteq \mathrm{Var}(\mathbf{F}_1 \vee \mathbf{F}_2)$, whence $\mathrm{Var}(\mathbf{F}_1) \vee \mathrm{Var}(\mathbf{F}_2) \subseteq \mathrm{Var}(\mathbf{F}_1 \vee \mathbf{F}_2)$. ∎

Remark 3.2. The analogue of Proposition 3.5 for the intersection does not hold. In fact, $\mathrm{Var}(\mathbf{Zr} \cap \mathbf{G}) \neq \mathrm{Var}(\mathbf{Zr}) \cap \mathrm{Var}(\mathbf{G})$, since $\mathrm{Var}(\mathbf{Zr} \cap \mathbf{G}) = \mathbf{I}$ and, as we will see in Proposition 3.8, $\mathrm{Var}(\mathbf{Zr}) \cap \mathrm{Var}(\mathbf{G}) = \mathbf{M} \cap \mathbf{G} = \mathbf{G}$.

3.3 Formations and varieties generated by a class of monoids

One of our next aims is to show that for formations of monoids the dichotomy between varieties that are of groups and varieties that contain $\mathbf{J}_1$ also holds. We start with a result that is an immediate consequence of Proposition 2.5.

Proposition 3.6. *Every formation of monoids is generated by its subdirectly irreducible monoids.*

Next we discuss $\mathbf{J}_1$. It is a well-known fact that $\mathrm{Var}(U_1) = \mathbf{J}_1$ [9, p. 120]. Here we reprove this fact and the stronger result that $\mathrm{Form}(U_1) = \mathbf{J}_1$.

Proposition 3.7. *The formation of monoids and the variety of monoids generated by* U_1 *are both equal to the variety* $\mathbf{J}_1$*.*

Proof. Since $\mathrm{Form}(U_1) \subseteq \mathrm{Var}(U_1) \subseteq \mathbf{J}_1$, it suffices to prove that $\mathbf{J}_1 \subseteq \mathrm{Form}(U_1)$. Let M be an idempotent and commutative monoid. For each element $e \in M$, the set $M_e = \{1, e\}$ is a submonoid of M isomorphic to U_1 and hence the direct product N of these submonoids is in $\mathrm{Form}(U_1)$. Moreover the map from N to M which takes each element of N to the product of its coordinates is a surjective morphism. Thus $M \in \mathrm{Form}(U_1)$ as required. ∎

Another easy but interesting result, observed at the end of the proof of Proposition 1.5 of [5], is the following.

Proposition 3.8. *The variety* $\mathbf{M}$ *is generated by the class of all monoids with zero.*

Proof. This follows from the fact that any monoid M is a submonoid of $M^\Box$. ∎

The following dichotomy result, mentioned above, extends to formations a classical property of varieties of monoids. It will be extensively used in this paper.

Proposition 3.9. *If* $\mathbf{F}$ *is a formation of monoids, then either* $\mathbf{F} \subseteq \mathbf{G}$ *or* $\mathbf{J}_1 \subseteq \mathbf{F}$.

Proof. Let $\mathbf{F}$ be a formation of monoids such that $\mathbf{F}$ is not contained in $\mathbf{G}$. Then, there exists a monoid M in $\mathbf{F}$ that is not a group. Let G be the group of units of M and let $I = M \setminus G$. Then $I \neq \emptyset$, and moreover, I is an ideal of M. Now the mapping $\varphi\colon M \to U_1$ defined by $\varphi(G) = \{1\}$ and $\varphi(I) = \{0\}$ is a surjective morphism. It follows that $U_1 \in \mathbf{F}$ and by Proposition 3.7, $\mathbf{J}_1 \subseteq \mathbf{F}$. ∎

This proposition shows that formations of monoids split into two categories: the formations of groups and the formations containing U_1, and hence $\mathbf{J}_1$. Formations of monoids are not necessarily closed under taking submonoids but formations containing U_1 enjoy an analogous property for zero-adjoined submonoids.

Proposition 3.10. *Let* $\mathbf{F}$ *be a formation of monoids containing* U_1*. If* $M \in \mathbf{F}$ *and* N *is a submonoid of* M*, then* $N^\Box \in \mathbf{F}$ *and* $N^0 \in \mathbf{F}$.

Proof. Since $\mathbf{F}$ contains U_1 and M, Proposition 2.7 gives that $\mathcal{U}_1(N, M) \in \mathbf{F}$. As $N^\Box$ is a quotient of $\mathcal{U}_1(N, M)$, one also gets $N^\Box \in \mathbf{F}$. Finally, Lemma 2.3 states that N^0 is a quotient of $N^\Box$ and thus $N^0 \in \mathbf{F}$. ∎

If $\mathbf{F}$ is a formation of monoids containing U_1 that is not a variety, then there exists a monoid M in $\mathbf{F}$ and a submonoid N of M such that $N \notin \mathbf{F}$. However, for such an N, Proposition 3.10 ensures that $N^\Box$ and N^0 are both in $\mathbf{F}$. This underlines the important role of the zero of a monoid in the study of formations of monoids.

Proposition 3.11. *Let* $\mathbf{F}$ *be a formation of monoids containing* U_1*. The following conditions are equivalent for a monoid* M*:*

(1) $M \in \mathrm{Var}(\mathbf{F})$,
(2) $M^\Box \in \mathbf{F}$,
(3) $M^0 \in \mathbf{F}$.

Proof. (1) implies (2). Let $M \in \mathrm{Var}(\mathbf{F})$. Since $\mathrm{Var}(\mathbf{F}) = \mathrm{HS}(\mathbf{F})$, there exist a monoid $R \in \mathbf{F}$, a submonoid N of R and a surjective morphism $\varphi\colon N \to M$. Proposition 3.10 shows that $N^\Box \in \mathbf{F}$. Now φ induces a surjective morphism from $N^\Box$ onto $M^\Box$ and thus $M^\Box \in \mathbf{F}$.

(2) implies (3). This follows from Lemma 2.3, which states that M^0 is a quotient of $M^\Box$.

(3) implies (1). If $M^0 \in \mathbf{F}$, then $M \in \mathrm{Var}(\mathbf{F})$, since M is a submonoid of M^0. ∎

Corollary 3.12. *If* $\mathbf{F}$ *is a formation of monoids containing* U_1*, then* $\mathrm{Var}(\mathbf{F}) = \mathrm{S}(\mathbf{F})$*.*

Proof. Let $\mathbf{V} = \mathrm{Var}(\mathbf{F})$. If $M \in \mathbf{V}$, then $M^\square \in \mathbf{F}$ by Proposition 3.11. As M is a submonoid of $M^\square$, it follows that $M \in \mathrm{S}(\mathbf{F})$. Hence $\mathbf{V} \subseteq \mathrm{S}(\mathbf{F})$, but $\mathrm{S}(\mathbf{F}) \subseteq \mathbf{V}$ since $\mathbf{V}$ is a variety, and thus $\mathrm{Var}(\mathbf{F}) = \mathrm{S}(\mathbf{F})$. ∎

3.4 Formations and varieties generated by a class of groups

In this section, we consider specifically formations of groups. In many cases, the formation generated by a class of groups is a variety. For instance, a difficult result of [15] states that every formation generated by a class of nilpotent groups is a variety (see [8, IV.1.16, p. 342] for an alternative proof). Proposition 3.3 also leads to further cases. For example, if n is a positive integer, then the formation generated by the class of all groups of order $\leqslant n$ is a variety and the formation generated by the class of all groups whose order divides n is also a variety.

However, the formation generated by a group is not always a variety. In particular, it follows from [8, II.2.13, p. 276] that the formation generated by a simple group is the set of finite direct powers of this group, which is not in general a variety. Moreover, Corollary 3.12 does not hold for formations of groups.

Proposition 3.13. *There exists a formation of groups* $\mathbf{H}$ *such that* $\mathrm{Var}(\mathbf{H})$ *is not equal to* $\mathrm{S}(\mathbf{H})$.

Proof. Let D_4 be the dihedral group of order 8, and let Q_8 be the quaternion group of order 8. These two groups generate the same formation [8, Exercise 9, p. 344] and this formation is actually a variety of groups by the result of Neumann [15] stated above.

Let G be the simple group $GL(3,2)$, which is isomorphic to $PSL(3,2)$ and to $PSL(2,7)$. As we have just mentioned, the formation $\mathbf{H}$ generated by G is equal to the set $\{G^n \mid n \geqslant 0\}$,

Recall that a *quasi-identity* (or a *universal Horn clause*) is an implication of the form

$$(u_1 = v_1 \wedge u_2 = v_2 \wedge \cdots \wedge u_n = v_n) \to u = v$$

where $u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u, v$ are words of the free group over some set of variables. For instance, an elementary computation using GAP [10] shows that G satisfies the quasi-identity

$$(y^2 = x^2) \wedge (xyx = y) \to x^2 = 1 \tag{3.1}$$

but Q_8 does not. Now, by a standard result of universal algebra (see [7, Theorem 2.25, p. 219] and [22, Theorem 9, p. 195]), the class of groups satisfying a given quasi-identity is closed under the operators S and P. In particular, since $\mathrm{S}(\mathbf{H}) = \mathrm{SP}(G)$, every group of $\mathrm{S}(\mathbf{H})$ satisfies (3.1) and thus $Q_8 \notin \mathrm{S}(\mathbf{H})$.

