COVERS FOR MONOIDS*

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ABSTRACT. A monoid M is an extension of a submonoid T by a group G if there is a morphism from M onto G such that T is the inverse image of the identity of G. Our first main theorem gives descriptions of such extensions in terms of groups acting on categories.

The theory developed is also used to obtain a second main theorem which answers the following question. Given a monoid M and a submonoid T, under what conditions can we find a monoid $\widehat{M}$ and a morphism θ from $\widehat{M}$ onto M such that $\widehat{M}$ is an extension of a submonoid $\widehat{T}$ by a group, and θ maps $\widehat{T}$ isomorphically onto T.

These results can be viewed as generalisations of two seminal theorems of McAlister in inverse semigroup theory. They are also closely related to Ash's celebrated solution of the Rhodes conjecture in finite semigroup theory.

McAlister proved that each inverse monoid admits an E-unitary inverse cover, and gave a structure theorem for E-unitary inverse monoids. Many researchers have extended one or both of these results to wider classes of semigroups. Almost all these generalisations can be recovered from our two main theorems.

INTRODUCTION

This paper is a contribution to the structure theory of semigroups, the object of which is to describe a semigroup by splitting it into simpler pieces. The theory of group extensions is a particular case of this general problem, and the questions we consider in this paper can be thought of as arising from an attempt to develop an analogous theory for monoids.

The situation is complicated by the fact that for monoids there are inequivalent analogues of the notion of normal subgroup. We are thus led to consider the following questions. First, given a monoid M and an appropriate submonoid T of M, find a group G and a monoid $\widehat{M}$ with submonoid $\widehat{T}$ and surjective morphisms α, β such that, in the diagram

$$\begin{tikzcd} \widehat{T} \arrow[r] \arrow[d, "\alpha"'] & \widehat{M} \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\ T \arrow[r] & M & G \end{tikzcd}$$

where the horizontal arrows are inclusion maps, the restriction of α to $\widehat{T}$ is an isomorphism and $1\beta^{-1} = \widehat{T}$. When such a monoid $\widehat{M}$ exists, we say that it is a T-cover of M. Secondly, what can we say about the structure of $\widehat{M}$ in terms of G and T? There are also the subsidiary questions of what conditions the submonoids T and $\widehat{T}$ must satisfy. In fact, the answer to the question about $\widehat{T}$ has been known since the 1940s (see [31, 32]).

When M is a group and T is a subgroup, it is natural to want $\widehat{M}$ to be a group. Then $\widehat{T}$ has to be a normal subgroup of $\widehat{M}$ and consequently, T is a normal subgroup of M. Thus we may take $\widehat{M}$ to be M, $\widehat{T}$ to be T and G to be M/T, and we are left with the problem of describing M in terms of G and T, that is, the synthesis problem in the theory of group extensions. These observations explain why there are no covering theorems in group theory.

In general, however, $\widehat{M}$ will be different from M. One of the first illustrations of this occurs in the work of McAlister [38, 39] in the mid 1970s. Groups and semilattices are the natural pieces into which to split an inverse monoid and this leads to considering the above situation with M inverse and T the commutative subsemigroup of idempotents of M, denoted by $E(M)$ in the sequel. McAlister obtained a covering theorem in which he showed the existence of an inverse monoid $\widehat{M}$ where we can take $\widehat{T}$ to be $E(\widehat{M})$ and G to be the maximum group homomorphic image of $\widehat{M}$. The monoid $\widehat{M}$ is said to be E-unitary because $E(\widehat{M})$ is a unitary subset of $\widehat{M}$ [45, Proposition III.7.2], and we say that $\widehat{M}$ is an E-unitary cover of M over G. In the cited papers, McAlister gave a description of E-unitary inverse monoids in terms of semilattices and groups, and we refer to this result as the structure theorem.

McAlister's work has been extended in various ways by many authors, and our aim is to answer the general questions posed above and thereby obtain almost all previous results as special cases. Our two main results are Theorem 4.5 and Theorem 5.1. In the latter we prove the existence of a T-cover of M when T is a "strongly dense" submonoid of M, and the former provides descriptions of the cover in terms of a category on which the group G acts freely and transitively. Our use of groups acting on categories generalises the pioneering work of Margolis and Pin [34, 35] who studied extensions of semilattices by groups. We need a notion of kernel of a monoid morphism (the weak derived category) which is rather more sophisticated than simply taking the inverse image of the identity of the codomain. The importance of categories when looking for such a concept was emphasised by Tilson [56] as well as in [34, 35]. We also use these ideas together with a method introduced by Fountain [18] to obtain the covering theorem of Section 5.

The first four sections are devoted to developing the necessary results on categories and groups acting on categories. Subgroupoids of groupoids have several properties which arbitrary subcategories of categories do not possess. In Section 1 we consider special types of subcategories which do enjoy some of these properties. In addition, we investigate various analogues in categories of normal subgroupoids of groupoids [25]. We use three of these notions, unitary, dense and planar subcategories in Section 2 to describe the kernels of certain morphisms of categories (functors), called quotient maps, from a category onto a groupoid, and to obtain a "fundamental theorem of homomorphisms". These results extend those for groupoids in [25]. The isomorphism theorems for a more restricted class of category morphisms are well known [56] and are related to congruences on categories. Using our results on quotient maps we characterise groupoid congruences on a category in Theorem 2.6, thereby generalising work of Levi [31, 32] and Gomes [21] on group congruences on semigroups.

Armed with these preliminaries we study group actions on categories in Sections 3 and 4. We introduce isotropic group actions in Section 3; with such an action, the orbits can be made into a category in a natural way. The main result of the section shows that an equivariant morphism of categories induces a unique morphism between the corresponding categories of orbits. This result underlies many of the key theorems in the rest of the paper. We use it immediately to show that if G acts isotropically and transitively on a category C, then the monoid of orbits C/G is a "universal" monoid for C in the sense that any equivariant morphism from C onto a monoid with trivial G-action factors through C/G. In Section 4 we consider free actions showing that when we specialise to this case some results of Section 3 can be coordinatised. The section concludes by giving one of the main results of the paper, Theorem 4.5 alluded to above, which gives several ways of describing an extension of a monoid by a group.

Section 5 is devoted to showing that if a submonoid T of a monoid M satisfies an appropriate condition, then M has a T-cover. The appropriate condition on T is that it is strongly dense in M, a notion introduced and discussed in Section 1. To find a T-cover $\widehat{M}$ of M, we look for a suitable group G and category C and take $\widehat{M}$ to be C/G. To find C we use a method introduced in [18] which amounts to finding a relational morphism from M to G. A *relational morphism* from a monoid M to another monoid Q is a mapping τ from M into the non-empty subsets of Q such that $1 \in 1\tau$ and $(a\tau)(b\tau) \subseteq (ab)\tau$, for all $a, b \in M$; τ is *surjective* if $Q = \bigcup\{a\tau \mid a \in M\}$ (see [46]). That this notion is closely related to the problems under discussion is seen from the fact that if τ is a surjective relational morphism from M to the group G, then its graph $\widehat{M} = \{(a, b) \mid b \in a\tau\}$ is a submonoid of $M \times G$, the projections $\alpha : \widehat{M} \to M$ and $\beta : \widehat{M} \to G$ are surjective morphisms, as in

$$\begin{tikzcd} & \widehat{M} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M \arrow[rr, "\tau"] & & G \end{tikzcd}$$

and τ is equal to the relation $\alpha^{-1}\beta$.

The last four sections of the paper are devoted to applying our main theorems to various special classes of monoids. For a monoid M in one of these classes, we are concerned with a particular submonoid T which has the property that it is mapped to the identity by any morphism from M onto a group. This choice of submonoid leads to the notion of *weak conjugacy* which we now describe. A *weak inverse* of an element a of a monoid is an element b such that $bab = b$. A submonoid T of M is closed under weak conjugation if for all elements a of M and t of T, and every weak inverse b of a, the elements bta and atb belong to T. The submonoid $D(M)$ is then defined to be the smallest submonoid of M which is closed under weak conjugation. It is well known that if α is a morphism from M onto a group G, then $D(M)\alpha = \{1\}$. We say that a monoid M is *D-unitary* if $D(M)$ is a unitary submonoid. A T-cover $\widehat{M}$ of a monoid M is a *D-unitary cover* if $T = D(M)$ and $\widehat{T} = D(\widehat{M})$. Such a cover is said to be an *E-unitary cover* if $D(\widehat{M}) = E(\widehat{M})$. We remark that in our definition of an E-unitary cover $\widehat{M}$ of a monoid M we have $E(\widehat{M})$ isomorphic to $E(M)$. This property is slightly more restrictive than the usual definition, but it holds in the case of regular semigroups by virtue of Lallement's lemma, and in more general cases, previous constructions of E-unitary covers have always enjoyed the property.

Each class considered has the property that the monoids in it have a minimum group congruence, and Section 6 is, in part, an investigation of when a monoid has such a congruence. We find a quite general condition to ensure that this is the case and, in addition, that the maximum group quotient is the fundamental group of the monoid. We provide an example — the details of which are given in an appendix — to show that a monoid may have a maximum group quotient which is not its fundamental group. We show that, when our general condition holds, the results of Section 4 give descriptions of a monoid which is an extension (of a monoid) by its maximum group quotient. We conclude the section by examining a specialisation of the covering theorem of Section 5 to the case of monoids in which the smallest weakly self-conjugate submonoid is dense.

Section 7 is devoted to E-dense monoids (also known as E-inversive monoids), that is, monoids in which, for every element a, there are elements b and c such that ab and ca are idempotents. The class of E-dense monoids is an extensive one which includes all regular monoids and all finite monoids. If a monoid is E-unitary, then it must be an E-monoid [19, Proposition 2.1], that is, a monoid in which the idempotents form a subsemigroup. Thus, in general, an E-dense monoid cannot have an E-unitary cover. However, after developing analogues for weak inverses in E-dense monoids of results about inverses in regular monoids, we prove that if a monoid M is E-dense, then so is $D(M)$, and this allows us to apply our main theorems to describe D-unitary E-dense monoids, and to show that every E-dense monoid has a D-unitary E-dense cover. We then specialise to recover results on E-dense E-monoids due to Almeida, Pin and Weil [1]. Specifically, we obtain their description of E-unitary E-dense E-monoids, and show that every E-dense E-monoid has an E-unitary E-dense cover.

We turn our attention to regular monoids in Section 8. We start by observing that if M is a regular monoid, then $D(M)$ is the least self-conjugate submonoid of M (often denoted by $C_\infty(M)$ in the regular semigroup literature). We then specialise our main theorems to regular monoids to give a description of D-unitary regular monoids, and a new approach to Trotter's covering theorem [58] for regular monoids, that is, that every regular monoid has an D-unitary regular cover.

Finally, in Section 9 we discuss the case of finite monoids. As finite monoids are E-dense, a structure theorem for D-unitary finite monoids follows immediately from that for D-unitary E-dense monoids. The corresponding covering theorem does not follow from the E-dense case because the proof of the general result produces an infinite cover. However, there is a finite covering theorem which follows from Ash's celebrated solution to the Rhodes conjecture. This fact has also been observed by Trotter and Zhonghao Jiang in [59]. It is a challenging problem to provide a unified framework that will give both the finite and infinite results.

We briefly mention the question of categories and monoids versus semigroupoids and semigroups. In the body of the paper we have chosen to state and prove all our results for monoids and categories. There are corresponding results for semigroups and semigroupoids. These can be obtained by slightly modifying the proofs we give; in many cases they can be deduced from those for monoids and categories by adjoining identities and then removing them.

1. Categories

We begin by reviewing some basic ideas about categories and monoids to establish notation and definitions. We refer the reader to [27] for further information about monoids and to [6, 25, 33, 56] for more details about categories.

We begin by recalling the definition of a category. In a departure from the standard notation we use the symbol $+$ for composition of morphisms in a category. The reason for this is that we will often have a group acting on a category and we believe that our notation, which follows [35], leads to increased clarity.

A (small) *category* C consists of a set of *objects* denoted by $\operatorname{Obj} C$ and a disjoint collection of sets $\operatorname{Mor}(u, v)$ (or $\operatorname{Mor}_C(u, v)$), one for each pair of objects u, v. The elements of the sets $\operatorname{Mor}(u, v)$ are called *morphisms* and the set of all morphisms of C is denoted by $\operatorname{Mor} C$. For each object u of C, there is a distinguished element 0_u of $\operatorname{Mor}(u, u)$, called the *identity morphism at u*. Finally, there is a partial operation on $\operatorname{Mor} C$, called *composition* and written $+$ which satisfies the following conditions:

(1) if p, q are morphisms of C, the *composite* $p + q$ of p and q is defined if and only if there exist objects u, v, w of C such that $p \in \operatorname{Mor}(u, v)$ and $q \in \operatorname{Mor}(v, w)$; in this case, $p + q \in \operatorname{Mor}(u, w)$;

(2) for any objects u, v, w of C and any morphisms $p \in \operatorname{Mor}(v, u)$, $q \in \operatorname{Mor}(u, w)$, we have

$$p + 0_u = p \text{ and } 0_u + q = q;$$

(3) for all objects u, v, w, x of C and for all morphisms $p \in \operatorname{Mor}(u, v)$, $q \in \operatorname{Mor}(v, w)$ and $r \in \operatorname{Mor}(w, x)$,

$$(p + q) + r = p + (q + r).$$

A morphism $p \in \operatorname{Mor}(u, v)$ is an *isomorphism* if there is a morphism $q \in \operatorname{Mor}(v, u)$ such that $p + q = 0_u$ and $q + p = 0_v$. Such a morphism, if it exists, is unique, and it is denoted by $-p$. A category is a *groupoid* if all its morphisms are isomorphisms.

If $p \in \operatorname{Mor}(u, v)$ for some objects u, v of a category C, then u is the *domain* of p and we write $u = \alpha(p)$, and v is the *codomain* of p and we write $v = \omega(p)$. Two morphisms having the same domain and the same codomain are said to be *coterminal.*

We use the term *morphism (of categories)* in preference to *functor* where a *morphism* $\varphi : C \to D$ between two categories C and D is given by:

(1) a function $\varphi : \operatorname{Obj} C \to \operatorname{Obj} D$ and

(2) for all objects u, v of C, a function $\varphi_{u,v} : \operatorname{Mor}_C(u, v) \to \operatorname{Mor}_D(u\varphi, v\varphi)$ such that for all $u, v, w \in \operatorname{Obj} C$ and all $p \in \operatorname{Mor}_C(u, v)$ and $q \in \operatorname{Mor}_C(v, w)$,

$$p\varphi_{u,v} + q\varphi_{v,w} = (p + q)\varphi_{u,w}.$$

Subscripts are usually omitted and the last formula is written as

$$p\varphi + q\varphi = (p + q)\varphi.$$

For each object u of a category C, the set of morphisms $\operatorname{Mor}(u, u)$ is a monoid under composition, called the *local monoid* of C at u. A category is said to be *locally commutative, idempotent,* etc. if all its local monoids are commutative, idempotent, etc. We note that if C is a groupoid, then each local monoid is actually a group.

A category is *connected* if for any pair of objects u, v, at least one of $\operatorname{Mor}(u, v)$ and $\operatorname{Mor}(v, u)$ is not empty. We are more interested in categories C in which $\operatorname{Mor}(u, v)$ is nonempty for all $u, v \in \operatorname{Mor} C$. Such categories are said to be *strongly connected* or in [56] to be *bonded.* At the opposite extreme we have *totally disconnected* categories in which $\operatorname{Mor}(u, v) \neq \emptyset$ if and only if $u = v$.

A category B is a *subcategory* of a category C if $\operatorname{Obj} B \subseteq \operatorname{Obj} C$, $\operatorname{Mor} B \subseteq \operatorname{Mor} C$ and composites and identity morphisms are the same in B as in C.

For any category C, let $\delta(C)$ be the subcategory with $\operatorname{Obj}\delta(C) = \operatorname{Obj} C$ and $\operatorname{Mor}\delta(C) = \bigcup\{\operatorname{Mor}(u,u) \mid u \in \operatorname{Obj} C\}$. Clearly $\delta(C)$ is a totally disconnected subcategory of C, and a subcategory of C is totally disconnected if and only if it is a subcategory of $\delta(C)$.

When a category C has just one object, then we may think of it as a monoid, namely the local monoid at the unique object. Thus many results for categories have immediate corollaries for monoids. For clarity, in the next two sections we occasionally make this explicit by stating the monoid version of a category result.

Following [25], by the intersection of a family $(D_i)_{i\in I}$ of subcategories of a category C, we mean the subcategory D with object set $\operatorname{Obj} D = \bigcap_i \operatorname{Obj} D_i$, and for all $u, v \in \operatorname{Obj} D$, $\operatorname{Mor}_D(u,v) = \bigcap_i \operatorname{Mor}_{D_i}(u,v)$. Similarly, we use the notation $D \subseteq C$ to indicate that D is a subcategory of C.

Remark. It is sometimes useful, and desirable, to work with semigroupoids rather than categories. Semigroupoids are defined like categories, dropping only those axioms which refer to the local identities: they are to categories what semigroups are to monoids. As a rule, the results and definitions in this paper will be given for categories and monoids, but they also hold for semigroupoids and semigroups. In a few places, adjustments must be made to definitions or proofs to fit the semigroupoid case. These are explicitly mentioned in the text.

Our main purpose in this section is to introduce five types of subcategory which play an important role in the next few sections. Subgroups of groups and subgroupoids of groupoids provide examples of the first two types, namely unitary and dense subcategories, and part of the motivation for introducing these notions is that they behave more like subgroupoids than general subcategories do. The definitions are straightforward extensions to categories of familiar ideas in semigroup theory.

A subcategory N of a category C is said to be *unitary* if for all $x, y \in \operatorname{Mor} C$,

(1) if $x + y, x \in \operatorname{Mor} N$, then $y \in \operatorname{Mor} N$, and
(2) if $x + y, y \in \operatorname{Mor} N$, then $x \in \operatorname{Mor} N$.

Note that, as well as subgroupoids of groupoids, the subcategory $\delta(C)$ is unitary for any category C.

We say that N is *dense in C* if for all objects u, v of C and all $x \in \operatorname{Mor}(u,v)$, there are elements $y, z \in \operatorname{Mor}(v,u)$ such that $x + y, z + x \in \operatorname{Mor} N$.

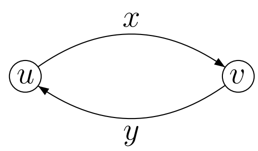
$$\begin{tikzcd} u \arrow[r, "x"] & v \arrow[l, bend right, "y"'] \arrow[l, bend left, "z"] \end{tikzcd}$$

Note that if N is dense in C, then $\operatorname{Obj} N = \operatorname{Obj} C$. All subgroupoids of groupoids are examples of dense subcategories, and if C is strongly connected, then $\delta(C)$ is a dense subcategory of C.

We next introduce reflexive, planar and strongly dense subcategories. Normal subgroups of groups and normal subgroupoids of groupoids enjoy all three properties. Reflexivity is an extension to categories of a well known idea in semigroup theory. Planarity is a generalisation of a less well known semigroup notion, that of a "normal subsemigroup" due to Levi [31, 32]. Strongly dense submonoids of a monoid were used in [19]. As we shall see in the next section, unitary, dense and planar subcategories arise naturally when we consider "kernels" of certain functors.