On the other hand, since D_4 is a subgroup of G, the variety $\mathrm{Var}(\mathbf{H})$ contains $\mathrm{Var}(D_4)$, which, as we have seen, contains Q_8. Consequently, $\mathrm{S}(\mathbf{H})$ is strictly contained in $\mathrm{Var}(\mathbf{H})$. ∎

We now turn to more positive results and show how to state the counterpart of Proposition 3.11 for a formation of groups.

Proposition 3.14. *Let* $\mathbf{H}$ *be a formation of groups. The following conditions are equivalent for a group* G*:*

(1) $G \in \mathrm{Var}(\mathbf{H})$,

(2) $G^\square \in \mathbf{J}_1 \vee \mathbf{H}$,

(3) $G^0 \in \mathbf{J}_1 \vee \mathbf{H}$.

Proof. The result is obvious if G is the trivial group. Otherwise, $G^\square = G^0$ and it suffices to prove the equivalence of (1) and (2).

(1) implies (2). If $G \in \mathrm{Var}(\mathbf{H})$, then $G^\square \in \mathbf{J}_1 \vee \mathbf{H}$ by Proposition 3.11 applied to the formation $\mathbf{J}_1 \vee \mathbf{H}$.

(2) implies (1). If $G^\square \in \mathbf{J}_1 \vee \mathbf{H}$, then by Proposition 3.11, G belongs to $\mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H})$, which equals $\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})$ by Proposition 3.5. As G is a group, $G \in \mathrm{Var}(\mathbf{H})$. ∎

Corollary 3.12 describes the variety generated by a formation containing U_1. Its counterpart for a formation of groups is more technical.

Corollary 3.15. *Let* $\mathbf{H}$ *be a formation of groups. The variety generated by* $\mathbf{H}$ *is* $\mathrm{S}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G}$.

Proof. Let $G \in \mathrm{Var}(\mathbf{H})$. Then by Proposition 3.14, we have $G^\square \in \mathbf{J}_1 \vee \mathbf{H}$. Since G is a subgroup of $G^\square$, we have $G \in \mathrm{S}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G}$. Therefore

$$\mathrm{Var}(\mathbf{H}) \subseteq \mathrm{S}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} \subseteq \mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} \tag{3.2}$$

Now, by Proposition 3.5, we get $\mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H}) = \mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})$. As every group in $\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})$ belongs to $\mathrm{Var}(\mathbf{H})$, we have $(\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})) \cap \mathbf{G} \subseteq \mathrm{Var}(\mathbf{H})$ and hence

$$\mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} = (\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})) \cap \mathbf{G} \subseteq \mathrm{Var}(\mathbf{H}) \tag{3.3}$$

Putting (3.2) and (3.3) together, we finally obtain $\mathrm{Var}(\mathbf{H}) = \mathrm{S}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G}$. ∎

Proposition 3.14 and Corollary 3.15 stress the importance of the formations of the form $\mathbf{J}_1 \vee \mathbf{H}$, where $\mathbf{H}$ is a formation of groups. Indeed, $\mathbf{J}_1 \vee \mathbf{H}$ is the unique minimal formation over $\mathbf{H}$ in the lattice of non-group formations of monoids. A precise description of these formations is given later in Theorem 6.5.

3.5 Varieties generated by formations

The next proposition shows in particular that for each formation of monoids $\mathbf{F}$ containing U_1, the traces of $\mathbf{F}$ and $\mathrm{Var}(\mathbf{F})$ on $\mathbf{Zr}$ are equal. For each class $\mathbf{C}$ of monoids, let $\mathbf{C}^\square$ denote the class of all monoids of the form $M^\square$ for $M \in \mathbf{C}$. Note that $\mathbf{C}$ is not in general contained in $\mathbf{C}^\square$, even if $\mathbf{C}$ is a formation.

Proposition 3.16. *Let* $\mathbf{F}$ *be a formation of monoids containing* U_1*. Then*

$$\mathrm{Form}(\mathbf{F}^\square) = \mathbf{F} \cap \mathbf{Zr} = \mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr} \tag{3.4}$$

and

$$\mathrm{Var}(\mathbf{F}) = \mathrm{Var}(\mathbf{F} \cap \mathbf{Zr}). \tag{3.5}$$

Proof. Since $U_1 \in \mathbf{F}$, Proposition 3.10 shows that if $M \in \mathbf{F}$, then $M^\square \in \mathbf{F}$. Therefore $\mathbf{F}^\square \subseteq \mathbf{F} \cap \mathbf{Zr}$, and since $\mathbf{F} \cap \mathbf{Zr}$ is a formation,

$$\mathrm{Form}(\mathbf{F}^\square) \subseteq \mathbf{F} \cap \mathbf{Zr} \subseteq \mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr}.$$

To prove the opposite inclusions, consider $M \in \mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr}$. Then $M = M^0$, and by Proposition 3.11, we get $M^\square \in \mathbf{F}^\square$. Now, by Lemma 2.3, M^0 is a quotient of $M^\square$, and since $M = M^0$, we have $M \in \mathrm{Form}(\mathbf{F}^\square)$. Therefore $\mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr} \subseteq \mathrm{Form}(\mathbf{F}^\square)$, which proves (3.4).

Clearly, $\mathrm{Var}(\mathbf{F} \cap \mathbf{Zr}) \subseteq \mathrm{Var}(\mathbf{F})$. If $M \in \mathbf{F}$, then $M \in \mathrm{Var}(\mathbf{F})$ and hence $M^0 \in \mathbf{F} \cap \mathbf{Zr}$ by Proposition 3.11. By the same proposition, but applied now to $\mathbf{F} \cap \mathbf{Zr}$, one gets $M \in \mathrm{Var}(\mathbf{F} \cap \mathbf{Zr})$. Thus $\mathbf{F} \subseteq \mathrm{Var}(\mathbf{F} \cap \mathbf{Zr})$, whence $\mathrm{Var}(\mathbf{F}) \subseteq \mathrm{Var}(\mathbf{F} \cap \mathbf{Zr})$, which gives (3.5). ∎

Corollary 3.17. *Let* $\mathbf{V}$ *be a variety of monoids containing* U_1. *The formation* $\mathbf{V} \cap \mathbf{Zr}$ *is the smallest formation that generates* $\mathbf{V}$ *as a variety.*

Proof. By (3.5), we know that $\mathbf{V} \cap \mathbf{Zr}$ generates $\mathbf{V}$. Let $\mathbf{F}$ be a formation such that $\mathbf{V} = \mathrm{Var}(\mathbf{F})$. Then by (3.4), one has $\mathbf{V} \cap \mathbf{Zr} = \mathbf{F} \cap \mathbf{Zr} \subseteq \mathbf{F}$, which concludes the proof. ∎

The next proposition considers several natural classes of groups that arise from a formation of monoids containing U_1. It shows, in particular, the equality between them and that they are varieties.

Proposition 3.18. *Let* $\mathbf{F}$ *be a formation of monoids containing* U_1. *Then the three following classes are equal to the variety of groups* $\mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$*:*

(1) *the class of all subgroups of the monoids of* $\mathbf{F}$*,*

(2) *the class of all maximal subgroups of monoids of* $\mathbf{F}$*,*

(3) *the class of all groups of units of the monoids of* $\mathbf{F}$*.*

Proof. Let $\mathbf{H}_1, \mathbf{H}_2$ and $\mathbf{H}_3$ be the classes of groups described by Conditions (1), (2) and (3), respectively. Clearly, $\mathbf{H}_3 \subseteq \mathbf{H}_2 \subseteq \mathbf{H}_1$.

Let $G \in \mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$. Then $G^{\square} \in \mathbf{F}$ by Proposition 3.11. Since G is the group of units of $G^{\square}$, we have $G \in \mathbf{H}_3$, whence $\mathrm{Var}(\mathbf{F}) \cap \mathbf{G} \subseteq \mathbf{H}_3$.

Let now G be a group in $\mathbf{H}_1$. Then G is a subgroup of a monoid of $\mathbf{F}$ and hence G belongs to $\mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$. Thus $\mathbf{H}_1 \subseteq \mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$. Consequently, $\mathrm{Var}(\mathbf{F}) \cap \mathbf{G} = \mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}_3$. ∎

Notice that this proposition does not apply when $\mathbf{F}$ is a formation of groups. In fact, in this case, the classes described by the items (2) and (3) are both equal to $\mathbf{F}$.

4 Building processes for formations

In this section, we study two building processes for formations. The first makes use of minimal ideals and the second is defined in terms of maximal subgroups.

4.1 Formations built on minimal ideals

We start by giving a general construction in Section 4.1.1, which is specialized in Section 4.1.2 and then further refined in Section 4.1.3.