We say that a subcategory N of a category C is *reflexive in* C if for all objects u, v of C and all $x \in \operatorname{Mor}(u, v)$ and $y \in \operatorname{Mor}(v, u)$, then

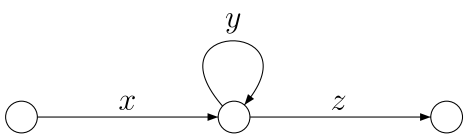
$$x + y \in \operatorname{Mor} N \text{ if and only if } y + x \in \operatorname{Mor} N.$$



Note that, for any category C, the subcategory $\delta(C)$ is clearly reflexive in C.

Next we define N to be *planar in* C if for any morphisms x, y, z of C such that $x + z$ and $x + y + z$ are both defined, the following condition holds:

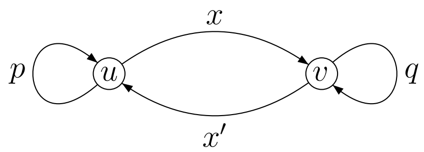
if any two of $y, x + z, x + y + z$ are in $\operatorname{Mor} N$, then so is the third.



It is easy to see that $\delta(C)$ is planar in C.

Finally, N is *strongly dense in* C if

(1) $\operatorname{Obj} N = \operatorname{Obj} C$, and
(2) for all objects u, v of C and all $x \in \operatorname{Mor}(u, v)$, there is a morphism $x' \in \operatorname{Mor}(v, u)$ such that $x + q + x', x' + p + x \in \operatorname{Mor} N$ whenever p and q are in the local monoids of N at u and v respectively.



We note that if C is strongly connected, then $\delta(C)$ is strongly dense in C.

It is clear that a strongly dense subcategory of C is also dense in C. The converse does not hold since any subgroup of a group G is dense in G but, as we see below, a subgroup of G is strongly dense in G if and only if it is normal in G. For semigroups, we have that a planar subsemigroup is unitary [11, Exercise 17, Section 10.2]. However, the corresponding result does not hold for categories as the following example shows.

Example. Let C be the category with three objects a, b, c and non-identity morphisms $x \in \mathrm{Mor}(a, b), y \in \mathrm{Mor}(b, c), z \in \mathrm{Mor}(a, c)$ and let $x + y = z$. Let N be the subcategory with the same objects and all the morphisms of C except x. It is easy to see that N is planar in C but it is not unitary since $y, x + y \in \mathrm{Mor}\, N$ but $x \notin \mathrm{Mor}\, N$.

$$\begin{tikzcd} a \arrow[r, "x"] \arrow[rr, bend right, "z"'] & b \arrow[r, "y"] & c \end{tikzcd}$$

Totally disconnected subcategories are essentially collections of submonoids of the local monoids of a category and so it is not surprising that they behave more like submonoids than arbitrary subcategories do. For example, a totally disconnected, planar subcategory is unitary as we see from the next lemma which is just the category version of [11, Exercise 17, Section 10] alluded to above.

Lemma 1.1. *Let T be a totally disconnected subcategory of a category C. Then T is planar if and only if it is unitary and reflexive.*

Proof. Suppose that T is planar. Then T is reflexive by Lemma 1.3. Let $x, y \in \mathrm{Mor}\, C$ be such that $x, x + y \in \mathrm{Mor}\, T$. Then, since T is totally disconnected, $x + x$ and $x + y + x$ are defined and in $\mathrm{Mor}\, T$ so that by planarity, $y \in \mathrm{Mor}\, T$. Similarly, if $x, y + x \in \mathrm{Mor}\, T$, then $y \in \mathrm{Mor}\, T$ and T is unitary.

Conversely, suppose that T is reflexive and unitary and let $x, y, z \in \mathrm{Mor}\, C$ be such that $y, x + z \in \mathrm{Mor}\, T$. Then by reflexivity, $y, z + x \in \mathrm{Mor}\, T$ so that $z + x + y \in \mathrm{Mor}\, T$ since T is a subcategory. Now reflexivity gives $x + y + z \in \mathrm{Mor}\, T$. The other two conditions for planarity are proved similarly. $\square$

Corollary 1.2. *A submonoid T of a monoid M is planar if and only if it is unitary and reflexive.*

The next lemma shows that when N is a planar subcategory of a category C, we have four equivalent definitions of denseness.

Lemma 1.3. *Let N be a planar subcategory of a category C. Then N is reflexive. In addition, the following conditions are equivalent:*

(1) *N is dense in C,*
(2) *for any morphism x of C, there is a morphism y of C such that $x+y, y+x \in \operatorname{Mor} N$,*
(3) *for any morphism x of C, there is a morphism y of C such that $x+y \in \operatorname{Mor} N$,*
(4) *for any morphism x of C, there is a morphism z of C such that $z+x \in \operatorname{Mor} N$.*

Proof. Let $u, v \in \operatorname{Obj} C$ be such that $x \in \operatorname{Mor}(u, v)$ and $y \in \operatorname{Mor}(v, u)$. If $x+y \in \operatorname{Mor} N$, then $x+y+x+y \in \operatorname{Mor} N$, and since N is planar, we also have $y+x \in \operatorname{Mor} N$. Therefore N is reflexive.

Now we prove the equivalence of conditions (1)–(4). Clearly, (2) implies (1) and (1) implies (3) and (4). By symmetry, it suffices to show that (3) implies (2). Let $u, v \in \operatorname{Obj} C$ be such that $x \in \operatorname{Mor}(u, v)$. By (3), there is a morphism $y \in \operatorname{Mor}(v, u)$ with $x+y \in \operatorname{Mor} N$. Thus $x+y$, $x+y+x+y$ and $y+x$ all are defined, and the first two are in $\operatorname{Mor} N$. As N is planar, we obtain $y+x \in \operatorname{Mor} N$ as required. □

When we combine denseness with planarity we do get a strongly dense subcategory as shown in the next lemma.

Lemma 1.4. *Any dense and planar subcategory is strongly dense.*

Proof. We have already observed that if N is a dense subcategory of a category C, then $\operatorname{Obj} N = \operatorname{Obj} C$. Now assume, in addition, that N is planar. For objects u, v of C, let $x \in \operatorname{Mor}_C(u, v)$, $p \in \operatorname{Mor}_N(u, u)$ and $q \in \operatorname{Mor}_N(v, v)$. By denseness and Lemma 1.3, there is a morphism y in C such that $x+y$ and $y+x$ are both in N. Hence by planarity,

$$x+q+y, y+p+x \in \operatorname{Mor} N$$

so that N is strongly dense in C. □

We recall from [25] that a subgroupoid N of a groupoid A is *normal* if

(1) N contains all the identity morphisms of A, and
(2) if $u, v \in \operatorname{Obj} A$ and $p \in \operatorname{Mor}_N(u, u), r \in \operatorname{Mor}_A(u, v)$, then $(-r)+p+r \in \operatorname{Mor}_N(v, v)$.

We can extend the notion and consider *normal subcategories of a groupoid.* The following lemma relates normality of subgroupoids with two of the concepts introduced above. First, we make the trivial observation that a subcategory C of a groupoid A with $\operatorname{Obj} C = \operatorname{Obj} A$ is dense in A.

Lemma 1.5. *For a subgroupoid N of a groupoid A, the following conditions are equivalent:*

(1) *N is normal in A,*
(2) *N is planar in A and $\operatorname{Obj} N = \operatorname{Obj} A$,*
(3) *N is strongly dense in A.*

Proof. Suppose that N is normal and let x, y, z be morphisms of A such that $x+z$ and $x+y+z$ are defined in A. If $x+z$ and y are in N, then $-z+y+z \in \operatorname{Mor} N$, so that $x+y+z = (x+z)+(-z+y+z) \in \operatorname{Mor} N$. If $x+y+z, y \in \operatorname{Mor} N$, then also $-y \in \operatorname{Mor} N$, and hence $-z-y+z \in \operatorname{Mor} N$. Therefore $x+z = x+y+z+(-z-y+z) \in \operatorname{Mor} N$. If $x+y+z, x+z \in \operatorname{Mor} N$, then $-z-x \in \operatorname{Mor} N$, so that $x+y-x = (x+y+z)+(-z-x)$ is in $\operatorname{Mor} N$. It follows that $y = -x+(x+y-x)+x \in \operatorname{Mor} N$. Certainly, $\operatorname{Obj} N = \operatorname{Obj} A$, and so (1) implies (2).

That (3) is a consequence of (2) is immediate from Lemma 1.4 since N is necessarily dense in A.

Finally, suppose that N is strongly dense in A. Certainly N contains all the identity morphisms. Let $x \in \operatorname{Mor} A$ and $u \in \operatorname{Obj} A$ be the domain of x. Then there is a morphism x' such that $x'+p+x \in \operatorname{Mor} N$ for all $p \in \operatorname{Mor}_N(u,u)$. Since $0_u \in \operatorname{Mor} N$, we have $x'+0_u+x = x'+x \in \operatorname{Mor} N$. Hence $-x-x' \in \operatorname{Mor} N$ and consequently,

$$-x+p+x = -x-x'+x'+p+x \in \operatorname{Mor} N$$

for all $p \in \operatorname{Mor}_N(u,u)$. Thus N is a normal subgroupoid of A. □

Specialising to groups we immediately have the following corollary.

Corollary 1.6. *For a subgroup H of a group G, the following conditions are equivalent:*

(1) *H is normal in G,*
(2) *H is strongly dense in G,*
(3) *H is planar in G.*

The equivalence of (1) and (2) explains why there are no covering theorems in the sense of this paper in group theory. The following examples show that the corollary does not extend to submonoids of groups or to inverse submonoids of inverse monoids.

Example. Any submonoid of an abelian group is strongly dense. However, the submonoid $\mathbb{N}$ of $\mathbb{Z}$ is not planar in $\mathbb{Z}$ since 2 and $1+2-2 = 1 \in \mathbb{N}$ but $1-2 \notin \mathbb{N}$.

Example. If x, y, z are integers, let

$$A(x,y,z) \;=\; \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

We let $G = \{A(x,y,z) \mid x,y,z \in \mathbb{Z}\}$ and $M = \{A(x,y,x) \mid x,y \in \mathbb{N}\}$. Then G is a group and M is a submonoid of G. It is easy to verify that $A(1,0,0)^{-1}A(1,0,1)A(1,0,0) \notin M$. Hence M is not normal in G. However, it can be readily shown that

$$A(p,q,r)A(a,b,a)A(x,y,z) \in M \text{ and } A(x,y,z)A(a,b,a)A(k,m,n) \in M$$

where $p = \max\{|x|, |z|\}$, $q = |y + pz|$, $r = p + x - z$, $n = \max\{|x|, |z|\}$, $k = n + z - x$ and $m = |y + xn|$. Thus M is strongly dense in G.

Example. Let M be any inverse monoid and let $E(M)$ be the semilattice of idempotents of M. For any $a \in M, e \in E(M)$ we have $aea^{-1}, a^{-1}ea \in E(M)$ and so $E(M)$ is strongly dense in M. However, as we see in Section 2, $E(M)$ is planar in M if and only if it is unitary in M, that is, if and only if M is E-unitary.

Remark. In the semigroupoid case we have to modify the definitions of dense and strongly dense subsemigroupoids. In the definition of dense subsemigroupoid we include the condition that the subsemigroupoid and the semigroupoid have the same set of objects. We say that a subsemigroupoid N of a semigroupoid C is strongly dense if N^1 is strongly dense in C^1. Here C^1 denotes the category obtained from C by adjoining identities where necessary.

2. Quotient maps

We introduce quotient maps of categories extending the notion of quotient map of groupoids [25]. In this section our concern will be with quotient maps from categories to groupoids but we will use the more general notion in Section 3. As observed in [25], many of the basic properties of group morphisms carry over to the class of quotient maps of groupoids.

Morphisms from semigroups onto groups provide another generalisation of group morphisms and they and the congruences they induce have been studied by several authors. See, for example, [14, 21, 31, 32] for the general case, [16, 29, 37] for regular semigroups and [23] for eventually regular semigroups. The work of Dubreil and Levi is reported in [11, Chapter 10].

In this section we give a common extension of some of the results of [25] and [21, 31, 32]. Following the terminology of [25], we define a morphism $\theta\colon C \to D$ of categories to be a *quotient map* if $\theta\colon \operatorname{Obj} C \to \operatorname{Obj} D$ is surjective and $\theta_{u,v}\colon \operatorname{Mor}(u, v) \to \operatorname{Mor}(u\theta, v\theta)$ is surjective for all $u, v \in \operatorname{Obj} C$. We note that a quotient map is necessarily surjective but, as pointed out in [25], the converse is not true. We warn the reader that in [56] the term *quotient morphism* is used to mean a morphism of categories which is *bijective* on the set of objects and *surjective* on morphism sets.

For a quotient map $\theta\colon C \to A$ from a category C to a groupoid A we define $\operatorname{Ker}\theta$, the *kernel* of θ, to consist of $\operatorname{Obj} C$ and all morphisms of C which map to identity morphisms of A. It is easy to see that $\operatorname{Ker}\theta$ is a subcategory of C. The following theorem describes precisely which subcategories are kernels of quotient maps.

Theorem 2.1. *A subcategory of a category C is the kernel of a quotient map from C to a groupoid if and only if it is dense, unitary and planar.*

Proof. First, we show that a kernel is dense, unitary and planar. Let $\theta\colon C \to A$ be a quotient map from a category C to a groupoid A. Let $x, y \in \operatorname{Mor} C$ and suppose that $x+y$ and y are in $\operatorname{Ker}\theta$. Then $(x+y)\theta = 0_u$ for some $u \in \operatorname{Obj} A$. It follows that $y\theta = 0_u$. Hence

$$x\theta = x\theta + 0_u = x\theta + y\theta = (x+y)\theta = 0_u$$

so that x is in $\operatorname{Ker}\theta$. Similarly, if the morphisms x and $x+y$ are in $\operatorname{Ker}\theta$, then so is y and so $\operatorname{Ker}\theta$ is unitary.

Let $x, y, z \in \operatorname{Mor} C$ be such that $x+z$ and $x+y+z$ are both defined. Then for some objects u, v, w of C we have $x \in \operatorname{Mor}(u, v), y \in \operatorname{Mor}(v, v)$ and $z \in \operatorname{Mor}(v, w)$.

If $x+z, x+y+z \in \operatorname{Ker}\theta$, then $u\theta = w\theta$ and $z\theta$ is the inverse of $x\theta$ so that $z\theta + x\theta = 0_{v\theta}$. Hence

$$\begin{aligned} y\theta &= 0_{v\theta} + y\theta + 0_{v\theta} = z\theta + x\theta + y\theta + z\theta + x\theta = z\theta + (x+y+z)\theta + x\theta \\ &= z\theta + 0_{u\theta} + x\theta = z\theta + x\theta = 0_{v\theta} \end{aligned}$$

and so $y \in \operatorname{Ker}\theta$ as required. Similar arguments give the other two conditions for planarity.

Now let $u, v \in \operatorname{Obj} C$ and let $x \in \operatorname{Mor}(u, v)$. Then $x\theta \in \operatorname{Mor}(u\theta, v\theta)$ and since A is a groupoid, $x\theta$ has an inverse in $\operatorname{Mor}(v\theta, u\theta)$. Now θ is quotient map and so there is a morphism y in $\operatorname{Mor}(v, u)$ such that $y\theta$ is the inverse of $x\theta$. Hence

$$(x+y)\theta = x\theta + y\theta = 0_{u\theta}$$

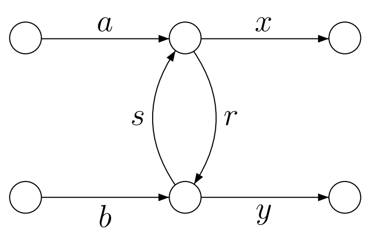
so that $x+y$ is a morphism of $\operatorname{Ker}\theta$. Since $\operatorname{Ker}\theta$ is planar, $\operatorname{Ker}\theta$ is dense by Lemma 1.3.

For the converse, let N be a unitary, dense, planar subcategory of a category C. Our aim is to construct a groupoid and a quotient map θ from C to the groupoid with $\operatorname{Ker}\theta = N$. First, we define relations on $\operatorname{Obj} C$ and on $\operatorname{Mor} C$, both denoted by ρ_N, by the following rules. For $u, v \in \operatorname{Obj} C$,

$$u \,\rho_N\, v \quad \text{if and only if} \quad \operatorname{Mor}(u, v) \cap \operatorname{Mor} N \neq \emptyset.$$

It follows from the fact that N is dense and unitary that ρ_N is an equivalence relation on $\operatorname{Obj} C$. For morphisms a, b of C,

$$\begin{aligned} a \,\rho_N\, b \quad \text{if and only if} \quad & b+s+x, a+r+y, a+x, b+y \in \operatorname{Mor} N \\ & \text{for some morphisms } r, s \text{ of } N \text{ and } x, y \text{ of } C. \end{aligned}$$



We now show that the relation ρ_N on $\operatorname{Mor} C$ is also an equivalence. First we note that if $a \,\rho_N\, b$, then $\alpha(a) \,\rho_N\, \alpha(b)$ and $\omega(a) \,\rho_N\, \omega(b)$. The first point follows from the relations $\alpha(a) \,\rho_N\, \omega(x)$ and $\alpha(b) \,\rho_N\, \omega(x)$. The second point is immediate since $r, s \in \operatorname{Mor} N$.

The following lemma simplifies the arguments in the proof.

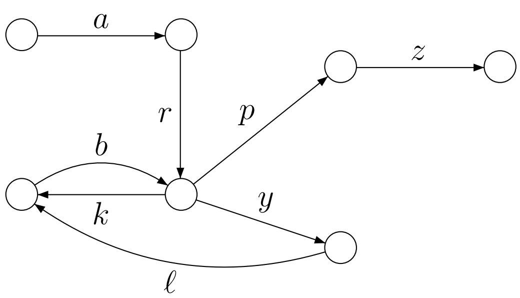
Lemma 2.2. *If $a \in \operatorname{Mor} C$, $r \in \operatorname{Mor} N$ and $a + r$ is defined, then $(a + r) \,\rho_N\, a$. Similarly, if $r + a$ is defined, then $(r + a) \,\rho_N\, a$.*

Proof. We prove only the first statement. Now N is dense and planar so that by Lemma 1.3, there is a morphism s such that $r + s, s + r \in \operatorname{Mor} N$. Note that $r + s \in \operatorname{Mor}(\omega(a), \omega(a))$. Moreover, N is unitary and so $s \in \operatorname{Mor} N$. Again using the fact that N is dense, there are morphisms x and y of C such that $a+x, (a+r)+y \in \operatorname{Mor} N$. Thus we have $(a+r)+s+x = a+(r+s+x) \in \operatorname{Mor} N$ because $r+s, a+x \in \operatorname{Mor} N$ and N is planar. The lemma follows. $\square$

Returning to the proof of Theorem 2.1, we now show that ρ_N is indeed an equivalence on $\operatorname{Mor} C$.

We have already observed that if N is dense in C, then $\operatorname{Obj} N = \operatorname{Obj} C$. Let $a \in \operatorname{Mor} C$. Then letting $r = s = 0_{\omega(a)}$ in the definition of ρ_N, we find that $a \,\rho_N\, a$, that is, ρ_N is reflexive.

It is clear from the definition that the relation ρ_N is symmetric. Now suppose that $a, b, c \in \operatorname{Mor} C$ with $a \,\rho_N\, b$ and $b \,\rho_N\, c$. Then there are morphisms p, q, r, s of N and x, y, z, t of C such that $b+s+x, a+r+y, a+x, b+y, b+p+z, c+q+t, c+z, b+t$ are all morphisms of N. As N is dense, there are morphisms k, ℓ of C such that $k + b$, $b + k$, $b + y + \ell$ and $\ell + b + y$ are all in N.



By planarity, $k+b+y+\ell+b \in \operatorname{Mor} N$ and hence $r+k+b+y+\ell+b+p \in \operatorname{Mor} N$. Now N is unitary and $b+y \in \operatorname{Mor} N$ so that $\ell \in \operatorname{Mor} N$. Also, $a+r+k+b+y \in \operatorname{Mor} N$ by planarity since $k+b$ and $a+r+y$ are in N. Hence

$$a+(r+k+b+y+\ell+b+p)+z=(a+r+k+b+y)+\ell+(b+p+z) \in \operatorname{Mor} N.$$

Similarly we can find a morphism d of N such that $c+d+x \in \operatorname{Mor} N$ and as $a+x$ and $c+z$ are both in $\operatorname{Mor} N$ we have $a \,\rho_N\, c$.

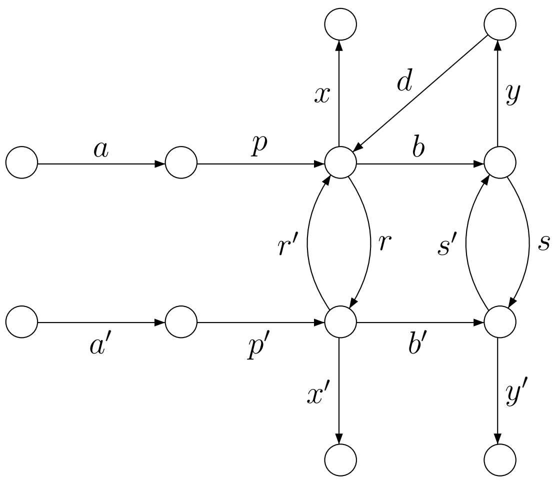
Thus ρ_N is an equivalence on $\operatorname{Mor} C$ as required.

Denote the ρ_N-equivalence class of an object or a morphism x of C by $[x]$. We now have a new graph with vertex set $\{[u] \mid u \in \operatorname{Obj} C\}$ and set of edges $\{[a] \mid a \in \operatorname{Mor} C\}$. To each edge $[a]$ we can assign unique initial and terminal vertices, $\alpha([a])$ and $\omega([a])$, by taking $\alpha([a])$ ($\omega([a])$) to be the ρ_N-class of the domain (codomain) of any morphism in $[a]$. We denote this graph by C/ρ_N.

We want to make C/ρ_N into a category. To this end, let $[a],[b] \in \operatorname{Mor}(C/\rho_N)$ with $\omega([a])=\alpha([b])$. Then $\omega(a)\,\rho_N\,\alpha(b)$ so that there is a morphism p of N in $\operatorname{Mor}(\omega(a),\alpha(b))$. We define composition of morphisms in C/ρ_N by the rule

$$[a]+[b]=[a+p+b].$$

To see that this is well-defined, let $a,a',b,b' \in \operatorname{Mor} C$ with $a\,\rho_N\,a', b\,\rho_N\,b'$ and let $p' \in \operatorname{Mor}(\omega(a'),\alpha(b')) \cap \operatorname{Mor} N$. Let $c=a+p$, $c'=a'+p'$. By Lemma 2.2, $a\,\rho_N\,c$ and $a'\,\rho_N\,c'$. Hence $c\,\rho_N\,c'$ and so there are morphisms r,r' in N and x,x' in C such that $c+x, c'+x', c+r+x'$ and $c'+r'+x$ are all in $\operatorname{Mor} N$.



Also $b\,\rho_N\,b'$ so that for some morphisms s,s' in N and y,y' in C, $b+y, b'+y', b+s+y'$ and $b'+s'+y$ are all in $\operatorname{Mor} N$. By Lemma 1.3, there is a morphism d such that $b+y+d$

and $d+b+y$ are both in N. Thus $b+y+d+r+r' \in \operatorname{Mor} N$ and by planarity,

$$c+b+y+d+r+r'+x \in \operatorname{Mor} N.$$

Since N is unitary and $b+y+d$ and $b+y$ are in $\operatorname{Mor} N$, we have $d \in \operatorname{Mor} N$. Moreover, $b'+s'+y, d, r, c'+r'+x$ are all in N so that by planarity,

$$c'+b'+s'+y+d+r+r'+x \in \operatorname{Mor} N.$$

Similarly, there is a morphism k such that $c'+b'+k$ and $c+b+s+k$ are both in N and hence, $(c+b)\,\rho_N\,(c'+b')$. Thus composition of morphisms in C/ρ_N is well-defined. It is clear that composition is associative and so C/ρ_N is a semigroupoid.

In fact, C/ρ_N is a category since it follows from the definition of composition and Lemma 2.2 that the morphism $[0_u]$ is the identity morphism at $[u]$ for each object $[u]$.

Next we show that C/ρ_N is a groupoid. First, if $u\,\rho_N\,u'$ and $a \in \operatorname{Mor}(u',u) \cap \operatorname{Mor} N$, then $a\,\rho_N\,0_u$. Since $u\,\rho_N\,u'$ there exist $p \in \operatorname{Mor}(u,u') \cap \operatorname{Mor} N$ and $q \in \operatorname{Mor}(u',u) \cap \operatorname{Mor} N$. Then, putting $x=r=q$ and $y=s=p$ in the definition of ρ_N, we find that $a\,\rho_N\,0_u$.

It is now easy to see that every morphism in C/ρ_N has an inverse. For, if $[b] \in \operatorname{Mor}([u],[v])$, then $b \in \operatorname{Mor}(u',v')$ for some $u' \in [u], v' \in [v]$ and by Lemma 1.3, there is a morphism $c \in \operatorname{Mor}(v',u')$ with $b+c, c+b \in \operatorname{Mor} N$. It follows that $[b]+[c]$ and $[c]+[b]$ are the identities at $[u]$ and $[v]$ respectively and consequently, C/ρ_N is a groupoid as required.

Finally, we define a functor $\theta : C \to C/\rho_N$ by putting $x\theta = [x]$ for any object or morphism x of C. It is easy to verify that θ is a functor, and it is obvious that θ is surjective on the set of objects.

Let $a \in \operatorname{Mor}(u,v)$ and let u' and v' be objects such that $u\,\rho_N\,u'$ and $v\,\rho_N\,v'$. To show that θ is surjective on morphism sets, we need to find a morphism $a' \in \operatorname{Mor}(u',v')$ such that $a\theta = a'\theta$. By definition, there exist $r \in \operatorname{Mor} N \cap \operatorname{Mor}(u',u)$ and $s \in \operatorname{Mor} N \cap \operatorname{Mor}(v,v')$. Now $a' = r+a+s \in \operatorname{Mor}(u',v')$ and, by Lemma 2.2, $a'\,\rho_N\,a$, that is, $a'\theta = a\theta$. Thus θ is a quotient map.

If $a \in \operatorname{Mor} N$ and $a \in \operatorname{Mor}(u',u)$, then $u'\,\rho_N\,u$ and we have already seen that $a\theta = [a] = [0_u] = 0_{[u]}$. On the other hand, if a is a morphism in $\operatorname{Ker}\theta$, then $[a] = a\theta = [0_u]$ for some object u of C, that is, $a\,\rho_N\,0_u$, and it is easy to see from the definition of ρ_N and the fact that N is unitary that $a \in \operatorname{Mor} N$. It follows that $N = \operatorname{Ker}\theta$ and this completes the proof of Theorem 2.1. □

There is a corresponding theorem for semigroupoids which we now state, indicating the changes needed in the proof.

Theorem 2.3. *Let N be a subsemigroupoid of a semigroupoid C. Then N is the kernel of a quotient map from C to a groupoid if and only if $\operatorname{Mor}_N(u,u) \neq \emptyset$ for every object u of C and N is dense, unitary and planar.*

Under the assumption that $\operatorname{Mor}_N(u,u) \neq \emptyset$ for every $u \in \operatorname{Obj} C$, the relations ρ_N on $\operatorname{Obj} C$ and $\operatorname{Mor} C$ are still equivalences. To see that ρ_N is reflexive on $\operatorname{Mor} C$ we note that if $a \in \operatorname{Mor} C$, then because N is dense and planar, there is a morphism x such that $a + x \in \operatorname{Mor} N$. Also $a + (x + a) + x \in \operatorname{Mor} N$ so that ρ_N is reflexive.

We note that it follows easily from Lemma 2.2 that for any object $[u]$ of C/N and $a \in \operatorname{Mor}_N(u,u)$, the morphism $[a]$ is the identity morphism at $[u]$.

By specialising Theorem 2.3 to semigroups, and using the semigroup version of Corollary 1.2, we recover Levi's description of morphisms from semigroups onto groups [31, 32].

Corollary 2.4. *Let N be a subsemigroup of a semigroup S. Then N is a dense and planar subsemigroup of S if and only if there is a surjective morphism $\theta\colon S \to G$ onto a group G with $N = 1\theta^{-1}$.*

Returning to categories, we observe that if a quotient map θ is bijective on the set of objects, then $\operatorname{Ker}\theta$ is totally disconnected. Moreover, if a subcategory N of a category C is totally disconnected, dense and planar (and hence also unitary by virtue of Lemma 1.1), then the description of ρ_N can be simplified considerably as we see in the following lemma. First, note that the equivalence on $\operatorname{Obj} C$ is simply the identity relation and so we need consider ρ_N only as an equivalence on $\operatorname{Mor} C$.

Lemma 2.5. *Let N be a dense, planar and totally disconnected subcategory of a category C. For $a, b \in \operatorname{Mor} C$, the following conditions are equivalent:*

(1) *$a \,\rho_N\, b$,*
(2) *there is a morphism x in C such that $a + x, b + x \in \operatorname{Mor} N$,*
(3) *there are morphisms $p, q \in \operatorname{Mor} N$ such that $p + a = b + q$.*

Proof. Suppose that $a \,\rho_N\, b$. Since N is totally disconnected, it follows from the definition of ρ_N that a and b are coterminal. Moreover, there are morphisms r, s in N and x, y in C such that $b + s + x, a + r + y, a + x, b + y \in \operatorname{Mor} N$. But N is planar, $b + x$ is defined and $s, b + s + x \in \operatorname{Mor} N$ so that $b + x \in \operatorname{Mor} N$. Thus (2) holds.

Suppose that $a, b \in \operatorname{Mor} C$ are such that $a + x, b + x \in \operatorname{Mor} N$ for some morphism x. Then, since N is totally disconnected, a, b are coterminal and by Lemma 1.1, $x + a \in \operatorname{Mor} N$. Putting $p = b + x$ and $q = x + a$, we have $p + a = b + q$ so that (3) holds.

Finally, suppose that $a, b \in \operatorname{Mor} C$ are such that $p + a = b + q$ for some p, q in N. Since N is totally disconnected, a and b must be coterminal, say $a, b \in \operatorname{Mor}(u, v)$. Hence $a + q$

is defined, and, by denseness, there is a morphism t such that $a + q + t \in \operatorname{Mor} N$. Put $x = q + t$ and note that

$$b + q + x = p + a + x \in \operatorname{Mor} N$$

so that by planarity, $b + x \in \operatorname{Mor} N$. Now $0_v \in \operatorname{Mor} N$ and we have

$$b + 0_v + x, a + 0_v + x, a + x, b + x \in \operatorname{Mor} N$$

so that $a \, \rho_N \, b$ and (1) holds. $\square$

We remark that items (2) and (3) of the lemma are extensions to categories of the definitions used in the semigroup case by Levi [31, 32] and Gomes [21] respectively.

When N is dense, planar and totally disconnected, it is easy to see, using the fact that N is reflexive, that ρ_N is a congruence in the sense of the following definition [33].

A *congruence* on a category C is an equivalence relation ρ on $\operatorname{Mor} C$ such that

(1) if $a \, \rho \, b$, then a and b are coterminal, and
(2) if $a \, \rho \, b$ and $p, q \in \operatorname{Mor} C$ are such that $p+a$ and $a+q$ are defined, then $(p+a) \; \rho \; (p+b)$ and $(a + q) \; \rho \; (b + q)$.

If ρ is a congruence on a category C, the *quotient category* C/ρ is defined as follows. The objects of C/ρ are the objects of C and if u, v are such objects, then

$$\operatorname{Mor}(u, v) = \{[a] \mid a \in \operatorname{Mor}(u, v)\}$$

where $[a]$ denotes the congruence class of a. Composition is given by the rule that for $[a] \in \operatorname{Mor}(u, v)$ and $[b] \in \operatorname{Mor}(v, w)$,

$$[a] + [b] = [a + b].$$

This composition is well defined and C/ρ is indeed a category. Furthermore, the functor θ from C onto C/ρ, given by $u\theta = u$ for all $u \in \operatorname{Obj} C$ and $a\theta = [a]$ for all $a \in \operatorname{Mor} C$, is a quotient map which is bijective on the set of objects, and called the *natural morphism* from C to C/ρ. The usual isomorphism theorems can be found in [56].

We say that a congruence ρ on a category C is a *groupoid congruence* if C/ρ is a groupoid. In this case, the *kernel* of ρ is the subcategory $\operatorname{Ker} \theta$, where θ is the natural morphism.

Now if N is a dense, planar and totally disconnected subcategory of C, then we can form the quotient category C/ρ_N. We claim that this is consistent with our definition of C/ρ_N in the proof of Theorem 2.1. Certainly, the sets of object are the same and so are the morphism sets. Also, if $[a], [b] \in \operatorname{Mor}(C/\rho_N)$, are such that $\omega([a]) = \alpha([b])$, then $\omega(a) = \alpha(b)$ (because $\operatorname{Obj} C/\rho_N = \operatorname{Obj} C$) and $0_{\omega(a)} \in \operatorname{Mor} N$. It follows that composition is the same in the two categories and that, in fact, they are identical. Thus ρ_N is a groupoid congruence.

We now have the following extension of the results of [31, 32, 21].

Theorem 2.6. *The mappings $N \mapsto \rho_N$ and $\rho \mapsto \operatorname{Ker}\rho$ are mutually inverse order isomorphisms between the set of all dense, planar, totally disconnected subcategories of C and the set all groupoid congruences on C.*

Proof. If N is a dense, planar, totally disconnected subcategory of C, then we have just noted that ρ_N is a groupoid congruence. The fact that $\operatorname{Ker}\rho_N = N$ follows from the proof of Theorem 2.1.

Conversely, if ρ is a groupoid congruence on C, then the natural morphism θ from C to C/ρ is bijective so that $\operatorname{Ker}\rho$ is totally disconnected. As θ is also a quotient map, it follows from Theorem 2.1 that $\operatorname{Ker}\rho$ is dense and planar. We claim that $\rho = \rho_{\operatorname{Ker}\rho}$.

Let a, b be coterminal morphisms in C with $a\,\rho\,b$, and let $z \in \operatorname{Mor} C$ be such that $[z] = -[a]$. Then $[b + z] = [a + z] = [0_{\alpha(a)}]$ so that $b + z, a + z \in \operatorname{Ker}\rho$. Hence by Lemma 2.5, $a\,\rho_{\operatorname{Ker}\rho}\,b$.

Conversely, if $a\,\rho_{\operatorname{Ker}\rho}\,b$, then by Lemma 2.5, $p + a = b + q$ for some $p, q \in \operatorname{Mor}(\operatorname{Ker}\rho)$. Hence $[a] = [p] + [a] = [p + a] = [b + q] = [b] + [q] = [b]$, that is, $a\,\rho\,b$.

Thus the two mappings are mutually inverse. It is straightforward to verify that they are order-preserving. □

Next we show that item (3) of Lemma 2.5 can be used to associate a groupoid congruence on a category C with any strongly dense, totally disconnected subcategory of C. This result extends [21, Lemma 4]. Let T be a strongly dense, totally disconnected subcategory of a category C. Since T is strongly dense, for each $a \in \operatorname{Mor} C$ there is at least one morphism a' such that $a' + p + a$ and $a + q + a'$ are in $\operatorname{Mor} T$ for all morphisms p, q of T such that $p + a$ and $a + q$ are defined. We say that a' is a *weak T-inverse* of a and the set of all weak T-inverses of a is denoted by $W_T(a)$.

We define a relation ρ_T on $\operatorname{Mor} C$ using the rule given in item (3) of Lemma 2.5.

Proposition 2.7. *Let T be a strongly dense, totally disconnected subcategory of a category C. Then the relation ρ_T is a groupoid congruence on C.*

Proof. Since T is strongly dense in C, we have $\operatorname{Obj} T = \operatorname{Obj} C$ so that $0_u \in \operatorname{Mor} T$ for all objects u of C. Hence, if $a \in \operatorname{Mor}(u, v)$, then $0_u + a = a + 0_v$ and so ρ_T is reflexive.

Suppose that $a, b \in \operatorname{Mor}(u, v)$ are such that $a\ \rho_T\ b$ and let $p, q \in \operatorname{Mor} T$ be such that $p + a = b + q$ so that $p \in \operatorname{Mor}(u, u)$, $q \in \operatorname{Mor}(v, v)$. Now let $a' \in W_T(a)$ and $b' \in W_T(b)$. Then

$$a + (a' + p + a) + (b' + b) = (a + a') + (b + q + b') + b.$$

Since a' and b' are weak T-inverses of a and b respectively, we have that the morphisms $(a' + p + a) + (b' + b)$ and $(a + a') + (b + q + b')$ are in $\operatorname{Mor} T$. Thus $y \ \rho_T \ x$ and ρ_T is symmetric.

If $a, b, c \in \operatorname{Mor}(u, v)$ and $p, q, r, s \in \operatorname{Mor} T$ are such that $p + a = b + q$ and $r + b = c + s$, then

$$(r + p) + a = r + (p + a) = r + (b + q) = (r + b) + q = (c + s) + q = c + (s + q)$$

and $r + p, s + q \in \operatorname{Mor} T$ so that $a \ \rho_T \ c$ and ρ_T is transitive.

Now we show that ρ_T is right compatible. Suppose that $a, b, c \in \operatorname{Mor} C$ are such that $a \ \rho_T \ b$ and $a+c, b+c$ are defined. Then there are morphisms p, q in T such that $p+a = b+q$. Let $b' \in W_T(b)$ and $c' \in W_T(c)$. Then

$$(b + c + c' + b' + p) + (a + c) = (b + c) + (c' + b' + b + q + c).$$

Since b' and c' are weak T-inverses of b and c respectively, it follows that ρ_T is right compatible and a similar argument shows that it is also left compatible.

Finally, to see that C/ρ_T is a groupoid, let $a \in \operatorname{Mor}(u, v)$ and $a' \in W_T(x)$. Then $a + a', a' + a \in \operatorname{Mor} T$. Now for any $p \in \operatorname{Mor}_T(u, u)$ we have $p \ \rho_T \ 0_u$ so that $[p]$ is the identity of $\operatorname{Mor}_{C/\rho_T}(u, u)$. Hence every morphism of C/ρ_T has an inverse. □

Corollary 2.8. *Let T be a strongly dense submonoid of a monoid M. Then the relation ρ_T on M, defined by the rule that $a \ \rho_T \ b$ if and only if $ta = bs$ for some $s, t \in T$, is a group congruence on M.*

Let T be a strongly dense, totally disconnected subcategory of a category C. Noting that a non-empty intersection of planar subcategories of a category is again planar, we let T_∞ be the least planar subcategory of C containing T. Since $\delta(C)$ is planar and contains T, the subcategory T_∞ is totally disconnected. We now show that the kernel of the groupoid congruence ρ_T is just T_∞.