4.1.1 Minimal ideals

Given a class $\mathbf{C}$ of monoids, let $\mathrm{I}(\mathbf{C})$ denote the class of all minimal ideals of members of $\mathbf{C}$. Since the minimal ideal of a monoid is a completely simple semigroup, I defines a correspondence between classes of monoids and classes of completely simple semigroups.

The inverse correspondence I^{-1} attaches to each class $\mathbf{C_s}$ of completely simple semigroups, the class $\mathrm{I}^{-1}(\mathbf{C_s})$ of monoids whose minimal ideal belongs to $\mathbf{C_s}$.

Proposition 4.1. *Let* $\mathbf{F}$ *be a formation of semigroups contained in* $\mathbf{CS}$. *Then* $\mathrm{I}^{-1}(\mathbf{F})$ *is a formation of monoids. Moreover,* $\mathrm{I}^{-1}(\mathbf{F})$ *is a variety if and only if* $\mathbf{CS} = \mathbf{F}$.

Proof. We first prove that $\mathrm{I}^{-1}(\mathbf{F})$ is closed under taking quotients. Let $M \in \mathrm{I}^{-1}(\mathbf{F})$ and let I be its minimal ideal. Then $I \in \mathbf{F}$. If $\varphi\colon M \to N$ is a surjective morphism, then $\varphi(I)$ is the minimal ideal of N and hence it belongs to $\mathbf{F}$. Thus $N \in \mathrm{I}^{-1}(\mathbf{F})$.

Next, let M be a subdirect product of a finite family $(M_j)_{j\in J}$ of monoids of $\mathrm{I}^{-1}(\mathbf{F})$ and let I be the minimal ideal of M. By Proposition 2.6, I is a subdirect product of the minimal ideals I_j of the monoids M_j. Since $M_j \in \mathrm{I}^{-1}(\mathbf{F})$, $I_j \in \mathbf{F}$ and thus $I \in \mathbf{F}$. Therefore $M \in \mathrm{I}^{-1}(\mathbf{F})$. Thus $\mathrm{I}^{-1}(\mathbf{F})$ is a formation.

If $\mathbf{F} = \mathbf{CS}$, then $\mathrm{I}^{-1}(\mathbf{F}) = \mathbf{M}$ and this is a variety. Conversely, assume that $\mathrm{I}^{-1}(\mathbf{F})$ is a variety. Since $\mathrm{I}^{-1}(\mathbf{F})$ contains all the monoids with zero, by Proposition 3.8 we get $\mathrm{I}^{-1}(\mathbf{F}) = \mathbf{M}$. Now, take a simple semigroup S. Then S is the minimal ideal of S^{I} and $S^{\mathrm{I}} \in \mathbf{M} = \mathrm{I}^{-1}(\mathbf{F})$, whence $S \in \mathbf{F}$. Therefore $\mathbf{CS} \subseteq \mathbf{F}$, but $\mathbf{F} \subseteq \mathbf{CS}$ to start with, hence $\mathbf{CS} = \mathbf{F}$. ∎

4.1.2 Minimal ideals that are groups

We now look at formations of monoids contained in $\mathrm{I}^{-1}(\mathbf{G})$, i.e. formations of monoids whose minimal ideals are groups. Examples include any formation of groups, the variety $\mathbf{ZE}$ of monoids with central idempotents and the formation $\mathbf{Zr}$ of monoids with zero.

Proposition 4.2. *Let $\mathbf{F}$ be a formation of monoids such that $\mathbf{F} \subseteq \mathrm{I}^{-1}(\mathbf{G})$. Then*

(1) $\mathrm{I}(\mathbf{F}) = \mathbf{F} \cap \mathbf{G}$,

(2) *if $\mathbf{C}$ is a generating class of $\mathbf{F}$, then the formation $\mathbf{F}\cap\mathbf{G}$ is generated by $\mathrm{I}(\mathbf{C})$.*

Proof. (1) Let $M \in \mathbf{F}$ and let G be the minimal ideal of M. Since $\mathbf{F} \subseteq \mathrm{I}^{-1}(\mathbf{G})$, we have that G is a group. Let e be its identity. Then the map $x \mapsto ex$ is a surjective morphism from M onto G. Therefore $G \in \mathbf{F}$.

(2) By (1), one has $\mathrm{I}(\mathbf{C}) \subseteq \mathrm{I}(\mathbf{F}) = \mathbf{F} \cap \mathbf{G}$. It remains to prove that $\mathrm{I}(\mathbf{C})$ generates $\mathbf{F} \cap \mathbf{G}$. Let G be a group belonging to $\mathbf{F}$. By Proposition 3.1(8), we know that G is a quotient of a subdirect product M of a finite family $(M_j)_{j\in J}$ of monoids of $\mathbf{C}$. According to Proposition 2.1 (4), the group G is also a quotient of the minimal ideal I of M. For each $j \in J$, let I_j be the minimal ideal of M_j. By Proposition 2.6, I is a subdirect product of the family $(I_j)_{j\in J}$. Moreover, $I_j \in \mathrm{I}(\mathbf{C})$, since $M_j \in \mathbf{C}$. It follows that I, as well as its quotient G, belong to the formation generated by $\mathrm{I}(\mathbf{C})$. Thus $\mathrm{I}(\mathbf{C})$ generates the formation $\mathbf{F} \cap \mathbf{G}$. ∎

The following result is a particular case of Proposition 4.1. It provides a method to obtain formations of monoids that are not varieties.

Proposition 4.3 [5, Prop. 1.5]**.** *Let $\mathbf{H}$ be a formation of groups. The class $\mathrm{I}^{-1}(\mathbf{H})$ is a formation of monoids containing $\mathbf{J}_1$ and $\mathbf{H}$. However, it is not a variety, even if $\mathbf{H}$ is a variety of groups.*

As usual, given an ideal J of a monoid M, we denote the associated Rees quotient by M/J.

Lemma 4.4. *Let $M \in \mathrm{I}^{-1}(\mathbf{G})$, and let G be its minimal ideal. Then M is a subdirect product of M/G and G.*

Proof. Take the canonical morphism $\pi\colon M \to M/G$ and the identity, say e, of the group G. Then M is a subdirect product of M/G and G under the morphism $M \to (M/G) \times G,\ x \mapsto \bigl(\pi(x), ex\bigr)$. ∎

Proposition 4.5. *Let $\mathbf{F}$ be a formation of monoids and $\mathbf{H}$ be a formation of groups. Then $\mathbf{F} \cap \mathrm{I}^{-1}(\mathbf{H}) = (\mathbf{F} \cap \mathbf{Zr}) \vee (\mathbf{F} \cap \mathbf{H})$.*

Proof. The inclusion $\mathbf{F} \cap \mathrm{I}^{-1}(\mathbf{H}) \subseteq (\mathbf{F} \cap \mathbf{Zr}) \vee (\mathbf{F} \cap \mathbf{H})$ follows from Lemma 4.4, and the other inclusion is clear, since $\mathbf{Zr}$ and $\mathbf{H}$ are included in $\mathrm{I}^{-1}(\mathbf{H})$. ∎

Corollary 4.6. *If $\mathbf{F}$ is a formation of monoids contained in $\mathrm{I}^{-1}(\mathbf{G})$, then $\mathbf{F} = (\mathbf{F} \cap \mathbf{Zr}) \vee (\mathbf{F} \cap \mathbf{G})$.*

Corollary 4.6 shows that a formation of monoids contained in $\mathrm{I}^{-1}(\mathbf{G})$ is determined by its monoids with zero and its groups. This is so since given formations $\mathbf{F}_1$ and $\mathbf{F}_2$ contained in $\mathrm{I}^{-1}(\mathbf{G})$, we have

$$\mathbf{F}_1 \subseteq \mathbf{F}_2 \iff \mathbf{F}_1 \cap \mathbf{Zr} \subseteq \mathbf{F}_2 \cap \mathbf{Zr} \quad \text{and} \quad \mathbf{F}_1 \cap \mathbf{G} \subseteq \mathbf{F}_2 \cap \mathbf{G},$$

and hence

$$\mathbf{F}_1 = \mathbf{F}_2 \iff \mathbf{F}_1 \cap \mathbf{Zr} = \mathbf{F}_2 \cap \mathbf{Zr} \quad \text{and} \quad \mathbf{F}_1 \cap \mathbf{G} = \mathbf{F}_2 \cap \mathbf{G}.$$

4.1.3 Monoids with zero

The purpose of this section is to study the formations of the form $\mathrm{I}^{-1}(\mathbf{C})$ when $\mathbf{C}$ is the trivial formation of groups.

It follows from [5, Cor. 1.6] (a corollary of Proposition 4.3) that $\mathbf{Zr}$ is a formation but not a variety. The formations of monoids contained in $\mathbf{Zr}$ admit a characterization that reminds the definition of variety.