Proposition 2.9. *Let T be a strongly dense, totally disconnected subcategory of a category C. Then $T_\infty = \operatorname{Ker} \rho_T$ and $\rho_T = \rho_{T_\infty}$.*

Proof. Clearly T is contained in $\operatorname{Ker} \rho_T$ and by Theorem 2.6, $\operatorname{Ker} \rho_T$ is planar and totally disconnected. Thus, by definition, T_∞ is contained in $\operatorname{Ker} \rho_T$.

If $a \in \operatorname{Ker} \rho_T$, then $a \ \rho_T \ 0_u$ for some object u and so there are morphisms p, q in $\operatorname{Mor} T$ such that $a + p = q + 0_u = q \in \operatorname{Mor} T$. Hence p, $a + p \in \operatorname{Mor} T_\infty$ and so $a \in \operatorname{Mor} T_\infty$ since, by Lemma 1.1, T_∞ is unitary in C. Thus $\operatorname{Ker} \rho_T = T_\infty$ and hence by Theorem 2.6, $\rho_T = \rho_{T_\infty}$. □

Remark. It may be useful, especially for computations, to note the following construction ("from below") of T_∞. If T is a totally disconnected subcategory of a category C, let

$$\begin{aligned} u(T) &= \{a \in \operatorname{Mor}\delta(C) \mid \ a+b \text{ or } b+a \text{ lies in } \operatorname{Mor} T \text{ for some } b \in \operatorname{Mor} T\ \}, \\ r(T) &= \{a \in \operatorname{Mor}\delta(C) \mid a = b+c \text{ for some } b, c \in \operatorname{Mor} C \\ &\qquad\qquad\qquad\qquad\qquad \text{such that } c+b \in \operatorname{Mor} T\}. \end{aligned}$$

Let $T_0 = T$, and for $k \geq 0$, let $T_{2k+1} = \langle u(T_{2k})\rangle$ and let $T_{2k+2} = \langle r(T_{2k+1})\rangle$ (where $\langle X\rangle$ denotes the subcategory generated by X). Since a totally disconnected subcategory is planar if and only if it is reflexive and unitary, it is not difficult to verify that $T_\infty = \bigcup_k T_k$.

Remark. The corresponding semigroupoid results for congruences are obtained under the blanket assumption that the subsemigroupoids T satisfy $\operatorname{Mor}_T(u,u) \neq \emptyset$ for each object u of the semigroupoid C.

To show that ρ_T is reflexive when T is a strongly dense subsemigroupoid, let $a \in \operatorname{Mor}_C(u,v)$, $t \in \operatorname{Mor}_T(v,v)$ and note that since T is strongly dense, there is a morphism $a' \in \operatorname{Mor}_C(v,u)$ such that $a+t+a', t+a'+a \in \operatorname{Mor} T$. Now $(a+t+a')+a = a+(t+a'+a)$ so that ρ_T is reflexive.

It is equally straightforward to show that for any $p \in \operatorname{Mor}_T(u,u)$, the morphism $[p]$ is the identity morphism at u in $C/\,\rho_T$.

3. Isotropic group actions

An action of a group G on a category C is given by a group morphism from G into the automorphism group of C. We want to consider left actions and so we assume that the automorphisms of C act from the left. When we have such an action we write gx for the result of the action of a group element g on an object or morphism x. We note that $\operatorname{Obj} C$ and $\operatorname{Mor} C$ are G-sets (sets on which G acts) and that the following identities hold:

(1) $g(p+q) = gp+gq$ for all $g \in G, u, v, w \in \operatorname{Obj} C, p \in \operatorname{Mor}(u,v)$ and $q \in \operatorname{Mor}(v,w)$,
(2) $g0_u = 0_{gu}$ for all $g \in G$ and $u \in \operatorname{Obj} C$.

We denote the *stabiliser* of an element x of a G-set X by $\operatorname{Stab}(x)$, that is,

$$\operatorname{Stab}(x) = \{g \in G \mid gx = x\}.$$

We say that G acts *isotropically* on X, or that the action of G on X is *isotropic*, if $\operatorname{Stab}(x) = \operatorname{Stab}(y)$ for all $x, y \in X$. Recall that in the special case where $\operatorname{Stab}(x) = \{1\}$ for all $x \in X$, G is said to act *freely* or *without fixed points* and X is said to be a *free G-set*. When G acts on a category C we say that the action is *isotropic* if G acts isotropically on $\operatorname{Mor} C$. It is easy to see that the action on $\operatorname{Obj} C$ is also isotropic. Indeed, it follows from property (2) above that $\mathrm{Stab}(u) = \mathrm{Stab}(p)$ for all objects u and morphisms p. If $\mathrm{Obj}\, C$ is a free G-set, then, clearly, G also acts freely on $\mathrm{Mor}\, C$ and we say that G acts *freely* on C.

Lemma 3.1. *Let G be a group acting isotropically on a category C. For every morphism $p \in \mathrm{Mor}\, C$ and every $u \in G\alpha(p)$ (resp. $u \in G\omega(p)$), there is exactly one element of Gp with domain (resp. codomain) u.*

Proof. Let $g \in G$ be such that $u = g\alpha(p)$. Then $gp \in Gp$ and $\alpha(gp) = u$. Let hp be an element of Gp with domain u. Then $u = h\alpha(p) = g\alpha(p)$, so $h^{-1}g\alpha(p) = \alpha(p)$. It follows that $h^{-1}g \in \mathrm{Stab}(0_{\alpha(p)})$. Since the action of G is isotropic, $h^{-1}g \in \mathrm{Stab}(p)$, and hence $hp = gp$. $\square$

Let G be a group acting on a category C. We use the orbits of the action to form a new category C/G. We define $\mathrm{Obj}(C/G)$ and the morphism sets as follows:

$$\mathrm{Obj}(C/G) = \{Gu \mid u \in \mathrm{Obj}\, C\},$$
$$\mathrm{Mor}(Gu, Gv) = \{Gp \mid p \in \mathrm{Mor}(u', v') \text{ for some } u' \in Gu, v' \in Gv\}.$$

The proposed law of composition is given by the rule that $Gp + Gq = G(p' + q')$ for any $p' \in Gp$, $q' \in Gq$ such that $p' + q'$ is defined. Unfortunately, this is not always well defined as the next example shows. However, we show in Proposition 3.2 that if the group action is isotropic, then we do get a category.

Example. Let $G = S_3$ be the symmetric group of degree 3 and let $\rho = (123)$ and σ_i denote the transposition which fixes i. We define a category C with $\mathrm{Obj}\, C = \{1, 2, 3\}$ and $\mathrm{Mor}(i, j) = \{(i, \lambda, j) \mid \lambda \in S_3, i\lambda = j\}$ for $i, j \in \mathrm{Obj}\, C$. The law of composition is given by

$$(i, \lambda, j) + (j, \mu, k) = (i, \lambda\mu, k).$$

Clearly, C is a category and we make G act on C as follows. First, the action of G on $\mathrm{Obj}\, C$ is given by $\sigma i = i\sigma^{-1}$ for all $\sigma \in G$ and $i \in \{1, 2, 3\}$. Next, the action of G on $\mathrm{Mor}\, C$ is given by the rule that $\sigma(i, \lambda, j) = (i\sigma^{-1}, \sigma\lambda\sigma^{-1}, j\sigma^{-1})$ for all $\sigma \in G$ and $(i, \lambda, j) \in \mathrm{Mor}\, C$.

There is only one orbit of objects but we have four orbits of morphisms. Note that $(1, \sigma_3, 2), (2, \sigma_3, 1), (2, \sigma_1, 3)$ are all in the same orbit since $(2, \sigma_3, 1) = \sigma_3(1, \sigma_3, 2)$ and $(2, \sigma_1, 3) = \rho^2(1, \sigma_3, 2)$. However,

$$(1, \sigma_3, 2) + (2, \sigma_3, 1) = (1, \sigma_3^2, 1) = (1, 1, 1)$$

and

$$(1, \sigma_3, 2) + (2, \sigma_1, 3) = (1, \sigma_3\sigma_1, 3) = (1, \rho^2, 3).$$

Clearly, $(1, 1, 1), (1, \rho^2, 3)$ are not in the same orbit so that composition is not well defined in this case.

Proposition 3.2. *Let G be a group which acts isotropically on a category C. Then with the law of composition defined above, C/G is a category. Furthermore, if C is a groupoid, then so is C/G.*

Proof. It is clear that C/G is a category if the law of composition is well defined. To see that this is the case, let $Gp \in \mathrm{Mor}(Gu, Gv)$, $Gq \in \mathrm{Mor}(Gv, Gw)$ and suppose that $p_1, p_2 \in Gp, q_1, q_2 \in Gq$ are such that $p_1 + q_1$ and $p_2 + q_2$ are defined. Then $p_2 = gp_1$ for some $g \in G$. Next q_2 and gq_1 are two elements of Gq with the same domain, so that by Lemma 3.1, $q_2 = gq_1$. Therefore $p_2 + q_2 = gp_1 + gq_1 = g(p_1 + q_1)$ and composition is well defined. $\square$

We now comment on equivariant versions of some of the results in Section 2. If G is a group which acts on a category C, we will say that C is a *G-category*. If C and D are G-categories, then a morphism $\theta\colon C \to D$ is *equivariant* or is a *G-morphism* if $(gx)\theta = g(x\theta)$ for all objects and morphisms x of C and elements g of G. We say that a subcategory N of C is a *G-subcategory* if the action of G on C restricts to an action of G on N. A congruence ρ on C is a *G-congruence* if $gp \;\rho\; gq$ whenever $p \;\rho\; q$ for any morphisms p, q of C. When we have a G-congruence ρ on C, the category C/ρ can be made into a G-category in the obvious way and then the natural morphism $C \to C/\rho$ is a G-morphism.

If θ is an equivariant quotient map from a G-category C to a G-groupoid A, then it is easy to see that $\mathrm{Ker}\,\theta$ is a G-subcategory of C. Also the relations ρ_N on $\mathrm{Obj}\,C$ and $\mathrm{Mor}\,C$ defined in Section 2 satisfy:

$$u \,\rho_N\, v \text{ implies } gu \,\rho_N\, gv \quad \text{and} \quad x \,\rho_N\, y \text{ implies } gx \,\rho_N\, gy$$

for all objects u, v of C and all morphisms x, y of C. It follows that the groupoid C/N becomes a G-groupoid if we define an action of G by $g[x] = [gx]$ for objects and morphisms $[x]$ of C/N. Thus we have an equivariant version of Theorem 2.1.

When T is a strongly dense, totally disconnected G-subcategory of a G-category C, it is clear that the congruence ρ_T defined in Section 2 is a G-congruence. It is then easy to verify that we have equivariant versions of all the results of Section 2.

If a group G acts isotropically on a category C, then there is an obvious surjective morphism of categories $\pi_C\colon C \to C/G$ which sends objects and morphisms to their G-orbits. Furthermore, π_C is equivariant if we make C/G into a G-category by letting G act trivially. In fact, π_C is a natural morphism. This is a consequence of the next theorem, which underlies much of the rest of the paper.

If $\gamma\colon G \to H$ is a morphism of groups and if H acts on a category D, then there is an induced action of G on D given by $gy = (g\gamma)y$ for all $g \in G$ and objects and morphisms y of

D. In this situation, a morphism $\varphi\colon C \to D$ is said to be *equivariant* if $(gx)\varphi = (g\gamma)(x\varphi)$ for all $g \in G$ and all objects and morphisms x of C.

Theorem 3.3. *Let $\gamma\colon G \to H$ be a morphism of groups. Let G and H act isotropically on categories C and D respectively and let $\varphi\colon C \to D$ be an equivariant morphism. Then there is a unique morphism $\psi\colon C/G \to D/H$ such that the square*

$$\begin{tikzcd}
C \arrow[r, "\varphi"] \arrow[d, "\pi_C"'] & D \arrow[d, "\pi_D"] \\
C/G \arrow[r, "\psi"'] & D/H
\end{tikzcd}$$

is commutative. Furthermore, if φ is surjective, then so is ψ.

Proof. Clearly, ψ is unique if it exists. We define ψ by putting $(Gx)\psi = H(x\varphi)$ if x is an object or a morphism of C. If x, y are both in $\operatorname{Obj} C$ or both in $\operatorname{Mor} C$, and if $Gx = Gy$, then $x = gy$ for some $g \in G$. Hence $x\varphi = (gy)\varphi = (g\gamma)(y\varphi)$ since φ is equivariant. Consequently, $H(x\varphi) = H(y\varphi)$ and ψ is well defined.

It is clear that the square is commutative and that ψ is surjective if φ is. It remains to be shown that ψ is a morphism.

If Gu, Gv are objects of C/G and $Gp \in \operatorname{Mor}(Gu, Gv)$, then $p \in \operatorname{Mor}(u', v')$ for some $u' \in Gu, v' \in Gv$. Now, $u' = au, v' = bv$ for some $a, b \in G$ and φ is equivariant so that $u'\varphi = (a\gamma)(u\varphi)$ and $v'\varphi = (b\gamma)(v\varphi)$ giving $H(u'\varphi) = H(u\varphi)$ and $H(v'\varphi) = H(v\varphi)$. Since $p\varphi \in \operatorname{Mor}(u'\varphi, v'\varphi)$ we have $H(p\varphi) \in \operatorname{Mor}(H(u'\varphi), H(v'\varphi))$ and so, by the commutativity of the square, $(Gp)\psi \in \operatorname{Mor}((Gu)\psi, (Gv)\psi)$.

Also, given morphisms Gp, Gq of C/G with $Gp + Gq$ defined, there are morphisms $p' \in Gp, q' \in Gq$ such that

$$\begin{aligned}
((Gp) + (Gq))\psi = (G(p' + q'))\psi = H((p' + q')\varphi) = H(p'\varphi + q'\varphi) \\
= H(p'\varphi) + H(q'\varphi) = H(p\varphi) + H(q\varphi) = (Gp)\psi + (Gq)\psi.
\end{aligned}$$

Thus ψ is a morphism. ☐

When $G = H$ and γ is the identity map we obtain the naturality of π_C. This particular case of Theorem 3.3 is the only one we use in this section and so for clarity we give the statement as a corollary.

Corollary 3.4. *Let G be a group acting isotropically on categories C and D and let $\varphi\colon C \to D$ be an equivariant morphism. Then there is a unique morphism $\psi\colon C/G \to D/G$ such* *that the square*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\pi_C"'] & D \arrow[d, "\pi_D"] \\ C/G \arrow[r, "\psi"'] & D/G \end{tikzcd}$$

is commutative. Furthermore, if φ is surjective, then so is ψ.

Let G be a group acting isotropically on a category C. We say that the action is *transitive* if $\operatorname{Obj} C$ is a transitive G-set. In this case the category C/G has only one object and we may regard it as a monoid (or a group if C is a groupoid). We note that G also acts isotropically and transitively on $\delta(C)$ so that we have a submonoid $\delta(C)/G$ of C/G. More generally, if $T \subseteq \delta(C)$ is a G-subcategory, then T/G is a submonoid of C/G. We look at the relationship between such a T and T/G in the next proposition.

Proposition 3.5. *Let G be a group acting isotropically on a category C and let T be a totally disconnected G-subcategory of C. Then T is planar in C if and only if T/G is planar in C/G. Also, T is dense in C if and only if T/G is dense in C/G.*

Proof. First we observe that, because T is a G-subcategory, the class Gq is in $\operatorname{Mor} T/G$ if and only if $q \in \operatorname{Mor} T$. With this observation, it is immediate that if T/G is planar, then T is planar.

Conversely, suppose that T is planar and let $p, q, r \in \operatorname{Mor} C$ be such that $Gp + Gq + Gr$ and $Gp + Gr$ are defined in C/G. By Lemma 3.1, there are uniquely determined elements q' of Gq and r' and r'' of Gr such that $\alpha(q') = \omega(p)$, $\alpha(r') = \omega(q')$ and $\alpha(r'') = \omega(p)$.

$$\begin{tikzcd} \bigcirc \arrow[r, "p"] & \bigcirc \arrow[r, "q'"] \arrow[d, "r''"] & \bigcirc \arrow[r, "r'"] & \bigcirc \\ & \bigcirc & & \end{tikzcd}$$

If $Gq \in \operatorname{Mor} T/G$, then q' is a morphism of T and hence of $\delta(C)$. Thus $\alpha(r') = \alpha(r'')$, and so $r' = r''$ by Lemma 3.1. If $Gp+Gq+Gr, Gp+Gr \in \operatorname{Mor} T/G$, then $p+q'+r', p+r'' \in \operatorname{Mor} T$, so that $\omega(r') = \omega(r'')$ and hence $r' = r''$ by Lemma 3.1. Thus, if two of $Gp+Gq+Gr$, Gq, $Gp+Gr$ lie in $\operatorname{Mor} T/G$, then $r' = r''$ and two of $p+q'+r'$, q', $p+r'$ lie in $\operatorname{Mor} T$. Since T is planar, all three lie in $\operatorname{Mor} T$ and the planarity of T/G follows.

The proof of the equivalence of the denseness of T and T/G is obtained in a similar manner. $\square$

The first part of the following corollary is immediate and the second part follows since $\delta(C)$ is dense in C (in fact, strongly dense) if C is strongly connected.

Corollary 3.6. *Let G be a group acting isotropically and transitively on a category C. Then $\delta(C)/G$ is a planar submonoid of the monoid C/G. If C is strongly connected, then $\delta(C)/G$ is dense.*

If C is a strongly connected G-category with G acting freely and transitively, then $\delta(C)$ is a G-subcategory which is strongly dense in C. Consequently, $\rho = \rho_{\delta(C)}$ is a groupoid G-congruence on C. The natural morphism $\varphi\colon C \to C/\rho$ is a G-morphism, where C/ρ has the induced G-action (which is isotropic). It is clear from the definition of ρ that for $p, q \in \operatorname{Mor} C$ we have $p \;\rho\; q$ if and only if p, q are coterminal. Now C is strongly connected and hence so is C/ρ. It follows that there is exactly one morphism in each set $\operatorname{Mor}_{(C/\rho)}(u, v)$ for objects u, v of C/ρ, that is, C/ρ is a simplicial groupoid. Since C/ρ is a groupoid, the monoid $(C/\rho)/G$ is a group and we have the following special case of Corollary 3.4 where we write π_ρ for $\pi_{C/\rho}$.

Corollary 3.7. *Let G be a group acting isotropically and transitively on a strongly connected category C. Then there is a unique surjective morphism ψ from C/G onto $(C/\rho)/G$ such that the diagram*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\pi_C"'] & C/\rho \arrow[d, "\pi_\rho"] \\ C/G \arrow[r, "\psi"'] & (C/\rho)/G \end{tikzcd}$$

is commutative and $1\psi^{-1} = \delta(C)/G$.

We now consider the universal nature of the monoid C/G where G is a group acting isotropically and transitively on a category C. Let M be a monoid with trivial G-action and let $\eta\colon C \to M$ be a surjective G-morphism. We say that M is a *universal monoid for C with universal map η* if for any monoid N with trivial G-action and any surjective G-morphism $\varphi\colon C \to N$, there is a unique morphism $\psi\colon M \to N$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\eta"'] & N \\ M \arrow[ur, "\psi"'] & \end{tikzcd}$$

is commutative.

Corollary 3.8. *Let G be a group acting isotropically and transitively on a category C. Then C/G is a universal monoid for C with universal map π_C.*

Proof. If $\varphi\colon C \to N$ is any surjective morphism onto a monoid N with trivial G-action, then, by Corollary 3.4, there is a unique morphism $\psi\colon C/G \to N/G$ such that the square

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & N \\ {\scriptstyle\pi_C}\big\downarrow & & \big\downarrow{\scriptstyle\pi_N} \\ C/G & \xrightarrow[\psi]{} & N/G \end{array}$$

is commutative. Since G acts trivially on N, the monoid N/G is just N and the morphism π_N is the identity map, whence the result. ☐

The next lemma follows in the usual way from commuting diagram arguments.