Proposition 4.7. *A class $\mathbf{F}$ of monoids contained in $\mathbf{Zr}$ is a formation of monoids if and only if the following conditions hold:*

(1) $\mathrm{H}(\mathbf{F}) \subseteq \mathbf{F}$,

(2) $\mathrm{P}(\mathbf{F}) \subseteq \mathbf{F}$,

(3) *if $M \in \mathbf{F}$ and N is a submonoid of M, then $N^0 \in \mathbf{F}$.*

Proof. Let $\mathbf{F}$ be a class of monoids contained in $\mathbf{Zr}$. If $\mathbf{F}$ is a formation, then, by definition, Conditions (1) and (2) hold. If $\mathbf{F}$ is the trivial formation, then Condition (3) is also satisfied since, if M is the trivial monoid, the unique submonoid of M is M itself and $M^0 = M$. If $\mathbf{F}$ is nontrivial and contained in $\mathbf{Zr}$, it is not contained in $\mathbf{G}$ and hence contains U_1 by Proposition 3.9. Thus Condition (3) is satisfied by Proposition 3.10.

Conversely, assume that $\mathbf{F}$ satisfies Conditions (1), (2) and (3). We only have to see that $\mathbf{F}$ is closed under taking subdirect products of finite families of monoids. Let $(M_i)_{i\in I}$ be a finite family of monoids of $\mathbf{F}$, let M be a monoid, and let $\varphi\colon M \to \prod_{i\in I} M_i$ be an embedding such that each morphism $\pi_j\varphi\colon M \to M_j$ is surjective. Then $\prod_{i\in I} M_i \in \mathbf{F}$ by (2), and $\varphi(M)$ is a submonoid of $\prod_{i\in I} M_i$. By (3), we have $(\varphi(M))^0 \in \mathbf{F}$. However, M and $\varphi(M)$ have a zero, since $\mathbf{F} \subseteq \mathbf{Zr}$ and $\mathbf{Zr}$ is a formation. Thus $M^0 = M$ and $(\varphi(M))^0 = \varphi(M)$. Then $M \in \mathbf{F}$, since M is isomorphic to $\varphi(M)$. ∎

Proposition 4.8. *Condition (3) of Proposition 4.7 can be replaced by any of the following conditions:*

(4) *if $M \in \mathbf{F}$ and N is a submonoid of M, then $N^{\Box} \in \mathbf{F}$,*

(5) *if $M \in \mathbf{F}$ and N is a submonoid of M with zero, then $N \in \mathbf{F}$.*

Proof. Let $\mathbf{F}$ be a class of monoids contained in $\mathbf{Zr}$. Assume that $\mathbf{F}$ satisfies Conditions (1), (2) and (3) of Proposition 4.7. Then $\mathbf{F}$ satisfies (5) by (3). Moreover, by Proposition 4.7, $\mathbf{F}$ is a formation, and hence it satisfies (4) by Proposition 3.10.

Conversely, if $\mathbf{F}$ satisfies (1) and (4), then it satisfies (3) by Lemma 2.3.

Suppose that $\mathbf{F}$ satisfies (1), (2) and (5). If $\mathbf{F}$ only contains the trivial monoid, then it satisfies (3). If $\mathbf{F}$ is nontrivial and contained in $\mathbf{Zr}$, it is not contained in $\mathbf{G}$ and hence contains U_1 by Proposition 3.9. Now, let $M \in \mathbf{F}$ and let N be a submonoid of M. If N has a zero, then $N \in \mathbf{F}$ by Condition (5). If N does not have a zero, then N^0 is a submonoid of M^0 and since $M \in \mathbf{Zr}$, $M^0 = M$. It follows by (5) that $N^0 \in \mathbf{F}$, which proves (3). ∎

4.2 Formations built on maximal subgroups

While in Proposition 3.18 we consider classes of groups arising from non-group formations of monoids, here we go in the opposite direction and look at some formations of monoids arising from classes of groups.

4.2.1 Maximal subgroups

Given a class $\mathbf{C}$ of groups, let $\overline{\mathbf{C}}$ denote the class of monoids whose maximal subgroups belong to $\mathbf{C}$. First we have

Lemma 4.9. *If* $\mathbf{C}$ *is a class of groups, then* $\overline{\mathbf{C}} \cap \mathbf{G} = \mathbf{C}$.

Proof. The result follows from the fact that if G is a group, then G itself is its only maximal subgroup. ∎

Proposition 4.10. *Let* $\mathbf{H}$ *be a class of groups. The following conditions are equivalent:*

(1) $\mathbf{H}$ *is a variety of groups,*

(2) *the class* $\overline{\mathbf{H}}$ *is a formation of monoids,*

(3) *the class* $\overline{\mathbf{H}}$ *is a variety of monoids.*

Proof. (1) $\implies$ (3) follows from [9, p. 145, Proposition 10.4]. Actually, Eilenberg shows that the class of monoids having their subgroups in $\mathbf{H}$ is a variety. But since $\mathbf{H}$ is a variety, this is equivalent to having their maximal subgroups in $\mathbf{H}$.

(3) $\implies$ (2) is trivial.

(2) $\implies$ (1). Suppose that $\overline{\mathbf{H}}$ is a formation of monoids. By Lemma 4.9, $\overline{\mathbf{H}} \cap \mathbf{G} = \mathbf{H}$. Thus $\mathbf{H}$ is a formation. Let G in $\mathbf{H}$ and let H be a subgroup of G. Observe that $U_1 \in \overline{\mathbf{H}}$ since the maximal subgroups of U_1 are trivial. It follows that $\mathcal{U}_1(H, G)$, which is a subdirect product of G and U_1 by Proposition 2.7, belongs to the formation $\overline{\mathbf{H}}$. Since H is a maximal subgroup of $\mathcal{U}_1(H, G)$, one has $H \in \mathbf{H}$ and thus $\mathbf{H}$ is closed under taking subgroups. It follows that $\mathbf{H}$ is a variety. ∎

4.2.2 Maximal subgroups of minimal ideals

We now combine the constructions of Sections 4.1.1 and 4.2.1.

Proposition 4.11. *Let* $\mathbf{H}$ *be a formation of groups. The class of simple semigroups whose maximal subgroups are in* $\mathbf{H}$ *is a formation of semigroups, which is a variety of semigroups if and only if* $\mathbf{H}$ *is a variety of groups.*

Proof. Let $\mathbf{C_H}$ be the class of simple semigroups whose maximal subgroups are in $\mathbf{H}$. Let $\varphi\colon S \to T$ be a surjective morphism of semigroups, with $S \in \mathbf{C_H}$. Then T is simple. By Proposition 2.1 (4), every maximal subgroup of T is the image by φ of a maximal subgroup of S, and therefore the maximal subgroups of T are in $\mathbf{H}$, which means that $T \in \mathbf{C_H}$.

Let $(S_i)_{i\in I}$ be a finite family of semigroups of $\mathbf{C_H}$, let S be a semigroup, and let $\varphi\colon S \to \prod_{i\in I} S_i$ be an embedding such that each morphism $\pi_i\varphi\colon S \to S_i$ is surjective. We aim to see that $S \in \mathbf{C_H}$. Let G be a maximal subgroup of S. Again by Proposition 2.1 (4), for each $i \in I$, we have that $\pi_i\varphi(G)$ is a maximal subgroup of S_i, and therefore $\pi_i\varphi(G) \in \mathbf{H}$. Since G is a subdirect product of $\prod_{i\in I} \pi_i\varphi(G)$, we get $G \in \mathbf{H}$. Thus $S \in \mathbf{C_H}$. Therefore $\mathbf{C_H}$ is a formation of semigroups.

It is clear that $\mathbf{C_H} \cap \mathbf{G} = \mathbf{H}$. It follows that if $\mathbf{C_H}$ is a variety, then $\mathbf{H}$ is a variety. The converse is also straightforward, as the simple semigroups form a variety. ∎

Corollary 4.12. *Let* $\mathbf{H}$ *be a formation of groups. The class of monoids whose maximal subgroups of the minimal ideal are in* $\mathbf{H}$ *is a formation. This formation is a variety if and only if* $\mathbf{H} = \mathbf{G}$*, in which case it is exactly* $\mathbf{M}$*.*

Proof. Let $\mathbf{F}$ be the class of monoids whose maximal subgroups of the minimal ideal are in $\mathbf{H}$. Let $\mathbf{C_H}$ be the class of simple semigroups whose maximal subgroups are in $\mathbf{H}$. Then $\mathbf{F} = \mathrm{I}^{-1}(\mathbf{C_H})$, whence applying successively Propositions 4.11 and 4.1, we conclude that $\mathbf{F}$ is a formation of monoids.

If $\mathbf{H} = \mathbf{G}$, then $\mathbf{F} = \mathbf{M}$ by definition of $\mathbf{F}$. Conversely, suppose that $\mathbf{F}$ is a variety. Then $\mathbf{C_H} = \mathbf{CS}$, by Proposition 4.1. Since $\mathbf{G} \subseteq \mathbf{CS}$ and $\mathbf{C_H} \cap \mathbf{G} = \mathbf{H}$, we have $\mathbf{H} = \mathbf{G}$. ∎

5 Two lattice isomorphisms

Let $\mathbf{H}$ be a formation of groups. We have seen at the end of Section 3.4 that $\mathbf{J}_1 \vee \mathbf{H}$ is the unique minimal non-group formation over $\mathbf{H}$. Describing the lattice of all formations over $\mathbf{H}$ seems to be out of reach at the moment, but we focus in this section on a specific sublattice for which we get a reasonable description.

5.1 H-suited formations

Let $\mathbf{H}$ be a formation of groups. Let us say that a formation of monoids $\mathbf{F}$ is $\mathbf{H}$-*suited* if

$$\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F} \subseteq \mathrm{I}^{-1}(\mathbf{H}).$$

In other words, a $\mathbf{H}$-suited formation of monoids contains $\mathbf{J}_1$ and $\mathbf{H}$ and the minimal ideals of its monoids are groups of $\mathbf{H}$.

The aim of this section is to give a complete description of the lattice of $\mathbf{H}$-suited formations. For the convenience of the reader, we give two versions of our next result, a short one and a detailed one.

Theorem 5.1 (Short version)**.** *Let* $\mathbf{H}$ *be a formation of groups. The lattice of* $\mathbf{H}$*-suited formations is isomorphic to the lattice of varieties containing* $\mathbf{J}_1 \vee \mathbf{H}$*.*

Theorem 5.1 (Detailed version)**.** *Given a formation of groups* $\mathbf{H}$*, the correspondence* $\mathbf{F} \longmapsto \mathrm{Var}(\mathbf{F})$ *that associates to each* $\mathbf{H}$*-suited formation of monoids* $\mathbf{F}$ *the variety generated by* $\mathbf{F}$*, and the correspondence* $\mathbf{V} \longmapsto \mathbf{V} \cap \mathrm{I}^{-1}(\mathbf{H})$ *that associates to each variety* $\mathbf{V}$ *of monoids such that* $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V}$ *the formation of monoids* $\mathbf{V} \cap \mathrm{I}^{-1}(\mathbf{H})$*, are two mutually inverse lattice isomorphisms between* $\mathbf{H}$*-suited formations* $\mathbf{F}$ *of monoids and varieties of monoids* $\mathbf{V}$ *such that* $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V}$*.*

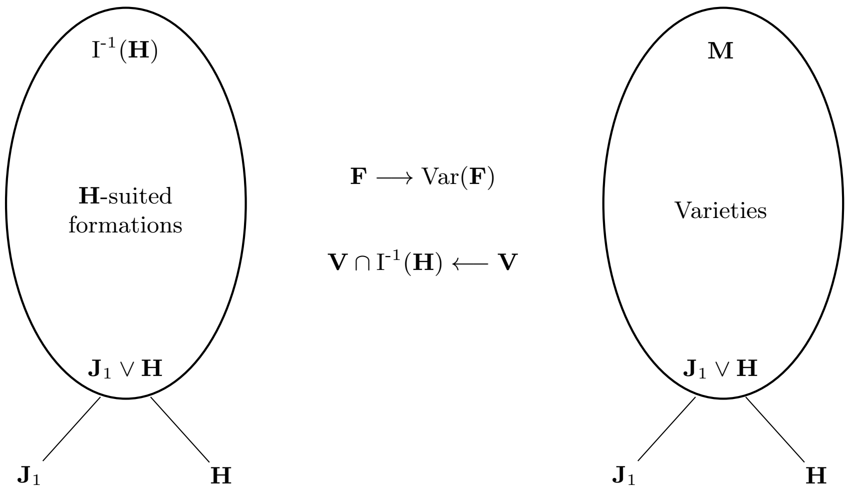


Figure 5.1: A lattice isomorphism.

Proof. On the one hand, it is obvious that if $\mathbf{F}$ is a formation of monoids such that $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F}$, then $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F} \subseteq \mathrm{Var}(\mathbf{F})$. On the other hand, if $\mathbf{V}$ is a variety of monoids such that $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V}$, then $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V} \cap \mathrm{I}^{-1}(\mathbf{H}) \subseteq \mathrm{I}^{-1}(\mathbf{H})$. Thus, both correspondences are well-defined.

Next, let $\mathbf{F}$ be a $\mathbf{H}$-suited formation of monoids. We aim to show that $\mathbf{F} = \mathrm{Var}(\mathbf{F}) \cap \mathrm{I}^{-1}(\mathbf{H})$. Clearly $\mathbf{F} \subseteq \mathrm{Var}(\mathbf{F}) \cap \mathrm{I}^{-1}(\mathbf{H})$. To prove the opposite inclusion, we will make use of Proposition 4.5 applied to $\mathrm{Var}(\mathbf{F})$ to get $\mathrm{Var}(\mathbf{F}) \cap \mathrm{I}^{-1}(\mathbf{H}) = (\mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr}) \vee (\mathrm{Var}(\mathbf{F}) \cap \mathbf{H})$. Now, the inclusion $\mathrm{Var}(\mathbf{F}) \cap \mathbf{Zr} \subseteq \mathbf{F}$ follows from (3.4) and clearly $\mathrm{Var}(\mathbf{F}) \cap \mathbf{H} \subseteq \mathbf{H} \subseteq \mathbf{F}$. Therefore $\mathrm{Var}(\mathbf{F}) \cap \mathrm{I}^{-1}(\mathbf{H}) \subseteq \mathbf{F}$.

Let $\mathbf{V}$ be a variety of monoids such that $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V}$. We claim that $\mathrm{Var}(\mathbf{V} \cap \mathrm{I}^{-1}(\mathbf{H})) = \mathbf{V}$. Corollary 3.17 shows that $\mathrm{Var}(\mathbf{V} \cap \mathbf{Zr}) = \mathbf{V}$. Since $\mathbf{Zr} \subseteq \mathrm{I}^{-1}(\mathbf{H})$, one gets

$$\mathbf{V} = \mathrm{Var}(\mathbf{V} \cap \mathbf{Zr}) \subseteq \mathrm{Var}(\mathbf{V} \cap \mathrm{I}^{-1}(\mathbf{H})) \subseteq \mathrm{Var}(\mathbf{V}) = \mathbf{V},$$

which proves the claim.

Concerning meets and joins, it is clear that the set of formations of monoids $\mathbf{F}$ such that $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F} \subseteq \mathrm{I}^{-1}(\mathbf{H})$ forms a sublattice of the lattice of formations, and also that the set of varieties of monoids $\mathbf{V}$ such that $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{V}$ is a sublattice of the lattice of varieties. The fact that both correspondences preserve meets and joins comes from the trivial implications

$$\mathbf{F}_1 \subseteq \mathbf{F}_2 \implies \mathrm{Var}(\mathbf{F}_1) \subseteq \mathrm{Var}(\mathbf{F}_2)$$

and

$$\mathbf{V}_1 \subseteq \mathbf{V}_2 \implies \mathbf{V}_1 \cap \mathrm{I}^{-1}(\mathbf{H}) \subseteq \mathbf{V}_2 \cap \mathrm{I}^{-1}(\mathbf{H})$$

that hold for any formations $\mathbf{F}_1, \mathbf{F}_2$ and any varieties $\mathbf{V}_1, \mathbf{V}_2$ of monoids. ∎

5.2 Formations of monoids with zero

From Theorem 5.1, letting $\mathbf{H}$ to be the trivial formation and taking into consideration Propositions 3.9 and Equation (3.4), we get the following result.

Theorem 5.2 (Short version)**.** *The lattice of non-trivial formations of monoids with zero is isomorphic to the lattice of varieties containing U_1.*

Theorem 5.2 (Detailed version)**.** *The correspondence* $\mathbf{F} \longmapsto \mathrm{Var}(\mathbf{F})$ *that associates to each non-trivial formation* $\mathbf{F}$ *of monoids with zero the variety of monoids generated by* $\mathbf{F}$*, and the correspondence* $\mathbf{V} \longmapsto \mathrm{Form}(\mathbf{V}^{\square}) = \mathbf{V} \cap \mathbf{Zr}$ *that associates to each variety* $\mathbf{V}$ *not contained in* $\mathbf{G}$ *the formation* $\mathbf{V} \cap \mathbf{Zr}$ *are two mutually inverse lattice isomorphisms between the non-trivial formations of monoids with zero and the varieties of monoids not contained in* $\mathbf{G}$*.*

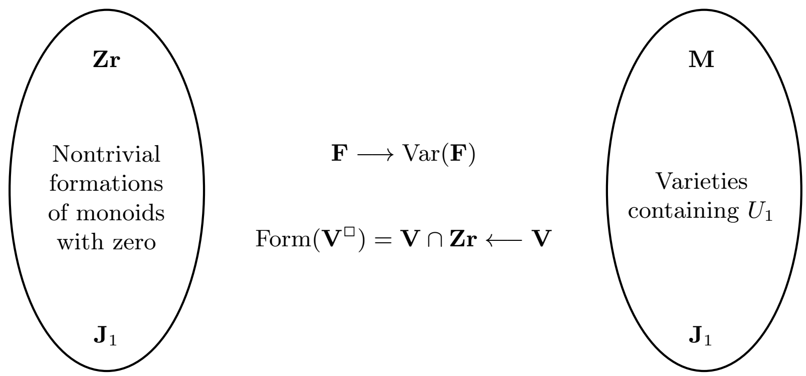


Figure 5.2: Another lattice isomorphism.