Lemma 3.9. *Let G be a group acting isotropically and transitively on a strongly connected category C. If M and N are universal monoids for C with universal maps η and θ respectively, then there is an isomorphism $\mu\colon M \to N$ with $\eta\mu = \theta$.*

Remark. We say that a group action on a semigroupoid is isotropic if the stabilisers of all objects and morphisms coincide. Similarly, for the action to be free, it must be free on both the set of objects and the set of morphisms.

For the semigroupoid versions of the results of this section we insist that all local semigroups of the subsemigroupoids are non-empty.

4. Free actions

In this section we specialise to the case of a group acting freely on a category and use this notion to describe extensions of monoids by groups. We begin by recalling a construction due to Margolis and Pin. In [35], they show how to "coordinatise" the monoid C/G when G is a group acting freely and transitively on a category C. We now describe this procedure. Let u be any object of C and let

$$C_u = \{(p, g) \mid g \in G, p \in \mathrm{Mor}(u, gu)\}.$$

Then C_u is a monoid under the multiplication defined by $(p, g)(q, h) = (p + gq, gh)$, and we have the following result from [35].

Proposition 4.1. *Let G be a group acting freely and transitively on a category C. Then, for all $u \in \mathrm{Obj}\, C$, the map $\zeta_u\colon C_u \to C/G$ given by $(p, g)\zeta_u = Gp$ is an isomorphism from the monoid C_u onto C/G.*

We note that

$$L_u = \{(p, 1) \mid p \in \mathrm{Mor}(u, u)\}$$

is a submonoid of C_u and that there is an obvious isomorphism $\lambda_u\colon \mathrm{Mor}(u, u) \to L_u$ given by $p\lambda_u = (p, 1)$. Now,

$$\delta(C)/G = \{Gp \mid p \in \delta(C)\} = \{Gp \mid p \in \mathrm{Mor}(u, u)\}$$

by Lemma 3.1 so that composing λ_u with the restriction of ζ_u to L_u gives an isomorphism δ_u from the local monoid $\mathrm{Mor}(u, u)$ onto the submonoid $\delta(C)/G$ of C/G. We record these observations in the following lemma.

Lemma 4.2. *Let G be a group acting freely and transitively on a category C. Then, for all $u \in \mathrm{Obj}\, C$, the monoids $\mathrm{Mor}(u, u)$, L_u and $\delta(C)/G$ are isomorphic via the isomorphisms $\lambda_u\colon \mathrm{Mor}(u, u) \to L_u$ and $\delta_u\colon \mathrm{Mor}(u, u) \to \delta(C)/G$.*

Next we give a "coordinate" version of Corollary 3.7 in the special case where the category C is strongly connected and the action is free.

Proposition 4.3. *Let G be a group acting freely and transitively on a strongly connected category C and let $\rho = \rho_{\delta(C)}$. Then, for each object u of C, there are surjective morphisms $\psi_u\colon C/G \to G$ and $\theta_u\colon C/\rho \to G$ such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C/\rho \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\theta_u} \\ C/G & \xrightarrow[\psi_u]{} & G \end{array}$$

is commutative.

Proof. Let $u \in \mathrm{Obj}\, C$. In view of Corollary 3.7, it is enough to find an isomorphism ξ_u from $(C/G)/\rho$ onto G and put $\psi_u = \psi\xi_u$ and $\theta_u = \pi_\rho\xi_u$. In fact, we define ψ_u and then obtain ξ_u as an induced mapping.

Since C is strongly connected, $\mathrm{Mor}(u, gu) \neq \emptyset$ for all $g \in G$ so that the map $\pi_u : C_u \to G$ defined by $(p, g)\pi_u = g$ is a surjective morphism. Hence by Proposition 4.1, we have a surjective morphism $\psi_u = \zeta_u^{-1}\pi_u$ from C/G onto G.

Note that for $p \in \mathrm{Mor}\, C$ there is, by Lemma 3.1, a unique morphism $p' \in Gp$ such that $\alpha(p') = u$ and further, that $(Gp)\psi_u = (Gp)\zeta_u^{-1}\pi_u = g$ where g is the unique element of G such that $p' \in \mathrm{Mor}(u, gu)$.

Next, we observe that $p\,\rho\,p'$ so that p and p' are coterminal, and hence $(Gp)\psi_u = (Gp')\psi_u$. Consequently, putting $G(p\rho)\xi_u = (Gp)\psi_u$, where $p\rho$ is the ρ-class of p, yields a well-defined mapping from $(C/\rho)/G$ into G, which is an isomorphism. As $\psi\xi_u = \psi_u$, this completes the proof. $\square$

Remark. In the context of the proposition we note that if u, v are objects of C, then $v = hu$ for some $h \in G$ and $\psi_v = \psi_u\theta_h$ where θ_h is the inner automorphism of G determined by h.

The *weak derived category* $C(\varphi)$ of a surjective morphism $\varphi\colon M \to N$ from a monoid M onto a monoid N has N as its object set and for all $n_1, n_2 \in N$, the set of morphisms $\mathrm{Mor}(n_1, n_2)$ is given by

$$\mathrm{Mor}(n_1, n_2) = \{(n_1, m, n_2) \in N \times M \times N \mid n_1(m\varphi) = n_2\}.$$

Composition is given by

$$(n_1, m, n_2) + (n_2, m', n_3) = (n_1, mm', n_3).$$

The *derived category* of φ as defined in [56] is a quotient of $C(\varphi)$ but we do not need this concept. When $N = G$ is a group we have an action of G on $C(\varphi)$ given by multiplication on the objects and by putting $a(g, m, h) = (ag, m, ah)$ for $a, g, h \in G, m \in M$. In this case it is clear that $C(\varphi)$ is a strongly connected category and that G acts freely and transitively on $C(\varphi)$. We have the following result from [35, Proposition 3.11 and the proof of Proposition 3.12].

Proposition 4.4. *Let $\varphi\colon M \to G$ be a surjective morphism onto a group G. Then there is an isomorphism $\psi\colon M \to C(\varphi)/G$ given by $m\psi = G(g, m, g(m\varphi))$.*

We are now in a position to give the first main theorem of the paper. It offers descriptions of extensions of a monoid T by a group G and gives some understanding of the structure of such an extension in terms of T and G

Theorem 4.5. *Let T be a submonoid of a monoid M. Then the following conditions are equivalent:*

1. *T is a dense, planar submonoid of M,*
2. *there is a surjective morphism $\varphi\colon M \to G$ from M onto a group G with $T = 1\varphi^{-1}$,*
3. *there is a group G acting freely and transitively on a strongly connected category C such that M is a universal monoid for C with universal map $\eta\colon C \to M$ mapping $\delta(C)$ onto T.*
4. *there is a group G acting freely and transitively on a strongly connected category C and an isomorphism from M onto C/G which maps T onto $\delta(C)/G$,*
5. *there is a group G acting freely and transitively on a strongly connected category C such that for any object u of C, there is an isomorphism from M onto C_u which maps T onto L_u.*

Proof. The equivalence of (1) and (2) is Levi's result given in Corollary 2.4.

Suppose that (2) holds. Then G acts freely and transitively on the strongly connected category $C(\varphi)$. By Proposition 4.4, there is an isomorphism $\psi\colon M \to C(\varphi)/G$ given by $m\psi = G(g, m, g(m\varphi))$. Since ψ is an isomorphism, it follows from Corollary 3.8 that M is universal for $C(\varphi)$ with universal map $\eta = \pi_{C(\varphi)}\psi^{-1}$. Now, $(g, m, g(m\varphi))\pi_{C(\varphi)}\psi^{-1} = G(g, m, g(m\varphi))\psi^{-1} = m$. Moreover, $(g, m, g(m\varphi))$ is in $\operatorname{Mor}\delta(C)$ if and only if $m\varphi = 1$, that is, if and only if $m \in T$. It is now clear that $\eta = \pi_{C(\varphi)}\psi^{-1}$ maps $\delta(C)$ onto T. Thus (2) implies (3).

If (3) holds, then by Corollary 3.8 and Lemma 3.9, there is an isomorphism $\mu\colon M \to C/G$ such that the triangle

$$\begin{tikzcd} & C \arrow[dl, "\eta"'] \arrow[dr, "\pi_C"] & \\ M \arrow[rr, "\mu"'] & & C/G \end{tikzcd}$$

is commutative. Now $(\delta(C))\eta = T$ so that $T\mu = (\delta(C))\eta\mu = (\delta(C))\pi_C = \delta(C)/G$.

If (4) holds, then, putting $\psi_u = \psi\zeta_u$, condition (5) follows from Proposition 4.1 and Lemma 4.2.

Finally, suppose that condition (5) holds. Define $\varphi\colon M \to G$ to be the composite of ψ_u and the projection of C_u onto G which sends (p, g) to g. It is now easy to see that (2) holds. $\square$

Remark. For the semigroupoid versions of the results of this section we insist that all local semigroups of the subsemigroupoids are non-empty.

5. Covers

We remind the reader that a T-*cover* of a monoid M with submonoid T is a monoid $\widehat{M}$ with a dense, planar submonoid $\widehat{T}$ and a surjective morphism $\theta\colon \widehat{M} \to M$ onto M such that the restriction of θ to $\widehat{T}$ is an isomorphism from $\widehat{T}$ onto T. Recall that, by Lemma 1.1, the submonoid $\widehat{T}$ is also unitary. The main result of this section is the following theorem which ensures the existence of a T-cover when T is a strongly dense submonoid of a monoid M.

Theorem 5.1. *If T is a strongly dense submonoid of a monoid M, then M has a T-cover.*

The proof is derived from [18] where it is shown that every E-dense semigroup S in which $E(S)$ is a semilattice has an $E(S)$-cover $\widehat{S}$ which is an E-unitary, E-dense semigroup with $E(\widehat{S})$ a semilattice. Variations of this proof were used in [1] and [19] to obtain more general results. We start by giving a sufficient condition for the existence of a T-cover in terms of a covering category. Let G be a group acting freely and transitively on a category C. Let M be a monoid and regard G as acting trivially on M. A G-morphism $\varphi\colon C \to M$ is a *G-covering* if φ is surjective and locally injective, that is, injective on each local monoid of C. We remark that if $\varphi\colon C \to M$ is a G-covering, then it is a covering in the sense of [35] in that for any object u of C, φ maps $\operatorname{Mor}(u, C)$ bijectively onto $\operatorname{Mor}(u\varphi, M)$ which, of course, is just M.

Proposition 5.2. *Let T be a submonoid of a monoid M and let $\varphi\colon C \to M$ be a G-covering. If $\delta(C)\varphi = T$, then the monoid C/G is a T-cover of M with $\widehat{T} = \delta(C)/G$.*

Proof. By Corollary 3.8, the monoid C/G is universal for C with universal map π_C. Hence, there is a morphism $\psi\colon C/G \to M$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\pi_C"'] & M \\ C/G \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. Clearly, ψ is surjective and maps $\delta(C)/G$ onto T. The submonoid $\delta(C)/G$ is dense and planar in C/G by Corollary 3.6 since C is strongly connected.

Suppose that $x, y \in \operatorname{Mor}\delta(C)$ and $(Gx)\psi = (Gy)\psi$. Let $u = \alpha(x)$. By Lemma 3.1 there exists $y' \in Gy$ such that $\alpha(y') = u$. But $\delta(C)$ is a G-subcategory since the action of G is free, so $y' \in \operatorname{Mor}\delta(C)$ and $x, y' \in \operatorname{Mor}(u, u)$. Now $x\varphi = y'\varphi$ and φ is locally injective, so $x = y'$, and hence $Gx = Gy$. Thus ψ is injective on $\delta(C)/G$. $\square$

To find a G-covering of a monoid M with strongly dense submonoid T, we have to find an appropriate group G and G-category C. In the following lemma we show that if there is a group G and a surjective relational morphism from M to G, then we can find a G-covering.

Lemma 5.3. *Let T be a strongly dense submonoid of a monoid M, and let $\tau : M \to G$ be a surjective relational morphism to a group G such that $T = 1\tau^{-1}$. Then there is a G-covering $\varphi : C \to M$ with $\delta(C)\varphi = T$.*

Proof. For each element g of G, put $M_g = g\tau^{-1} = \{m \in M \mid g \in m\tau\}$. We now define a category C as follows. First, $\operatorname{Obj} C = G$. Next, for all $g, h \in G$, let

$$\operatorname{Mor}(g, h) = \{(g, m, h) \in G \times M \times G \mid m \in M_{g^{-1}h}\},$$

with composition given by

$$(g, m, h) + (h, n, k) = (g, mn, k).$$

To see that the composition of (g, m, h) and (h, n, k) is actually a morphism, note that $m \in M_{g^{-1}h}$ and $n \in M_{h^{-1}k}$, that is, $g^{-1}h \in m\tau$ and $h^{-1}k \in n\tau$ so that

$$g^{-1}k = g^{-1}hh^{-1}k \in m\tau n\tau \subseteq (mn)\tau.$$

Hence $mn \in M_{g^{-1}k}$ and so $(g, mn, k) \in \mathrm{Mor}(g, k)$ as required. Clearly, the composition is associative and $0_g = (g, 1, g)$. Thus C is a category.

Next we observe that the category C is strongly connected since τ is surjective and so $M_g \neq \emptyset$ for all $g \in G$.

We make G act on C as follows. The multiplication in G gives an action of G on the objects of C, and for g in G and (h, m, k) in $\mathrm{Mor}(h, k)$ we put $g(h, m, k) = (gh, m, gk)$. Certainly, G acts freely and transitively. It follows immediately from the definitions of the action of G on C and composition in C that the map $\varphi\colon \mathrm{Mor}(C) \to M$ given by $(g, m, h)\varphi = m$ determines an equivariant morphism from C onto M. Furthermore,

$$\mathrm{Mor}\,\delta(C) = \bigcup_{g \in G} \{(g, m, g) \mid m \in M_1\}$$

so that $(\mathrm{Mor}\,\delta(C))\varphi = M_1 = T$. Finally, it is clear that φ is locally injective so that $\varphi\colon C \to M$ is a G-covering. $\square$

Corollary 5.4. *Let T be a strongly dense submonoid of a monoid M, and let $\tau : M \to G$ be a surjective relational morphism to a group G such that $T = 1\tau^{-1}$. Then M has a T-cover.*

Proof. By the lemma, there is a G-covering $\varphi\colon C \to M$ with $(\mathrm{Mor}\,\delta(C))\varphi = T$. Hence by Proposition 5.2, M has a T-cover $\widehat{M}$ where $\widehat{M} = C/G$ and $\widehat{T} = \delta(C)/G$. $\square$

In view of the corollary, to prove Theorem 5.1, it suffices to find a group G and a surjective relational morphism $\tau : M \to G$ with $1\tau^{-1} = T$. We take G to be the free group on the set M.

We construct a relational morphism as follows. First, let $a \in M$ and put

$$W_T(a) = \{b \in M \mid bta, atb \in T \text{ for all } t \in T\}.$$

We remark that $W_T(a)$ is not empty for any $a \in M$ since T is strongly dense in M. For each $a \in M$ let $\gamma_T(a)$ be any non-empty subset of $W_T(a)$ and put

$$\mathscr{C} = \{\gamma_T(a) \mid a \in M\}.$$

Any such $\mathscr{C}$ gives rise to a relational morphism $\tau_{\mathscr{C}}$. If we choose different collections $\mathscr{C}$ and $\mathscr{D}$ of nonempty subsets of the sets $W_T(a)$, we get different relational morphisms $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$ but it is not clear whether or not the associated covers of M are non-isomorphic.

Let $\overline{M} = \{\overline{x} \mid x \in M\}$ be a set disjoint from M and such that $x \mapsto \overline{x}$ is a bijection. Let $X = M \cup \overline{M}$ and let X^* be the free monoid on X. For each word w in X^* we define a non-empty subset M_w of M. First, let $M_1 = T$. Here 1 denotes the empty word in X^*, rather than the identity of M. Next, for $a \in M$, let $M_a = TaT$ and $M_{\overline{a}} = T\gamma_T(a)T$. Finally, if $v = x_1 \dots x_n$ where $x_1, \dots, x_n$ are in X, we put $M_v = M_{x_1} \dots M_{x_n}$. Clearly, $M_v M_w = M_{vw}$ for any nonempty words v and w and by the following lemma, the same is true if one of v, w is empty.

Lemma 5.5. *Let $a \in M$. Then*

(1) $M_a T = TM_a = M_a$,
(2) $M_{\overline{a}} T = TM_{\overline{a}}$,
(3) $M_a M_{\overline{a}} \subseteq T$ *and* $M_{\overline{a}} M_a \subseteq T$.

Proof. The first two parts follow easily from the fact that $T^2 = T$.

If $m \in M_a$ and $n \in M_{\overline{a}}$, then $m = xay$ and $n = zbt$ for some $x, y, z, t \in T$ and $b \in \gamma_T(a)$. Now $yz \in T$ so that $ayzb \in T$ since $\gamma_T(a)$ is contained in $W_T(a)$. Consequently, $mn = xayzbt \in T$. Similarly, $nm \in T$ and hence (3) holds. □

Note that we may regard the free group G on M as the quotient of X^* by the congruence generated by the relation $\{x\overline{x} \mid x \in X\}$ where we adopt the convention that $\overline{\overline{a}} = a$ for each $a \in N$. In each congruence class $[w]$ there is a unique *reduced* word $r(w)$ that contains no occurrence of $x\overline{x}$ for any $x \in X$. If $w = ux\overline{x}v$, then by Lemma 5.5, we have

$$M_w = M_u M_x M_{\overline{x}} M_v \subseteq M_u T M_v = M_u M_v = M_{uv}$$

and so an easy induction argument gives that $M_w \subseteq M_{r(w)}$ for any $w \in X^*$. For an element g of G we define M_g to be $M_{r(w)}$ for any w in X^* such that $g = [w]$.

Lemma 5.6. *If $g, h \in G$, then $M_g M_h \subseteq M_{gh}$.*

Proof. Let $v, w \in X^*$ be such that $g = [v]$ and $h = [w]$. Then

$$M_g M_h = M_{r(v)} M_{r(w)} = M_{r(v)r(w)} \subseteq M_{r(r(v)r(w))} = M_{r(vw)} = M_{gh}.$$

□

It follows immediately from the lemma that if, for each $m \in M$, we put

$$m\tau = \{g \in G \mid m \in M_g\},$$

then $\tau : M \to G$ is a relational morphism. Clearly, $1\tau^{-1} = T$, and τ is surjective since each M_g is nonempty. Thus the proof of Theorem5.1 is complete. $\square$

It is sometimes useful to have a coordinate version of the cover and so we record that by Proposition 4.1, C/G is isomorphic to C_1 where

$$C_1 = \{((1, m, g), g) \in \{1\} \times M \times G \mid m \in M_g\},$$

and that under this isomorphism, $\delta(C)/G$ corresponds to $\{((1, m, 1), 1) \mid m \in M_1\}$. The covering morphism $C_1 \to M$ now maps $((1, m, g), g)$ to m.

To conclude this section we point out an interesting property of the above construction of a T-cover by giving an analogue of [1, Proposition 2.7]. Let $\alpha_{M,T} \colon \widehat{M} \to M$ be the T-covering constructed in the proof of Theorem 5.1.