6 Clifford formations

In this section, we come back once again to formations of the form $\mathbf{J}_1 \vee \mathbf{H}$, where $\mathbf{H}$ is a formation of groups, and give a complete description of them (Theorem 6.5). But in order to state this result conveniently, a study of the variety $\mathbf{C}\ell$ of Clifford monoids is in order.

Let us first recall some classical properties of Clifford monoids (see [14, Section 4.2, p. 107]).

Proposition 6.1. *Let* M *be a monoid. The following conditions are equivalent:*

(1) M *is a Clifford monoid,*

(2) *every* $\mathcal{J}$*-class of* M *is a group,*

(3) M *is a semilattice of groups.*

A *Clifford formation* is a formation of monoids contained in $\mathbf{C}\ell$ and similarly, a *Clifford variety* is a variety of monoids contained in $\mathbf{C}\ell$.

6.1 Subdirectly irreducible Clifford monoids

Proposition 3.6 shows the relevance of subdirectly irreducible monoids in the study of formations. The description of the subdirectly irreducible monoids of $\mathbf{C}\ell$ relies on an elementary decomposition result. Recall that $\mathbf{ZE}$ denotes the variety of monoids with central idempotents.

Lemma 6.2. *Let* $M \in \mathbf{ZE}$ *and let* e *be an idempotent of* M*. Then* Me *is a monoid with* e *as identity, which is also an ideal of* M*. Moreover,* M *is a subdirect product of* M/Me *and* Me*.*

Proof. Let $\pi_1 \colon M \to M/Me$ be the canonical morphism, and let $\pi_2 \colon M \to Me$ be the surjective map defined by $\pi_2(x) = xe$. Since $M \in \mathbf{ZE}$, the mapping π_2 is also a morphism and $\pi_1 \times \pi_2 \colon M \to (M/Me) \times Me$ is an embedding. The projections are clearly surjective and thus M is a subdirect product of M/Me and Me. ∎

Proposition 6.3. *Any subdirectly irreducible monoid of* $\mathbf{C}\ell$ *is either a group or equal to* $G^{\square}$ *for some group* G.

Proof. Let M be a subdirectly irreducible monoid of $\mathbf{C}\ell$ and let G be the group of units of M. If $M \neq G$, then M contains an idempotent $e \neq 1$ and Me is a proper submonoid of M. By Lemma 6.2, M is a subdirect product of M/Me and Me. Since M is subdirectly irreducible, Me is necessarily trivial and thus e is a zero of M. Thus 1 and the zero are the unique idempotents of M and since M is a Clifford monoid, $M = G \cup \{0\} = G^{\square}$. ∎

6.2 Minimal formations outside of group formations

The variety $\mathbf{J}_1$ and the group formations are Clifford formations. Let $\mathbf{H}$ be a group formation. We now describe the minimal non-group formation above $\mathbf{H}$, which turns out to be also a Clifford formation.

Proposition 6.4. *Let* $\mathbf{F}$ *be a formation containing* U_1 *and let* $\mathbf{H} = \mathbf{F} \cap \mathbf{G}$. *Then* $\mathbf{J}_1 \vee \mathbf{H}$ *is the smallest formation of monoids contained in* $\mathbf{F}$ *and containing* $\mathbf{H}$ *that is not a formation of groups.*

Proof. Let $\mathbf{F}'$ be formation of monoids contained in $\mathbf{F}$ and containing $\mathbf{H}$ that is not a formation of groups. Then $\mathbf{F}'$ contains U_1 by Proposition 3.9 and thus $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F}'$. ∎

In particular, every non-group formation above a group formation $\mathbf{H}$ contains the formation $\mathbf{J}_1 \vee \mathbf{H}$. We now characterize the formations of this form. The subtle point of this characterization is that it simultaneously uses $\mathbf{H}$ and the variety generated by $\mathbf{H}$.

Theorem 6.5. *Let* $\mathbf{H}$ *be a formation of groups. The formation* $\mathbf{J}_1 \vee \mathbf{H}$ *is the class of Clifford monoids whose maximal subgroups are in the variety of groups generated by* $\mathbf{H}$ *and whose minimal ideal is in* $\mathbf{H}$. *That is,*

$$\mathbf{J}_1 \vee \mathbf{H} = \mathbf{C}\ell \cap \overline{\mathrm{Var}(\mathbf{H})} \cap \mathrm{I}^{-1}(\mathbf{H}). \tag{6.1}$$

Proof. Let $\mathbf{F} = \mathbf{C}\ell \cap \overline{\mathrm{Var}(\mathbf{H})} \cap \mathrm{I}^{-1}(\mathbf{H})$. Since both formations $\mathbf{H}$ and $\mathbf{J}_1$ are contained in $\mathbf{F}$, one gets $\mathbf{J}_1 \vee \mathbf{H} \subseteq \mathbf{F}$.

Let us show that $\mathbf{F} \subseteq \mathbf{J}_1 \vee \mathbf{H}$. According to Proposition 3.6, it is enough to prove that the subdirectly irreducible monoids of $\mathbf{F}$ belong to $\mathbf{J}_1 \vee \mathbf{H}$. Let M be such a subdirectly irreducible monoid. Since $M \in \mathbf{C}\ell$, Proposition 6.3 shows that M is either a group or a monoid of the form $G^{\square}$, where G is a group. In the former case, the condition $M \in \mathrm{I}^{-1}(\mathbf{H})$ implies that $M \in \mathbf{H}$, and hence trivially $M \in \mathbf{J}_1 \vee \mathbf{H}$. In the latter case, $M = G^{\square}$ and so $G \in \mathrm{Var}(\mathbf{H})$, as $M \in \overline{\mathrm{Var}(\mathbf{H})}$. Since $\mathrm{Var}(\mathbf{H}) = \mathrm{HS}(\mathbf{H})$ by Corollary 3.2, the group G is a quotient of a subgroup K of a group H of $\mathbf{H}$. Then $G^{\square}$ is a quotient of $K^{\square}$. On the other hand, by the second part of Proposition 2.7, the monoid $\mathcal{U}_1(K, H)$ is a subdirect product of H and U_1, thus it belongs to $\mathbf{J}_1 \vee \mathbf{H}$. But $K^{\square}$ is a quotient of $\mathcal{U}_1(K, H)$, thus $K^{\square} \in \mathbf{J}_1 \vee \mathbf{H}$. It follows that $M = G^{\square} \in \mathbf{J}_1 \vee \mathbf{H}$. ∎

An immediate consequence of Theorem 6.5 is the following result, which extends a result of [1, p. 56]. Note that, contrary to the results of [3] on joins involving a variety of groups, here we do not make any assumption on the variety of groups.

Corollary 6.6. *If* $\mathbf{W}$ *is a variety of groups, the variety* $\mathbf{J}_1 \vee \mathbf{W}$ *is the class of Clifford monoids whose maximal subgroups are in* $\mathbf{W}$*; that is,* $\mathbf{J}_1 \vee \mathbf{W} = \mathbf{C}\ell \cap \overline{\mathbf{W}}$. *Furthermore* $(\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{G} = \mathbf{W}$.

Proof. It is clear from Proposition 6.1 that the minimal ideal of a Clifford monoid is one of its maximal subgroups. Consequently, (6.1) simplifies to $\mathbf{J}_1 \vee \mathbf{W} = \mathbf{C}\ell \cap \overline{\mathbf{W}}$. It follows that $\mathbf{W} \subseteq (\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{G} \subseteq \overline{\mathbf{W}} \cap \mathbf{G} = \mathbf{W}$ and hence $(\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{G} = \mathbf{W}$. ∎

Theorem 6.7. *The correspondence $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{G}$ that associates to each Clifford variety of monoids $\mathbf{V}$ containing $\mathbf{J}_1$ the variety of groups $\mathbf{V} \cap \mathbf{G}$ and the correspondence $\mathbf{W} \longrightarrow \mathbf{J}_1 \vee \mathbf{W}$ that associates to each variety of groups $\mathbf{W}$ the variety of monoids $\mathbf{J}_1 \vee \mathbf{W}$ are two mutually inverse lattice isomorphisms between varieties of groups and Clifford varieties containing $\mathbf{J}_1$.*

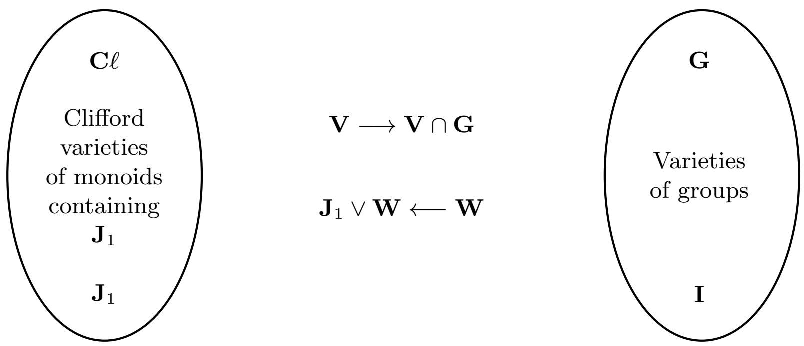


Figure 6.1: A third lattice isomorphism.