Proposition 5.7. *Let T, U be strongly dense submonoids of the monoids M, N respectively. Let $\varphi \colon M \to N$ be a morphism with $T\varphi \subseteq U$ and $\gamma_T(a)\varphi \subseteq \gamma_U(a\varphi)$ for all $a \in M$. Then there is a morphism $\widehat{\varphi} \colon \widehat{M} \to \widehat{N}$ such that the square*

$$\begin{tikzcd} \widehat{M} \arrow[r, "\widehat{\varphi}"] \arrow[d, "\alpha_{M,T}"'] & \widehat{N} \arrow[d, "\alpha_{N,U}"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

is commutative. If $T\varphi = U$, $\gamma_T(a)\varphi = \gamma_U(a\varphi)$ for all $a \in M$ and φ is surjective, then $\widehat{\varphi}$ is surjective.

Proof. Let $X = M \cup \overline{M}$, let G be the free group on M and let H be the free group on N. First we extend φ, in a natural way, to a morphism $\varphi \colon (M \cup \overline{M})^* \to (N \cup \overline{N})^*$. Next we consider the morphism $\varphi^* \colon G \to H$ such that $[a]\varphi^* = a\varphi$ for each $a \in M$.

Note that for $a \in M$,

(1) $M_1\varphi = T\varphi \subseteq U = N_1$,
(2) $M_a\varphi = (TaT)\varphi = (T\varphi)(a\varphi)(T\varphi) \subseteq U(a\varphi)U = N_{a\varphi}$, and
(3) $M_{\overline{a}}\varphi = (T\gamma_T(a)T)\varphi = (T\varphi)(\gamma_T(a)\varphi)(T\varphi) \subseteq U\gamma_U(a\varphi)U = N_{\overline{a\varphi}}$.

Hence for $w \in X^*$ we have $M_w\varphi \subseteq N_{w\varphi}$. It follows that for $g \in G$, we have $M_g\varphi \subseteq N_{g\varphi^*}$.

Hence, if we denote the categories in the constructions for M and N by C and D respectively, we can define a morphism $\widetilde{\varphi} \colon C \to D$ by putting $g\widetilde{\varphi} = g\varphi^*$ for $g \in \operatorname{Obj} C$ and putting $(a, m, b)\widetilde{\varphi} = (a\varphi^*, m\varphi, b\varphi^*)$ for $a, b \in G$ and $(a, m, b) \in \operatorname{Mor}(a, b)$. That $\widetilde{\varphi}$ is a morphism is immediate from the fact that φ and φ^* are morphisms. It is clear that $(a(b, m, c))\widetilde{\varphi} = (a\varphi^*)((b, m, c)\widetilde{\varphi})$ for $a \in G$ and $(b, m, c) \in \operatorname{Mor} C$, that is, $\widetilde{\varphi}$ is equivariant.

Let $\chi\colon C \to M$ and $\xi\colon D \to N$ be the covering maps in the construction, given by $(a,m,b)\chi = m$ and $(h,n,k)\xi = n$ respectively. Then it is obvious that the square

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\chi"'] & D \arrow[d, "\xi"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

is commutative. By Theorem 3.3, there is a morphism $\widehat{\varphi}\colon C/G \to D/H$ such that the square

$$\begin{tikzcd} C \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi_C"'] & D \arrow[d, "\pi_D"] \\ C/G \arrow[r, "\widehat{\varphi}"'] & D/H \end{tikzcd}$$

is commutative. From the construction we know that the triangle

$$\begin{tikzcd} C \arrow[r, "\chi"] \arrow[d, "\pi_C"'] & M \\ C/G \arrow[ur, "\alpha_{M,T}"'] & \end{tikzcd}$$

is commutative and so is the corresponding triangle for N. It follows that the square

$$\begin{tikzcd} C/G \arrow[r, "\widehat{\varphi}"] \arrow[d, "\alpha_{M,T}"'] & D/H \arrow[d, "\alpha_{N,U}"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

is commutative as required.

If $T\varphi = U$, $\gamma_T(a)\varphi = \gamma_U(a\varphi)$ for all $a \in M$ and φ is surjective, then it is clear from the definitions above that φ^* is surjective and hence that $\widetilde{\varphi}$ is surjective. The surjectivity of $\widehat{\varphi}$ is now immediate from Theorem 3.3. $\square$

Remark. We deduce the semigroup versions of Theorem 5.1 and Proposition 5.7 directly from these results by adding new identities (whether or not identities are already present) and then restricting morphisms to non-identity elements.

6. Monoids which have a minimum group congruence

We investigate a class of monoids which have a minimum group congruence and examine special cases of the results of Sections 4 and 5 for such monoids. It is well known that not every monoid has a minimum group congruence. An example of one which does not is the infinite cyclic monoid [10, Exercise 6 of Section 1.6]. On the other hand, it is far from clear under exactly what conditions a monoid does have a minimum group congruence. It is worth noting, however, that all monoids in the classes mentioned in the introduction in connection with covering theorems are E-dense. The existence of a minimum group congruence on an E-dense monoid is proved in [22], and an explicit description is given in [41]. In fact, all E-dense monoids are in the class we consider in this section. Thus when we come to constructions of D-unitary, E-dense monoids, and covering theorems for E-dense monoids in Section 7 and for regular monoids in Section 8, we will be applying the results of this section.

For a monoid, having a minimum group congruence is, of course, equivalent to having a maximum group quotient. Given any monoid M, there is a group $\pi_1(M)$ which is universal with respect to morphisms from M into groups. This is known variously as the *fundamental group* of M [25], the *universal group* of M [12] or the *free group on the monoid M* [11]. In general, if M has a minimum group congruence, $\pi_1(M)$ need not be the maximum group quotient of M. We can, however, give a sufficient condition for $\pi_1(M)$ to be the maximum group quotient by using the "least weakly self-conjugate, planar submonoid" $\widetilde{D}(M)$ of M. If $\widetilde{D}(M)$ is dense in M, then M has a minimum group congruence and $\pi_1(M)$ is the maximum group quotient of M, as we show in Proposition 6.9 below.

We begin by defining a submonoid $K(M)$ of M by

$$K(M) = \{k \in M \mid k\theta = 1 \text{ for all surjective morphisms } \theta \text{ from } M \text{ onto a group}\}.$$

Lemma 6.1. *The submonoid $K(M)$ is a planar submonoid of M. If M has a minimum group congruence σ, then $K(M) = 1\sigma$.*

Proof. By the monoid version of Theorem 2.6, $K(M)$ is the intersection of a family of planar submonoids. It follows immediately that $K(M)$ is planar as well. The second part of the statement is immediate. ☐

In general, $K(M)$ is not dense in M. For example, if M is the infinite cyclic monoid, then $K(M) = \{1\}$.

Lemma 6.2. *A monoid M has a minimum group congruence σ if and only if $K(M)$ is dense in M. Moreover, when σ exists, $\sigma = \rho_{K(M)}$.*

Proof. Suppose that $K(M)$ is dense in M. In view of Lemmas 6.1 and 1.4, $K(M)$ is strongly dense in M and so by Corollary 2.8, there is a group congruence $\rho_{K(M)}$ on M determined by $K(M)$. Furthermore, since $K(M)$ is dense and planar, it follows from Theorem 2.6 that $K(M) = 1\rho_{K(M)}$. If ρ is any group congruence on M, then it follows from the definition of

$K(M)$ that $K(M) \subseteq 1\rho$, that is, $1\rho_{K(M)} \subseteq 1\rho$. Consequently, $\rho_{K(M)} \subseteq \rho$ and thus $\rho_{K(M)}$ is the minimum group congruence on M.

Conversely, suppose that M has a minimum group congruence σ. Then $K(M)$ is dense in M by Lemma 6.1 and Theorem 2.6. $\square$

An element a of a monoid M is a *weak inverse* of the element b if $aba = a$. We denote the set of all weak inverses of the element b by $W(b)$. A submonoid T of M is said to be *weakly self-conjugate* if $aTb \cup bTa \subseteq T$ for all $b \in M$ and all $a \in W(b)$. It is clear that the intersection of a family of weakly self-conjugate submonoids is itself weakly self-conjugate. Hence, given any monoid M, we can define $D(M)$ to be the least (under inclusion) weakly self-conjugate submonoid of M. Similarly, we can define $\widetilde{D}(M)$ to be the least submonoid of M which is planar and weakly self-conjugate. For a category C, we can define $D(C)$ and $\widetilde{D}(C)$ in a similar way: $D(C)$ is the least weakly self-conjugate subcategory of C with the same set of objects as C and $\widetilde{D}(C)$ is the least subcategory of C, with the same set of objects, which is planar and weakly self-conjugate.

If we are dealing with semigroupoids rather than categories, we take $D(C)$ (resp. $\widetilde{D}(C)$) to be the least weakly self-conjugate (resp. planar and weakly self-conjugate) subsemigroupoid of C with the same object set and containing $E(C)$. In the sequel, we will continue to work with categories.

It is clear that if e is an idempotent element of $\operatorname{Mor} C$, then $e \in W(e)$. It follows easily that

$$E(C) \subseteq D(C) \subseteq \widetilde{D}(C).$$

In addition, $\delta(C)$ is obviously planar and weakly self-conjugate, so that $\widetilde{D}(C) \subseteq \delta(C)$ and hence, $D(C)$ and $\widetilde{D}(C)$ are totally disconnected.

To further illustrate these ideas, consider an E-dense monoid M in which $E(M)$ is a submonoid. Then it follows from [19, Proposition 2.1] that $E(M)$ is weakly self-conjugate, that is, $D(M) = E(M)$. By [1, Proposition 1.2], $E(M)$ is reflexive so that by Corollary 1.2, $E(M)$ is planar if and only if it is unitary. Thus $\widetilde{D}(M) = E(M)$ if and only if M is E-unitary so that we can have $D(M) \neq \widetilde{D}(M)$. In the case when M has a zero, for example, $D(M) = E(M)$ but $\widetilde{D}(M) = M$.

Lemma 6.3. *Let $\varphi\colon C_1 \to C_2$ be a morphism from a category C_1 to a category C_2. Then $D(C_1)\varphi$ is a subcategory of $D(C_2)$ and $\widetilde{D}(C_1)\varphi$ is a subcategory of $\widetilde{D}(C_2)$.*

Proof. We prove the result for $\widetilde{D}$; the proof for D is similar. Let $T = \widetilde{D}(C_2)\varphi^{-1}$. Then $\operatorname{Obj} T = \operatorname{Obj} C_1$ and T is immediately seen to be a planar, weakly self-conjugate subcategory of C_1. Thus $\widetilde{D}(C_1) \subseteq T$ and $\widetilde{D}(C_1)\varphi \subseteq \widetilde{D}(C_2)$. $\square$

In the particular case of a morphism onto a group, we have the following.

Corollary 6.4. *Let $\varphi\colon M \to G$ be a morphism from a monoid M onto a group G. Then $\widetilde{D}(M)\varphi = \{1\}$. Moreover, $\widetilde{D}(M) \subseteq K(M)$.*

Proof. It is immediate that $\widetilde{D}(G) = \{1\}$ so the first statement follows from Lemma 6.3. Thus $\widetilde{D}(M)$ is contained in $1\varphi^{-1}$ for each morphism φ from M onto a group. By definition of $K(M)$, this implies that $\widetilde{D}(M) \subseteq K(M)$. □

The following example shows that it is possible to have $\widetilde{D}(M) \neq K(M)$.

Example. Let S be an idempotent-free, congruence-free semigroup. That such semigroups exist follows from [50] where it is shown that any idempotent-free semigroup can be embedded in a semigroup with the same property which is also congruence-free. It is easy to see that the only group congruence on the monoid S^1 is the universal congruence and hence that $K(S^1) = S^1$. On the other hand, it is equally easy to see that $\widetilde{D}(S^1) = \{1\}$.

Let G be a group and let D be a subcategory of a G-category C. Let $D_G = \bigcap_{g \in G} gD$.

Lemma 6.5. *Let D be a subcategory of a G-category C. Then D_G is a G-subcategory of C. Furthermore, if D is weakly self-conjugate (resp. unitary, reflexive, planar) in C, then so is D_G.*

Proof. The first statement is straightforward. By definition of a group action, $x \mapsto gx$ is an automorphism of C, so that gD is weakly self-conjugate (resp. unitary, reflexive, planar) in C if and only if D is. Since these properties are preserved under intersection, it follows that they hold for D_G if they hold for D. □

The following corollary is important.

Corollary 6.6. *For any G-category C, the subcategories $D(C)$ and $\widetilde{D}(C)$ are G-subcategories.*

It follows that if G acts isotropically and transitively on C, then we can form monoids $D(C)/G$ and $\widetilde{D}(C)/G$. We have already noted that $D(C)$ and $\widetilde{D}(C)$ are contained in $\delta(C)$, and that $\delta(C)$ is planar and weakly self-conjugate. Hence $D(C)/G$ and $\widetilde{D}(C)/G$ are submonoids of $\delta(C)/G$.

Proposition 6.7. *Let G be a group acting freely and transitively on a category C. Then $\widetilde{D}(C/G) = \widetilde{D}(C)/G$ and $D(C/G) = D(C)/G$.*

Proof. We give the proof for $\widetilde{D}$, the proof for D being similar and easier. First, by Lemma 6.3, we have $\widetilde{D}(C)/G \subseteq \widetilde{D}(C/G)$.

To prove the opposite inclusion, it suffices to establish that $T = \widetilde{D}(C)/G$ is unitary, reflexive and weakly self-conjugate in C/G. Indeed, this will prove that T is planar and weakly self-conjugate in C/G, by Lemma 1.1, and hence contains $\widetilde{D}(C/G)$.

Let $p, q \in C/G$ with $p, p+q \in T$. Then $Gx = p$ for some $x \in \operatorname{Mor} \widetilde{D}(C)$ and $x \in \operatorname{Mor}(u, u)$ for some $u \in \operatorname{Obj} C$. By Lemma 3.1, there exists $y \in \operatorname{Mor} C$ such that $\alpha(y) = u$ and $Gy = q$. Furthermore, there exists $z \in \operatorname{Mor} \widetilde{D}(C)$ such that $Gz = p+q$. Let $v = \alpha(z) = \omega(z)$. Since the action of G is transitive, there exists $g \in G$ such that $u = gv$. Then $gz \in \widetilde{D}(C)$ by Corollary 6.6 and thus, by Lemma 3.1 again, $gz = x + y$. Since $\widetilde{D}(C)$ is unitary, $y \in \widetilde{D}(C)$ and hence $q = Gy \in T$. It follows that T is unitary.

Let $p, q \in C/G$ and suppose that $p + q \in T$. Then $p = Gx$ for some $x \in \operatorname{Mor} C$, say $x \in \operatorname{Mor}(u, v)$. By Lemma 3.1, there exists $y \in \operatorname{Mor}(v, C)$ such that $Gy = q$. Then $G(x + y) = p + q$. Now $p + q = Gz$ for some $z \in \operatorname{Mor} \widetilde{D}(C)$. It follows that $x + y \in Gz$ and by Corollary 6.6, $x + y \in \operatorname{Mor} \widetilde{D}(C)$. Since $\widetilde{D}(C)$ is totally disconnected, $y + x$ is also defined in C, and since $\widetilde{D}(C)$ is reflexive, $y + x \in \operatorname{Mor} \widetilde{D}(C)$ and hence $q + p \in T$. Thus T is reflexive.

Let $p, q, r \in C/G$ be such that $p \in T$ and $q + r + q = q$. Then $p = Gx$ for some $x \in \operatorname{Mor} \widetilde{D}(C)$, say $x \in \operatorname{Mor}(u, u)$. By Lemma 3.1, there exist y, y' and z such that $Gy = q = Gy'$, $Gz = r$, $\omega(y) = u = \alpha(z)$ and $\alpha(y') = \omega(z)$. Now $G(y + z + y') = q = Gy$ and so by Lemma 3.1, $y = y+z+y'$. It follows that $\omega(y) = \omega(y')$ and, by Lemma 3.1 again, $y = y'$. In particular, $\omega(z) = \alpha(y)$. Since $\widetilde{D}(C)$ is weakly self-conjugate, $y + x + z \in \operatorname{Mor} \widetilde{D}(C)$ and hence $q + p + r \in T$. Finally, let x' be the element of Gx such that $\alpha(x) = \omega(y)$. By Corollary 6.6, $x' \in \operatorname{Mor} \widetilde{D}(C)$, so $z + x' + y \in \operatorname{Mor} \widetilde{D}(C)$ and hence $r + p + q \in T$. Thus T is weakly self-conjugate, which completes the proof. $\square$

Remark. Hey, wait a minute, what happens for semigroupoids, we hear you ask; we need T to be full, right? OK, OK, chill out, reader, there is no problem.

The analogue of Proposition 6.7 for a semigroupoid C is obtained by adjoining identities as necessary to get a category C^1 on which G acts in the obvious way. Then $\widetilde{D}(C^1/G) = \widetilde{D}(C^1)/G$ and since $\widetilde{D}(C) = \widetilde{D}(C^1) \cap C$, we see that every idempotent of C/G is contained in $\widetilde{D}(C)/G$.

Next we consider the maximum group quotient of a monoid M when $\widetilde{D}(M)$ is dense in M. The *fundamental group* $\pi_1(M)$ of M is described as follows. Let $F(M)$ be the free group with basis M. Then there is a natural injection $\iota\colon M \to F(M)$ which is not, of course, a monoid morphism. Now $\pi_1(M)$ is defined to be the group with presentation

$$\operatorname{gp}(M \mid (m\iota)(n\iota) = (mn)\iota;\ m, n \in M).$$

Let $\pi\colon F(M) \to \pi_1(M)$ be the surjective group morphism onto $\pi_1(M)$ extending ι. Then from [35, Proposition 3.6] we have that the map $\eta\colon M \to \pi_1(M)$ where $\eta = \iota\pi$ has the following universal property.

Proposition 6.8. *The map η is a monoid morphism and for each monoid morphism $\gamma\colon M \to G$ into a group G, there is a unique group morphism $\varphi\colon \pi_1(M) \to G$ such that the following triangle*

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[d, "\eta"'] & G \\ \pi_1(M) \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is commutative.

It follows from the proposition that if $\pi_1(M)$ is a quotient of M, then it is the maximum group quotient of M. However, as the next example demonstrates, it is possible for a monoid M to have a maximum group quotient but for $\pi_1(M)$ not to be a quotient of M, that is, for η not to be surjective.

Example. Let M be the monoid with presentation

$$\mathrm{Mon}(a, t_i, t_i^{-1} \mid t_i^{-1}at_i = a^{p_i}, t_it_i^{-1} = t_i^{-1}t_i = 1, t_it_j = t_jt_i \text{ for all } i, j \in \mathbb{N})$$

where p_i is the ith prime. In an appendix we prove that M is a reversible cancellative monoid. Thus M has a group of quotients G (see [10]) and we show in the appendix that G is isomorphic to $\pi_1(M)$ and that $\pi_1(M)$ is not a quotient of M. On the other hand, we also show that M does have a maximum group quotient. It follows from the next result that we must have $K(M) \neq \widetilde{D}(M)$.