Proof. Let us show that, for every Clifford variety of monoids $\mathbf{V}$ containing $\mathbf{J}_1$, one has

$$\mathbf{J}_1 \vee (\mathbf{V} \cap \mathbf{G}) = \mathbf{V}. \tag{6.2}$$

Clearly, $\mathbf{J}_1 \vee (\mathbf{V} \cap \mathbf{G}) \subseteq \mathbf{V}$. Next, observe that $\mathbf{V} \subseteq \mathbf{C}\ell \cap \overline{(\mathbf{V} \cap \mathbf{G})}$ and apply Corollary 6.6 to the variety of groups $\mathbf{V} \cap \mathbf{G}$ to get $\mathbf{C}\ell \cap \overline{(\mathbf{V} \cap \mathbf{G})} = \mathbf{J}_1 \vee (\mathbf{V} \cap \mathbf{G})$. It follows that $\mathbf{V} \subseteq \mathbf{J}_1 \vee (\mathbf{V} \cap \mathbf{G})$. We have proved (6.2).

Furthermore, Corollary 6.6 shows that the formula

$$(\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{G} = \mathbf{W} \tag{6.3}$$

holds for every variety of groups $\mathbf{W}$. Formulas (6.2) and (6.3) show that the two correspondences are mutually inverse bijections. It remains to show that they are lattice homomorphisms between varieties of groups and Clifford varieties containing $\mathbf{J}_1$.

It is clear that the correspondence $\mathbf{W} \longrightarrow \mathbf{J}_1 \vee \mathbf{W}$ preserves joins. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be two varieties of groups. Corollary 6.6 shows that $\mathbf{J}_1 \vee (\mathbf{W}_1 \cap \mathbf{W}_2) = \mathbf{C}\ell \cap \overline{\mathbf{W}_1 \cap \mathbf{W}_2}$. Since it is clear that $\overline{\mathbf{W}_1 \cap \mathbf{W}_2} = \overline{\mathbf{W}_1} \cap \overline{\mathbf{W}_2}$, applying Corollary 6.6 again, one gets

$$\begin{aligned}\mathbf{J}_1 \vee (\mathbf{W}_1 \cap \mathbf{W}_2) &= \mathbf{C}\ell \cap \overline{\mathbf{W}_1} \cap \overline{\mathbf{W}_2} = (\mathbf{C}\ell \cap \overline{\mathbf{W}_1}) \cap (\mathbf{C}\ell \cap \overline{\mathbf{W}_2}) \\ &= (\mathbf{J}_1 \vee \mathbf{W}_1) \cap (\mathbf{J}_1 \vee \mathbf{W}_2).\end{aligned}$$

It follows that the correspondence $\mathbf{W} \longrightarrow \mathbf{J}_1 \vee \mathbf{W}$ is a lattice homomorphism.

The correspondence $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{G}$ obviously preserves meets. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two Clifford varieties of monoids containing $\mathbf{J}_1$ and let G be a group in $\mathbf{V}_1 \vee \mathbf{V}_2$. Then G is a quotient of a submonoid M of a product $M_1 \times M_2$, for some $M_1 \in \mathbf{V}_1$ and $M_2 \in \mathbf{V}_2$. By Proposition 2.1 (4), G is a quotient of a maximal subgroup H of M. Let H_1 and H_2 be the projections of H on M_1 and M_2, respectively. Then $H_1 \in \mathbf{V}_1 \cap \mathbf{G}$ and $H_2 \in \mathbf{V}_2 \cap \mathbf{G}$ and, since H is a subgroup of $H_1 \times H_2$, one has $H \in (\mathbf{V}_1 \cap \mathbf{G}) \vee (\mathbf{V}_2 \cap \mathbf{G})$, and hence $G \in (\mathbf{V}_1 \cap \mathbf{G}) \vee (\mathbf{V}_2 \cap \mathbf{G})$. Thus $(\mathbf{V}_1 \vee \mathbf{V}_2) \cap \mathbf{G} \subseteq (\mathbf{V}_1 \cap \mathbf{G}) \vee (\mathbf{V}_2 \cap \mathbf{G})$. Since the opposite inclusion is clear, the correspondence $\mathbf{V} \longrightarrow \mathbf{V} \cap \mathbf{G}$ preserves joins. ∎

It follows that Clifford varieties of monoids are easy to describe.

Corollary 6.8. *A Clifford variety of monoids is either a variety of groups or a variety of monoids of the form $\mathbf{J}_1 \vee \mathbf{W}$, with $\mathbf{W}$ a variety of groups.*

We now return to formations of the form $\mathbf{J}_1 \vee \mathbf{H}$, where $\mathbf{H}$ is a formation of groups. The hope would be to get a result similar to Corollary 6.8 for formations. The next proposition, which extends Formula (6.3) to group formations, is an encouraging result in this direction.

Proposition 6.9. *Let $\mathbf{H}$ be a formation of groups. Then $(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} = \mathbf{H}$.*

Proof. Let $G \in (\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G}$. Since $\mathbf{H}, \mathbf{J}_1 \subseteq \mathrm{I}^{-1}(\mathbf{H})$, we have $(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} \subseteq \mathrm{I}^{-1}(\mathbf{H})$ by Proposition 4.1, whence $G \in \mathrm{I}^{-1}(\mathbf{H})$. Since G is a group, its minimal ideal is itself, and hence $G \in \mathbf{H}$. Therefore $(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{G} \subseteq \mathbf{H}$. The opposite inclusion is obvious. ∎

But now comes the bad news. Contrarily to the variety case, not every Clifford formation $\mathbf{F}$ containing U_1 is of the form $\mathbf{J}_1 \vee \mathbf{H}$, with $\mathbf{H}$ a formation of groups, but this even worse if $\mathbf{F}$ is contained in $\mathbf{Zr}$, as the following result shows.

Proposition 6.10.

(1) *A Clifford formation $\mathbf{F}$ such that $\mathbf{J}_1 \subsetneq \mathbf{F} \subseteq \mathbf{Zr}$ can not be of the form $\mathbf{J}_1 \vee \mathbf{H}$, with $\mathbf{H}$ a formation of groups.*

(2) *There exists a Clifford formation $\mathbf{F}$ such that $\mathbf{J}_1 \subsetneq \mathbf{F} \not\subseteq \mathbf{Zr}$, but which is not of the form $\mathbf{J}_1 \vee \mathbf{H}$, with $\mathbf{H}$ a formation of groups.*

Proof. (1) Let $\mathbf{F}$ be a Clifford formation such that $\mathbf{J}_1 \subsetneq \mathbf{F} \subseteq \mathbf{Zr}$. Suppose there exists a formation of groups $\mathbf{H}$ such that $\mathbf{F} = \mathbf{J}_1 \vee \mathbf{H}$. Since $\mathbf{F} \neq \mathbf{J}_1$, the formation $\mathbf{H}$ is non-trivial. But on the other hand, $\mathbf{H} \subseteq \mathbf{F} \subseteq \mathbf{Zr}$, a contradiction since $\mathbf{H}$, as a nontrivial formation of groups, is not contained in $\mathbf{Zr}$.

(2) Take the symmetric group S_3. The cyclic group $C_2 = \{-1, 1\}$ of order 2 is a quotient of S_3, via the morphism $\pi\colon S_3 \to C_2$ that assigns to each permutation in S_3 its signature. Let $M = \mathcal{U}_1(S_3, C_2, \pi)$. Then M is a Clifford monoid with S_3 as group of units and C_2 as minimal ideal. By Proposition 3.9, the formation of monoids $\mathbf{F}$ generated by M satisfies $\mathbf{J}_1 \subsetneq \mathbf{F} \not\subseteq \mathbf{Zr}$. Notice that, as M is Clifford, $\mathbf{F}$ is a Clifford formation of monoids. Suppose now that $\mathbf{F} = \mathbf{J}_1 \vee \mathbf{H}$ for some formation $\mathbf{H}$ of groups. Then $S_3 \in \mathrm{Var}(\mathbf{H})$ by Theorem 6.5 and $\mathbf{H} = \mathbf{F} \cap \mathbf{G} = \mathrm{Form}(C_2)$ by Propositions 6.9 and 4.2.

Thus $\mathrm{Var}(\mathbf{H}) = \mathrm{Var}(C_2)$, and hence $\mathrm{Var}(\mathbf{H})$ is a variety of commutative groups, which contradicts the fact that $S_3 \in \mathrm{Var}(\mathbf{H})$. ∎

One can prove, in a similar way to the proof of Proposition 6.3, that if M is a subdirectly irreducible monoid of $\mathbf{ZE}$, then either M is a group or $M = G \cup N$, where G is a group and N is a nilpotent semigroup[1]. One may wonder whether, in this last case, M belongs to the formation generated by G and N^1. The following counterexample shows that it is not the case.

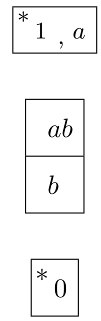
Example 6.11. Let M be the transformation monoid generated by the following generators:

[1] Recall that a semigroup with zero is *nilpotent* if 0 is its unique idempotent.

	1	2	3
a	2	1	3
b	3	0	0

It can be presented by the relations $a^2 = 1$, $ba = b$ and $b^2 = 0$. Its idempotents are 1 and 0. Its elements and its $\mathcal{D}$-class structure are represented below.

	1	2	3
$*\,1$	1	2	3
a	2	1	3
b	3	0	0
ab	0	3	0
$*\,0$	0	0	0



This monoid is not commutative, since $ab \neq ba$, but it belongs to $\mathbf{ZE}$. Furthermore, $M = G \cup N$, where $G = \{1, a\}$ is the cyclic group of order 2 and $N = \{ab, b, 0\}$ is a nilpotent commutative semigroup. Consequently, the monoids of the formation $\mathrm{Form}(\{G, N^1\})$ are commutative and thus M is not an element of this formation.

Although the description of formations of Clifford monoids remains an open problem, a much more satisfying result holds for formations of Clifford monoids with zero.

Theorem 6.12 (Short version)**.** *The lattice of non-trivial formations of Clifford monoids with zero is isomorphic to the lattice of varieties of groups.*

Theorem 6.12 (Detailed version)**.** *The correspondence* $\mathbf{F} \longmapsto \mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$ *that associates to each non-trivial formation* $\mathbf{F}$ *of Clifford monoids with zero the variety of groups* $\mathrm{Var}(\mathbf{F}) \cap \mathbf{G}$*, and the correspondence* $\mathbf{W} \longmapsto \mathrm{Form}(\mathbf{W}^{\square}) = (\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{Zr}$ *that associates to each variety of groups* $\mathbf{W}$ *the non-trivial formation* $(\mathbf{J}_1 \vee \mathbf{W}) \cap \mathbf{Zr}$*, are mutually inverse lattice isomorphims between non-trivial formations of Clifford monoids with zero and varieties of groups.*

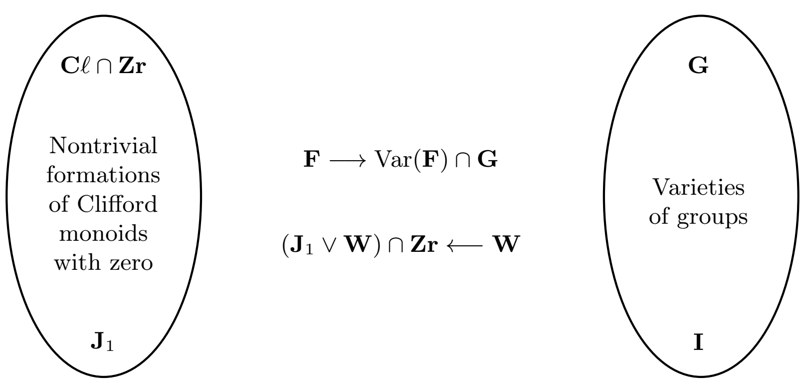


Figure 6.2: A fourth lattice isomorphism.

Proof. This is a consequence of Theorems 5.2 and 6.7. Indeed, the right hand side of Figure 6.3 is a copy of Figure 6.1 and its left handside is the restriction to Clifford varieties of Figure 5.2.

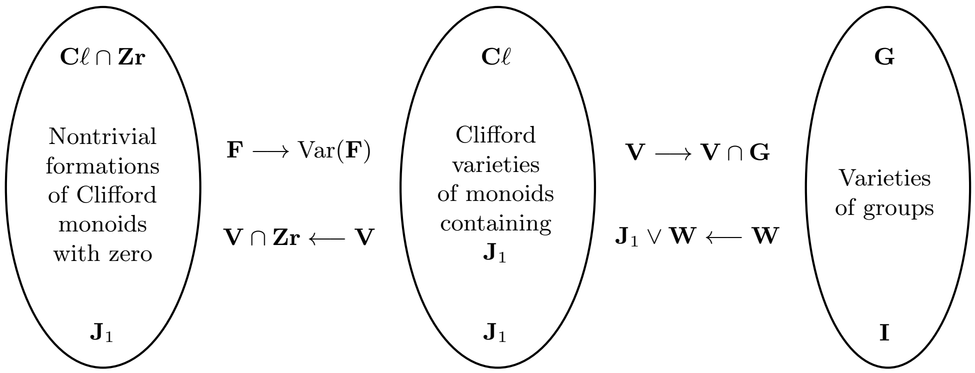


Figure 6.3: Composition of two lattice isomorphisms.

The composition of the lattice isomorphisms of Figure 6.3 yields the lattice isomorphisms of Figure 6.2. ∎

We can now complete Proposition 3.16 by giving a complete description of the formations of the form $\mathrm{Form}(\mathbf{F}^\Box)$. We start by considering group formations, for which we need a result to be compared with Corollary 3.15.

Proposition 6.13. *Let* $\mathbf{H}$ *be a formation of groups. Then* $\mathrm{Var}(\mathbf{H}) = \mathrm{Var}(\mathbf{H}^\Box) \cap \mathbf{G}$.

Proof. Let H be a group of $\mathbf{H}$. Then H is a subsemigroup of $H^\Box$, and hence $H \in \mathrm{Var}(\mathbf{H}^\Box) \cap \mathbf{G}$. It follows that $\mathrm{Var}(\mathbf{H}) \subseteq \mathrm{Var}(\mathbf{H}^\Box) \cap \mathbf{G}$. The opposite inclusion follows from the fact that $\mathrm{Var}(\mathbf{H}^\Box) \cap \mathbf{G} \subseteq \overline{\mathrm{Var}(\mathbf{H})} \cap \mathbf{G} = \mathrm{Var}(\mathbf{H})$. ∎

Theorem 6.14. *Let* $\mathbf{H}$ *be a group formation. Then*

$$\mathrm{Form}(\mathbf{H}^\Box) = \mathrm{Form}\big((\mathbf{J}_1 \vee \mathbf{H})^\Box\big) = (\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{Zr} = \big(\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})\big) \cap \mathbf{Zr}.$$

Proof. Applying Theorem 6.12 to $\mathbf{F} = \mathrm{Form}(\mathbf{H}^\Box)$ gives

$$\begin{aligned}\mathrm{Form}(\mathbf{H}^\Box) &= \Big(\mathbf{J}_1 \vee \Big(\mathrm{Var}\big(\mathrm{Form}(\mathbf{H}^\Box)\big) \cap \mathbf{G}\Big)\Big) \cap \mathbf{Zr} \\ &= \Big(\mathbf{J}_1 \vee \big(\mathrm{Var}(\mathbf{H}^\Box) \cap \mathbf{G}\big)\Big) \cap \mathbf{Zr} \\ &= \big(\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})\big) \cap \mathbf{Zr} \quad \text{by Proposition 6.13}\end{aligned}$$

On the other hand, applying (3.4) to the formation $\mathbf{J}_1 \vee \mathbf{H}$, one gets

$$\begin{aligned}\mathrm{Form}\big((\mathbf{J}_1 \vee \mathbf{H})^\Box\big) &= (\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{Zr} = \mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H}) \cap \mathbf{Zr} \\ &= (\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})) \cap \mathbf{Zr},\end{aligned}$$

as $\mathrm{Var}(\mathbf{J}_1 \vee \mathbf{H}) = \mathbf{J}_1 \vee \mathrm{Var}(\mathbf{H})$ by Proposition 3.5. The result follows. ∎

Corollary 6.15. *Let* $\mathbf{F}$ *be a formation of monoids. Then*

$$\mathrm{Form}(\mathbf{F}^\Box) = \mathrm{Form}\big((\mathbf{J}_1 \vee \mathbf{F})^\Box\big) = (\mathbf{J}_1 \vee \mathbf{F}) \cap \mathbf{Zr} = \big(\mathbf{J}_1 \vee \mathrm{Var}(\mathbf{F})\big) \cap \mathbf{Zr}.$$

Proof. If $\mathbf{F}$ contains $\mathbf{J}_1$, then $\mathbf{F} = \mathbf{J}_1 \vee \mathbf{F}$ and $\mathrm{Var}(\mathbf{F}) = \mathbf{J}_1 \vee \mathrm{Var}(\mathbf{F})$ and the result follows immediately from (3.4). Thus by Proposition 3.9, we are left with the case where $\mathbf{F}$ is a group formation, which follows from Theorem 6.14. ∎

7 References

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