Proposition 6.9. *Let M be a monoid in which $\widetilde{D}(M)$ is dense. Then $\rho_{\widetilde{D}(M)}$ is the minimum group congruence on M, $\widetilde{D}(M) = K(M)$, and $\pi_1(M)$ is the maximum group quotient of M.*

Proof. By Corollary 6.4, $\widetilde{D}(M) \subseteq K(M)$. It follows that $K(M)$ is dense and hence, by Lemma 6.2, $\rho_{K(M)}$ is the minimal group congruence of M. Now Lemma 1.4 shows that $\widetilde{D}(M)$ is strongly dense, and hence $\rho_{\widetilde{D}(M)}$ is a group congruence by Lemma 2.7. But $\widetilde{D}(M) \subseteq K(M)$ implies $\rho_{\widetilde{D}(M)} \subseteq \rho_{K(M)}$, so that $\rho_{\widetilde{D}(M)} = \rho_{K(M)}$.

Now Theorem 2.6 shows that $\widetilde{D}(M) = K(M)$. The proof of the final assertion is essentially that of [35, Proposition 3.7]. As noted above we have simply to show that $\pi_1(M)$ is a quotient of M, that is, η is surjective. Let $m \in M$. Since $\widetilde{D}(M)$ is dense in M, there is an element n of M such that $mn \in \widetilde{D}(M)$. By Corollary 6.4, $(mn)\eta = 1$ and so $(m\eta)^{-1} = n\eta \in M\eta$. Thus the set $\{(m\eta)^{-1} \mid m \in M\}$ is contained in $M\eta$ and since $M\eta$ generates $\pi_1(M)$ as a group, it also generates it as a monoid. Hence η is surjective. $\square$

We now use Theorem 4.5 to give several characterisations of monoids M in which $\widetilde{D}(M)$ is dense.

Theorem 6.10. *For a monoid M, the following conditions are equivalent:*

(1) *M has a minimum group congruence and $\widetilde{D}(M) = K(M)$,*
(2) *there is a group G and a surjective morphism $\varphi\colon M \to G$ with $1\varphi^{-1} = \widetilde{D}(M)$,*
(3) *$\widetilde{D}(M)$ is dense in M,*
(4) *the morphism $\eta\colon M \to \pi_1(M)$ is surjective and $1\eta^{-1} = \widetilde{D}(M)$,*
(5) *M is isomorphic to C/G where G is a group acting freely and transitively on a strongly connected category C with $\widetilde{D}(C) = \delta(C)$,*
(6) *M is isomorphic to C/G where G is a group acting freely and transitively on a strongly connected category C where $\widetilde{D}(C)$ is strongly dense in C and $C/\rho_{\widetilde{D}(C)}$ is a simplicial groupoid.*

Proof. Conditions (1) and (3) are equivalent by Proposition 6.9 and Lemma 6.2. Conditions (2) and (3) are equivalent by Theorem 4.5 and obviously, (4) implies (2). If (3) holds, then by Proposition 6.9, $\widetilde{D}(M) = K(M)$ and $\pi_1(M)$ is the maximum group quotient of M. Hence $1\eta^{-1} = \widetilde{D}(M)$ by Lemma 6.1 and condition (4) holds. Thus (1), (2), (3) and (4) are equivalent.

If (3) holds, then by Theorem 4.5, there is a group G acting freely and transitively on a strongly connected category C such that C/G is isomorphic to M via an isomorphism which restricts to an isomorphism between $\widetilde{D}(M)$ and $\delta(C)/G$. Thus $\delta(C)/G = \widetilde{D}(C/G)$ and so by Proposition 6.7, $\delta(C)/G = \widetilde{D}(C)/G$. Hence $\delta(C) = \widetilde{D}(C)$ and (5) holds.

If (5) holds, then (6) follows from the fact that $\delta(C)$ is strongly dense in C and the remarks preceding Corollary 3.7.

If (6) holds, then for any objects u, v of C and morphisms $p, q \in \mathrm{Mor}(u, v)$ we have $p \; \rho_{\widetilde{D}(C)} \; q$ so that $\rho_{\delta(C)} \subseteq \rho_{\widetilde{D}(C)}$ and hence $\delta(C) \subseteq \widetilde{D}(C)$ by Theorem 2.6. But $\widetilde{D}(C)$ is totally disconnected so that $\delta(C) = \widetilde{D}(C)$ and (5) holds.

Finally, suppose that (5) holds. Then using Proposition 6.7, $\widetilde{D}(C/G) = \widetilde{D}(C)/G = \delta(C)/G$ and hence the the isomorphism from M onto C/G maps $\widetilde{D}(M)$ onto $\delta(C)/G$. Therefore, by Theorem 4.5, condition (3) holds. $\square$

We now consider two covering results, the first for monoids M in which $D(M)$ is strongly dense, and the second for monoids M in which $\langle E\rangle$ is strongly dense. In both cases, $\widetilde{D}(M)$ is necessarily dense and so M has a minimum group congruence.

Theorem 6.11. *Let M be a monoid in which $D(M)$ is strongly dense. Then M has a $D(M)$-cover $\widehat{M}$ with $\widetilde{D}(\widehat{M}) = \widehat{D(M)} = D(\widehat{M})$.*

Proof. We use the notation of the construction in the proof of Theorem 5.1, with $T = D(M)$. For each $a \in M$ we have $W(a) \subseteq W_{D(M)}(a)$ since $D(M)$ is weakly self-conjugate, and we choose $\gamma_{D(M)}(a)$ with $W(a) \subseteq \gamma_{D(M)}(a)$. Then we have a group G – the free group on M – which acts freely and transitively on a strongly connected category C and a G-covering $\chi\colon C \to M$. In view of Propositions 5.2 and 6.7 it suffices to prove that $\delta(C) = D(C)$. Since $D(C)$ is contained in $\delta(C)$ by definition, this amounts to proving that $\mathrm{Mor}(g,g)$ is contained in $\mathrm{Mor}\, D(C)$ for all $g \in G$. By Corollary 6.6, $D(C)$ is a G-subcategory and so it is enough to show that $\mathrm{Mor}\, D(C)$ contains $\mathrm{Mor}(1,1)$.

By definition of C,

$$\mathrm{Mor}(1,1) = \{(1,m,1) \mid m \in M_1\} = \{(1,m,1) \mid m \in D(M)\}.$$

Now let $N = \{m \in M \mid (1,m,1) \in \mathrm{Mor}\, D(C)\}$. Clearly N is a submonoid of M. We claim that N is weakly self-conjugate. Let $a \in M$ and $b \in W(a)$. Since $W(a) \subseteq \gamma_{D(M)}(a)$, the triples $(1,a,[a])$, $([\overline{a}],a,1)$, $(1,b,[\overline{a}])$ and $([a],b,1)$ are all in $\mathrm{Mor}\, C$. Furthermore, $([a],b,1)$ is a weak inverse of $(1,a,[a])$ since

$$([a],b,1) + (1,a,[a]) + ([a],b,1) \;=\; ([a],bab,1) \;=\; ([a],b,1).$$

Let $n \in N$. Then $(1,n,1) \in \mathrm{Mor}\, D(C)$ and hence $([a],n,[a]) \in \mathrm{Mor}\, D(C)$ by Corollary 6.6. Since $D(C)$ is weakly self-conjugate,

$$(1,a,[a]) + ([a],n,[a]) + ([a],b,1) = (1,anb,1) \in \mathrm{Mor}\, D(C),$$

and hence $anb \in N$. Similarly, one verifies that $bna \in N$. Thus N is weakly self-conjugate. Consequently, $D(M) \subseteq N$ and hence $\mathrm{Mor}(1,1) \subseteq \mathrm{Mor}\, D(C)$. It follows that $\delta(C) = D(C)$ so that $D(C) = \widetilde{D}(C)$. □

In [19] a monoid M is said to be $\langle E\rangle$-*dense* if the submonoid $\langle E\rangle$ generated by the idempotents of M is strongly dense in M. Such a monoid is said to be *strongly $\langle E\rangle$-unitary dense* if $\langle E\rangle$ is unitary and reflexive. It is now easy to recover the following result from [19].

Theorem 6.12. *Every $\langle E\rangle$-dense monoid has a strongly $\langle E\rangle$-unitary dense cover.*

Proof. As in the proof of Theorem 6.11 we use the construction in the proof of Theorem 5.1 with $T = \langle E(C)\rangle$. We do not impose any extra condition on the sets $\gamma_{\langle E(C)\rangle}(a)$. All we need show is that $\mathrm{Mor}(1,1) \subseteq \mathrm{Mor}\, D(C)$, that is, $\langle E(C)\rangle \subseteq \mathrm{Mor}\, D(C)$, but this is obvious. $\square$

Remark. We now outline a well known recursive construction for $D(C)$, which is similar to that for T_∞ at the end of Section 2. First, for any subcategory D of C, put

$$q(D) = \{a + d + b, b + d + a \mid d \in \mathrm{Mor}\, D, b \in \mathrm{Mor}\, C, a \in W(b)\}$$

and $Q(D) = \langle q(D)\rangle$. Now define $D_0(C)$ to be the subcategory consisting only of the identity morphisms of C and for each non-negative integer i, put $D_{i+1}(C) = Q(D_i(C))$. Clearly, we have an ascending chain

$$D_0(C) \subseteq D_1(C) \subseteq \cdots \subseteq D_i(C) \subseteq \dots$$

and it is easy to see that $D(C) = \bigcup_{i\geqslant 0} D_i(C)$.

There is a similar construction for $\widetilde{D}(C)$ which we now describe. First, as in Section 2, if D is a subcategory of a category C, then

$$u(D) = \{x \in \mathrm{Mor}\, C \mid x + y \in \mathrm{Mor}\, D \text{ or } y + x \in \mathrm{Mor}\, D \text{ for some } y \in \mathrm{Mor}\, D\},$$

$$r(D) = \{x \in \mathrm{Mor}\, C \mid x = y + z \text{ for some } y, z \in \mathrm{Mor}\, C \text{ such that } z + y \in \mathrm{Mor}\, D\},$$

$R(D) = \langle r(D)\rangle$ and $U(D) = \langle u(D)\rangle$. Now put

$$\widetilde{D}_0 = \widetilde{D}_0(C) = D_0(C)$$

and for each non-negative integer k, put $\widetilde{D}_{3k+1} = U(\widetilde{D}_{3k})$, $\widetilde{D}_{3k+2} = R(\widetilde{D}_{3k+1})$ and $\widetilde{D}_{3k+3} = Q(\widetilde{D}_{3k+2})$. It is clear that we again have an ascending chain

$$\widetilde{D}_0 \subseteq \widetilde{D}_1 \subseteq \cdots \subseteq \widetilde{D}_i \subseteq \dots$$

so that $\bigcup_{i\geqslant 0} \widetilde{D}_i$ is a subcategory of C. The construction is designed to ensure that $\bigcup_{i\geqslant 0} \widetilde{D}_i$ is reflexive, unitary and weakly self-conjugate. Hence $\widetilde{D}(C) \subseteq \bigcup_{i\geqslant 0} \widetilde{D}_i$. An easy induction argument shows that $\widetilde{D}_i \subseteq \widetilde{D}(C)$ for all i, so that $\widetilde{D}(C) = \bigcup_{i\geqslant 0} \widetilde{D}_i$.

7. E-dense monoids

The definition of a dense submonoid can be applied to give the notion of a dense subset of a monoid; thus a subset T of a monoid M is *dense* in M if for any element a of M there are elements b, c of M such that $ab, ca \in T$. A monoid is *E-dense* (or *E-inversive*) if the set $E(M)$ of idempotents of M is dense in M. There is an analogous definition of dense subsets of $\mathrm{Mor}\, C$ for a category C and we say that C is *E-dense* if $\{p \in \mathrm{Mor}\, C \mid p + p = p\}$

is dense in $\operatorname{Mor} C$. Thus C is E-dense if for all morphisms p there are morphisms q, r such that $p+q, r+p$ are defined and are idempotent.

The concept of an E-dense monoid was introduced by Thierrin [55] and was studied by Petrich [43, 44], Lallement and Petrich [28] and Mitsch [41]. The latter provides several examples of E-dense monoids and notes, in particular, that regular, eventually regular and periodic monoids are all E-dense. An E-dense monoid in which the idempotents form a commutative submonoid is said to be *E-commutative dense*.

Margolis and Pin [34, 35] showed that extensions of semilattices by groups are precisely the E-unitary, E-commutative dense monoids and that McAlister's structure theorem can be recovered by specialising their description of these monoids to the regular case. In [18] it was shown that every E-commutative dense monoid has an E-unitary, E-commutative dense cover. This result and those of [34, 35] were generalised by Almeida, Pin and Weil [1] and independently by Zhonghao Jiang [60] to the case of E-dense monoids in which the idempotents form a submonoid. A survey of this work is given in [48].

In this section we extend these results to arbitrary E-dense monoids, by applying the main theorems of the previous section. In particular, we show that every E-dense monoid M has a D-unitary E-dense cover, and we describe E-dense D-unitary monoids in terms of groups acting on categories. The results of [1] and [60] are then obtained as corollaries.

First, we need to develop some general theory for E-dense monoids and categories. In particular, we show that, in an E-dense monoid, the submonoid $D(M)$ is also E-dense. To do this we examine weak inverses and obtain some analogues of results about inverses in regular monoids. We start by considering the connections between C and C/G for a G-category C, and between M and the weak derived category $C(\varphi)$ of a surjective morphism from a monoid M onto a group G.

There are several possible first-order languages for categories discussed briefly in [20, Chapter 11]. For further development of one of them based on work of Lawvere [30], the reader can consult [24, Chapter 8]. If we were to write the definition of E-dense in one of the first-order languages for categories, then clearly it would be built up from atomic formulæ using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$. That is, the definition is expressed by a positive sentence. Now by [9, Corollary 3.2.5], positive sentences are preserved by homomorphisms and hence any quotient of an E-dense category is E-dense. We are interested in the transfer of properties from a category C to the monoid C/G where G is a group acting freely and transitively on C. By virtue of the cited result we have the following proposition.

Proposition 7.1. *Let G be a group acting freely and transitively on a category C. If C is E-dense or regular, then so is C/G.*

In the opposite direction we show that the weak derived category of a morphism from a monoid onto a group inherits some properties of the monoid.

Proposition 7.2. *Let $\varphi\colon M \to G$ be a morphism from a monoid M onto a group G. If M is E-dense or regular, then so is $C(\varphi)$.*

Proof. Let $(g, m, h) \in \operatorname{Mor} C(\varphi)$ so that $g(m\varphi) = h$. If $mn \in E(M)$, then $(m\varphi)(n\varphi) = (mn)\varphi = 1$ so that $g = h(n\varphi)$ and $(h, n, g) \in \operatorname{Mor} C(\varphi)$. Hence $(g, m, h) + (h, n, g) = (g, mn, g)$ is idempotent. Similarly, if $m'm \in E(M)$, then $(h, m', g) + (g, m, h)$ is idempotent.

If m' is an inverse of m, then it is easy to check that (h, m', g) is an inverse of (g, m, h). $\square$

The next result summarises some elementary properties of E-dense monoids (compare with Lemma 1.3 on planar subcategories).

Proposition 7.3. *Let M be a monoid and let $E = E(M)$. Then the following conditions are equivalent:*

(1) *M is E-dense,*
(2) *for every $a \in M$, there is an element b of M such that $ab \in E$ and $ba \in E$,*
(3) *for every $a \in M$, there is an element c of M such that $ac \in E$,*
(4) *for every $a \in M$, there is an element d of M such that $da \in E$,*
(5) *every element of M has a weak inverse.*

Proof. The equivalence of (1) to (4) can be found in [1] or [41] and the equivalence of (5) with the rest is in [8] but for completeness we give a short proof.

If a' is a weak inverse of a, then aa' and $a'a$ are idempotent and so (5) implies (2). Clearly, (2) implies (1) and (1) implies (3) and (4). By symmetry, it is enough to show that (3) implies (5). Let $a \in M$ and let $c \in M$ be such that $ac \in E$. Then clearly, cac is a weak inverse of a, proving (5). $\square$

There is an obvious analogue for categories and the following corollary is an immediate consequence of condition (5) of the proposition.

Corollary 7.4. *For any E-dense monoid M (category C), the submonoid $D(M)$ (subcategory $D(C)$) is strongly dense.*

We now give an alternative proof of the following result of Mitsch [41].

Proposition 7.5. *The minimum group congruence on an E-dense monoid M is the relation $\rho_{D(M)}$.*

Proof. By Corollary 7.4, $D(M)$ is strongly dense in M so that by Corollary 2.8, $\rho_{D(M)}$ is a group congruence. Now $\widetilde{D}(M)$ is dense in M (since $E \subseteq \widetilde{D}(M)$) so that by Proposition 6.9, $\rho_{\widetilde{D}(M)}$ is the minimum group congruence. But $D(M) \subseteq \widetilde{D}(M)$ so that $\rho_{D(M)} \subseteq \rho_{\widetilde{D}(M)}$ and the result follows. □

There is, of course, a category version of this result which tells us that the minimum groupoid congruence on an E-dense category C is $\rho_{D(C)}$. Furthermore, if C is a G-category, then $\rho_{D(C)}$ is equivariant.

Recall from [42] that for idempotents e, f of a monoid M, the set $\mathscr{M}(e, f)$ is defined by

$$\mathscr{M}(e, f) = \{g \in E(M) \mid ge = g = fg\}.$$

For an E-dense monoid, this set plays a similar role to that of the sandwich set in a regular monoid (which we define in the next section). For example, it is easy to verify that if x is a weak inverse of fe, then $fxe \in \mathscr{M}(e, f)$ and so we have the following lemma.

Lemma 7.6. *If the monoid M is E-dense, then $\mathscr{M}(e, f)$ is non-empty for all idempotents e, f of M.*

In fact, the analogy is much closer and although we do not use it, we mention the following result since it is of interest for its own sake.

Proposition 7.7. *If E is a biordered set such that $\mathscr{M}(e, f) \neq \emptyset$ for all $e, f \in E$, then there is an E-dense semigroup S with $E(S) \cong E$.*

Rather than give a detailed proof of Proposition 7.7, we content ourselves with observing that a proof can be obtained by extracting the appropriate parts from Easdown's proof [15] of the corresponding result for regular semigroups.

The proof of the next lemma is a simple computation.

Lemma 7.8. *Let $a' \in W(a)$, $b' \in W(b)$ and $g \in \mathscr{M}(a'a, bb')$ where a, b are elements of an E-dense monoid M. Then $b'ga' \in W(ab)$.*

We also need the following analogue of a result for regular monoids due to FitzGerald [17] (see also [27, Exercise 2.6.23]). The proof is essentially the same as in the regular case.

Lemma 7.9. *If M is an E-dense monoid and $m \in M$ is a product of n idempotents, then $W(m) \subseteq E(M)^{n+1}$.*

Proof. By definition, $W(1) = E(M) = E$, so the result holds for $n = 0$. Now assume that it holds for some n. Let $z \in E^n$, $e \in E$ and $x \in W(ze)$. Then $x = xzex$. It follows that $ex \in W(z)$, so $ex \in E^{n+1}$. Now $x = (xze)(ex) \in EE^{n+1} = E^{n+2}$ and the result follows by induction. $\square$

We can now prove the following result which is important in the sequel.

Proposition 7.10. *If M is an E-dense monoid, then the submonoid $D(M)$ is E-dense.*

Proof. The proof relies on the constructive description of $D(M)$ given at the end of Section 6. With the notation of that remark, $D(M)$ is the union of the increasing sequence $(D_i(M))_{i\geq 0}$, where $D_0(M) = \{1\}$ and $D_{i+1}(M) = Q(D_i(M))$. It is immediately verified that $D_1(M) = \langle E(M)\rangle$ and that, to show that $D(M)$ is E-dense, it suffices to show that each $D_i(M)$ $(i \geq 1)$ is E-dense. Thus the problem reduces to showing that $\langle E(M)\rangle$ is E-dense, and that if T is a full, E-dense submonoid of M, then $Q(T)$ is E-dense. We put $E = E(M)$.

By Lemma 7.9, if $z \in \langle E\rangle$, then $W(z) \subseteq \langle E\rangle$, and hence $\langle E\rangle$ is E-dense by Proposition 7.3.

We now assume that T is a full, E-dense submonoid of M and show that $Q(T)$ is E-dense. By Proposition 7.3 again, it suffices to show that each element of $Q(T)$ has a weak inverse in $Q(T)$.

Let $q \in Q(T)$. Then $q = q_1 \cdots q_n$ for some $q_1, \ldots, q_n \in q(T)$. For each $1 \leq i \leq n$, we have $q_i = a_i t_i b_i$ or $q_i = b_i t_i a_i$ for some $t_i \in T$, $a_i \in M$ and $b_i \in W(a_i)$. Let t'_i be a weak inverse of t_i in T and let $b'_i \in W(b_i)$. By Lemma 7.8, there exist idempotents e_i, f_i such that $q'_i = b'_i e_i t'_i f_i b_i$ (resp. $b_i e_i t'_i f_i b'_i$) is a weak inverse of q_i. Now $e_i t'_i f_i \in T$ since T is full and so $q'_i \in q(T)$.

Finally, applying Lemma 7.8 $n-1$ times, there exist idempotents $g_1, \ldots, g_{n-1}$ such that $q'_n g_{n-1} q'_{n-1} \cdots g_1 q'_1$ is a weak inverse of q, which lies in $Q(T)$. This concludes the proof. $\square$

Proposition 7.11. *Let M be an E-dense monoid and let T be a full, weakly self-conjugate submonoid. Then T is planar if and only if T is unitary.*

Proof. In view of Corollary 1.2, it is enough to show that if T is unitary, then it is reflexive. Let $a, b \in M$ be such that $ab \in T$. Since M is E-dense, a has a weak inverse c by Proposition 7.3. Now T is weakly self conjugate and so $caba \in T$. But ca is idempotent and hence is in T, and T is unitary so that $ba \in T$. Thus T is reflexive. $\square$

When the monoid M is E-dense, the proposition applies to $D(M)$ and we say that M is *D-unitary* if $D(M)$ is unitary in M. The following result is immediate from the proposition and the definitions of $D(M)$ and $\widetilde{D}(M)$.

Corollary 7.12. *If M is an E-dense, D-unitary monoid, then $D(M) = \widetilde{D}(M)$.*

We now have the following special case of Theorem 6.10.

Theorem 7.13. *For a monoid M, the following conditions are equivalent:*

(1) *M is E-dense and there is a group G and a surjective morphism $\varphi\colon M \to G$ with $1\varphi^{-1} = D(M)$,*
(2) *M is E-dense and D-unitary,*
(3) *M is E-dense and the morphism $\eta\colon M \to \pi_1(M)$ is surjective with $1\eta^{-1} = D(M)$,*
(4) *M is isomorphic to C/G where G is a group acting freely and transitively on a strongly connected E-dense category C with $D(C) = \delta(C)$.*

Proof. We know that $D(M) \subseteq \widetilde{D}(M)$ and so if (1) or (3) hold, then by Corollary 6.4, $D(M) = \widetilde{D}(M)$. Hence (1) and (3) are equivalent by Theorem 6.10. Also, if these conditions hold, then $D(M)$ is planar and so (2) holds. If (2) holds, then $D(M) = \widetilde{D}(M)$ by Corollary 7.12 and so (1) and (3) hold by Theorem 6.10.

If (1) holds, then by Proposition 7.2, $C(\varphi)$ is E-dense. Further, by Proposition 4.4, M is isomorphic to $C(\varphi)/G$ and $D(M) = \widetilde{D}(M)$ so that using Proposition 6.7,

$$D(C(\varphi))/G = D(C(\varphi)/G) = \widetilde{D}(C(\varphi)/G) = \widetilde{D}(C(\varphi))/G$$

and consequently, $D(C(\varphi)) = \widetilde{D}(C(\varphi))$. Condition (4) now follows from the proofs of Theorems 4.5 and 6.10.

If (4) holds, then $D(C) = \widetilde{D}(C) = \delta(C)$ so that $D(C/G) = \widetilde{D}(C/G)$ by Proposition 6.7 and hence $D(M) = \widetilde{D}(M)$. Moreover, M is E-dense by Proposition 7.1. Condition (3) now follows by Theorem 6.10. □

Before we specialise our results to important subclasses of E-dense monoids, we note the following result which strengthens [19, Proposition 1.2].

Lemma 7.14. *If M is an E-dense, E-unitary monoid, then $E(M)$ is a weakly self-conjugate submonoid of M, and $E(M) = D(M)$.*

Proof. First we verify that $E(M)$ is a submonoid. Let $e, f \in E(M)$. By denseness there exists an element $x \in M$ such that $(ef)x \in E(M)$. Since $E(M)$ is unitary, we have $fx \in E(M)$, so $x \in E(M)$, and hence $ef \in E(M)$.

Next we consider $a \in M$, $b \in W(a)$ and $e \in E(M)$. Then $bab = b$ and $ba \in E(M)$. So

$$(aeb)^2 = aebaeb = aebaebab = a(eba)^2b = aebab = aeb.$$

Thus $aeb \in E(M)$. Similarly, $bea \in E(M)$ so that $E(M)$ is a weakly self-conjugate submonoid. Then $E(M) = D(M)$ by definition of $D(M)$. □

When we specialise to *E-monoids*, that is, monoids in which the idempotents form a subsemigroup, we obtain the following corollary which combines two theorems of [1]. Note that each condition in the corollary forces $E(M)$ to be a submonoid of M.

Corollary 7.15. *For a monoid M, the following conditions are equivalent:*

(1) *there is a group G and a surjective morphism $\varphi\colon M \to G$ with $1\varphi^{-1} = E(M)$,*
(2) *M is E-dense and E-unitary,*
(3) *the morphism $\eta\colon M \to \pi_1(M)$ is surjective and $1\eta^{-1} = E(M)$,*
(4) *M is isomorphic to C/G where G is a group acting freely and transitively on a strongly connected, locally idempotent category C.*

Proof. Certainly (3) implies (1) and if (1) holds, then $E(M)$ is a submonoid of M since $E(M) = 1\varphi^{-1}$. By the monoid version of Theorem 2.1, $E(M)$ is unitary. Further, for every $a \in M$ there is an element b of M such that $(a\varphi)(b\varphi) = 1$. Hence $ab \in 1\varphi^{-1}$ and so $E(M)$ is dense in M. Thus (2) holds.

If (2) holds, then by Lemma 7.14, $D(M) = E(M)$, and hence (1) and (3) follow by Theorem 7.13.

If (1)–(3) hold, then by Theorem 7.13, there is a group G acting freely and transitively on a strongly connected category C such that M is isomorphic to C/G and $D(C) = \delta(C)$. But $D(C)/G = D(C/G) \cong D(M) = E(M)$ so that if $x \in \operatorname{Mor}\delta(C)$, then $Gx = G(x+x)$. Lemma 3.1 then implies that $x = x + x$, that is, $\delta(C)$ consists of idempotent morphisms and C is locally idempotent.

If (4) holds, then $\delta(C)$ consists of idempotent morphisms so that certainly $D(C) = \delta(C)$. Since C is strongly connected, $\delta(C)$ is strongly dense so that C is E-dense. Hence by Theorem 7.13, M is E-dense and D-unitary. Now by Proposition 6.7,

$$D(C/G) = D(C)/G = E(C)/G$$

so that $D(M) \subseteq E(M)$ and hence $D(M) = E(M)$, that is, condition (2) holds. $\square$

We now turn our attention to covering theorems. We say that an E-dense monoid $\widehat{M}$ is a *D-unitary cover* of an E-dense monoid M if $\widehat{M}$ is D-unitary and there is a surjective morphism $\theta\colon \widehat{M} \to M$ such that the restriction of θ to $D(\widehat{M})$ is an isomorphism of $D(\widehat{M})$ onto $D(M)$. In other words, a D-unitary cover of an E-dense monoid M is a $D(M)$-cover $\widehat{M}$ which is E-dense and such that $\widehat{D(M)} = D(\widehat{M})$.

Theorem 7.16. *Every E-dense monoid has an E-dense, D-unitary cover.*

Proof. Since M is E-dense, $D = D(M)$ is strongly dense in M by Corollary 7.4. Also, by Proposition 7.3, $W(a)$ is nonempty for each a in M. Hence in the proof of Theorem 6.11 we may choose $\gamma_{D(M)}(a)$ to be $W(a)$ and with this choice we have a monoid $\widehat{M}$ and a surjective morphism $\theta\colon \widehat{M} \to M$ such that $\widehat{D(M)} = D(\widehat{M}) = \widetilde{D}(\widehat{M})$. Hence $\widehat{M}$ is D-unitary and it remains to be shown that $\widehat{M}$ is E-dense.

By the definition of $D(M)$-cover, we have that $\widehat{D(M)}$, that is $D(\widehat{M})$, is dense in $\widehat{M}$. By Proposition 7.10, $D(M)$ is E-dense and hence so is $D(\widehat{M})$ since $D(M)$ is isomorphic to $D(\widehat{M})$. Now $E(M)$ is dense in $D(\widehat{M})$ and $D(\widehat{M})$ is dense in $\widehat{M}$. It is not difficult to infer that $\widehat{M}$ is E-dense. $\square$

We now easily obtain the covering results of [1], [60] and [18], as a consequence of Theorem 7.16 and Lemma 7.14.

Theorem 7.17. *Every E-dense E-monoid has an E-unitary E-dense cover.*

To conclude this section we give the specialisation to the E-dense case of Proposition 5.7. Given an E-dense monoid M, let $\widehat{M}$ be the E-dense, D-unitary cover of Theorem 7.16 and let $\alpha_M\colon \widehat{M} \to M$ be the covering map.

Proposition 7.18. *Let M, N be E-dense monoids and let $\varphi\colon M \to N$ be a morphism. Then there is a morphism $\widehat{\varphi}\colon \widehat{M} \to \widehat{N}$ such that the square*

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\varphi}} & \widehat{N} \\ {\scriptstyle \alpha_M}\downarrow & & \downarrow{\scriptstyle \alpha_N} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

is commutative. If $D(M)\varphi = D(N)$, $W(a)\varphi = W(a\varphi)$ and φ is surjective, then $\widehat{\varphi}$ is surjective.

Proof. It is clear that $W(a)\varphi \subseteq W(a\varphi)$ for all $a \in M$ and that $E(M)\varphi \subseteq E(N)$. In addition, Lemma 6.3 shows that $D(M)\varphi \subseteq D(N)$. The proposition is now immediate from Proposition 5.7. $\square$

8. Regular monoids

We apply the results of previous sections to regular monoids to obtain some new results and recover some old ones. In the class of regular monoids, McAlister's theorems for inverse semigroups were extended first to orthodox semigroups by McAlister [40], Szendrei [51, 52] and Takizawa [53, 54]. Independently, in [40], [51] and [54], they provided a generalisation of the covering theorem. The structure theorem was extended to E-unitary $\mathscr{R}$-unipotent monoids in [53] and to E-unitary orthodox monoids in [52]. As in the E-dense case, an arbitrary regular monoid cannot have an E-unitary regular cover. Recently, however, the covering theorem has been extended to regular monoids by Trotter [58]. Rather than being E-unitary, the covers are D-unitary regular monoids. Moreover, $D(M)$ is equal to the self conjugate core of M, defined as the least full self conjugate submonoid of M. In [58], Trotter proves that any regular monoid has a D-unitary regular cover. Indeed, he shows that the cover can be chosen to be in the same variety or e-variety as the semigroup to be covered provided that the variety contains all groups.

As usual, if M is a regular monoid and $a \in M$, we denote the set of inverses of a in M by $V(a)$. We adopt the same notation for categories. Thus, if C is a regular category and $p \in \operatorname{Mor} C$, then

$$V(p) = \{q \in \operatorname{Mor} C \mid p + q + p = p \text{ and } q + p + q = q\}.$$

Notice that $V(p) \subseteq \operatorname{Mor}(\omega(p), \alpha(p))$ for any morphism p of C. A submonoid N (subcategory D) of a regular monoid M (category C) is *self-conjugate* if $aNa' \subseteq N$ for all $a \in M$ and all $a' \in V(a)$ ($p + r + q \in \operatorname{Mor} D$ for all $p \in \operatorname{Mor} C$, $r \in \operatorname{Mor} D \cap \operatorname{Mor}(\omega(p), \omega(p))$ and $q \in V(p)$). The least self-conjugate submonoid (subcategory) of a regular monoid M (category C) is denoted by $C_\infty(M)$ ($C_\infty(C)$). (If we want to deal with semigroups and semigroupoids instead of monoids and categories, $C_\infty(M)$ ($C_\infty(C)$) must be defined as the least *full*, self-conjugate subsemigroup (subsemigroupoid) of the semigroup M (semigroupoid C).)

Lemma 8.1. *Let C be a regular category and D be a subcategory with* $\operatorname{Obj} D = \operatorname{Obj} C$. *Then D is weakly self-conjugate if and only if it is self-conjugate. In particular $C_\infty(C) = D(C)$.*

Proof. Clearly, D is self-conjugate if it is weakly self-conjugate. Suppose that D is self-conjugate and let $p \in \operatorname{Mor} C$, $\overline{p} \in W(p)$, $r \in \operatorname{Mor} D \cap \operatorname{Mor}(\omega(p), \omega(p))$ and $p^* \in V(p)$. Then

$$\begin{aligned} p + r + \overline{p} &= p + r + \overline{p} + p + \overline{p} = p + r + \overline{p} + p + p^* + p + \overline{p} \\ &= p + (r + \overline{p} + p) + p^* + (p + \overline{p}) \end{aligned}$$

and since $E(C) \subseteq D$ we have $p + \overline{p} \in \operatorname{Mor} D$ and $r + \overline{p} + p \in \operatorname{Mor} D$. But D is self-conjugate and so it follows that $p + r + \overline{p} \in \operatorname{Mor} D$. Similarly, if $s \in \operatorname{Mor} D \cap \operatorname{Mor}(\alpha(p), \alpha(p))$, then $\overline{p} + s + p \in \operatorname{Mor} D$ so that D is weakly self-conjugate. □

We record the monoid version of Lemma 8.1 (see [7, Fact 2.4]).

Corollary 8.2. *Let T be a submonoid of a regular monoid M. Then T is weakly self-conjugate if and only if it is self-conjugate. In particular, $C_\infty(M) = D(M)$.*

A regular monoid M is C_∞*-unitary* if $C_\infty(M)$ is unitary in M. In other words, M is C_∞-unitary if it is D-unitary. We specialise Theorem 7.13 to obtain the following result. The equivalence of (2) and (3) is essentially [58, Lemma 1.2 (ii)].

Theorem 8.3. *For a monoid M, the following conditions are equivalent:*

1. *M is regular and there is a group G and a surjective morphism $\varphi\colon M \to G$ with $1\varphi^{-1} = C_\infty(M)$,*
2. *M is regular and C_∞-unitary,*
3. *M is regular and the morphism $\eta\colon M \to \pi_1(M)$ is surjective with $1\eta^{-1} = C_\infty(M)$,*
4. *M is isomorphic to C/G where G is a group acting freely and transitively on a strongly connected regular category C with $C_\infty(C) = \delta(C)$.*

Proof. The equivalence of (1), (2) and (3) is immediate from Theorem 7.13 and Corollary 8.2.

If (1) holds, then by Proposition 7.2, $C(\varphi)$ is regular and so (4) follows from Theorem 7.13 by taking $C = C(\varphi)$ since $C_\infty(C) = D(C)$ by Corollary 8.2.

If (4) holds, then by Proposition 7.1, C/G is regular so that M is regular and hence (1) holds by Theorem 7.13 and Lemma 8.1 and Corollary 8.2. □

Before we proceed, we need a few facts about $C_\infty(M)$.

Lemma 8.4. *Let M be a regular monoid. Then*

1. [57] *$C_\infty(M)$ is a full regular submonoid of M which contains all the inverses of its elements.*
2. [29] *If $\varphi\colon M \to N$ is a morphism of regular monoids, then $C_\infty(N) = (C_\infty(M))\varphi$.*

Turning now to covering theorems, we first recall the notion of a *sandwich set* originally due to Nambooripad [42]. The sandwich $S(e, f)$ of two idempotents e, f in a monoid M is a subset of the set $\mathscr{M}(e, f)$ defined in the previous section. To be precise,

$$\begin{aligned} S(e, f) &= \mathscr{M}(e, f) \cap V(ef) \\ &= \{g \in V(ef) \cap E(M) \mid ge = fg = g\}. \end{aligned}$$

In a regular monoid all sandwich sets are non-empty and Nambooripad [42] (see also [27]) proved the following lemma, an analogue of Lemma 7.8.

Lemma 8.5. *Let a, b be elements of a regular monoid and let $a' \in V(a)$, $b' \in V(b)$ and $g \in S(a'a, bb')$. Then $b'ga' \in V(ab)$.*

We can now prove the following theorem due to Trotter [58]. He defines a regular monoid $\widehat{M}$ to be a C_∞-*unitary cover* for a regular monoid M if $\widehat{M}$ is C_∞-unitary and there is a surjective morphism $\theta\colon \widehat{M} \to M$ such that θ maps $C_\infty(\widehat{M})$ isomorphically onto $C_\infty(M)$. In view of Corollary 8.2, a C_∞-unitary cover is a regular D-unitary cover and vice-versa.

Theorem 8.6. *Every regular monoid has a regular C_∞-unitary cover.*

Proof. We use the construction of Theorem 7.16 to give an E-dense, D-unitary cover $\widehat{M}$ for M. Since $D(M) = C_\infty(M)$, we have only to show that $\widehat{M}$ is regular. To do this, we prove that the category C used in the construction is regular.

Let (g, m, h) be a morphism of C. If $g = h$, then $m \in M_1$ and $M_1 = C_\infty(M)$ is regular by Lemma 8.4 (1) so that there is an inverse m' of m in M_1. Hence $(g, m', h) \in \operatorname{Mor} C$ is an inverse of (g, m, h).

If $g \neq h$, then $m \in M_{g^{-1}h}$ so that if $w = x_1 \dots x_n$ where $x_i \in X$ is the reduced word representing $g^{-1}h$, then $m = m_1 \dots m_n$ for some $m_i \in M_{x_i}$. By Lemma 8.5 applied $n-1$ times, if $m_i' \in V(x_i)$ for $i = 1, \dots, n$, then there are idempotents $e_1, \dots, e_{n-1}$ such that $m' = m_n' e_{n-1} m_{n-1}' \dots e_1 m_1'$ is an inverse of m.

If $x_i \in M$, then $m_i = c x_i d$ for some $c, d \in C_\infty(M)$ so that if $x_i' \in V(x_i)$ and c' and d' are inverses of c and d in $C_\infty(M)$, then there are idempotents f_i, g_i such that $m_i' = d' f_i x_i' g_i c'$ is an inverse of m_i. Now $d' f_i, g_i c' \in C_\infty(M)$ so that $m_i' \in M_{\overline{x_i}}$.

If $x_i = \overline{a}$ for some $a \in M$, then $m_i = cbd$ for some $b \in W(a)$ and $c, d \in C_\infty(M)$. Hence, choosing inverses c', d' in C_∞ of c, d and any inverse b' of b, we can take $m_i' = d' f b' g c'$ for some idempotents f, g. Hence

$$m_i' = d' f b' b b' g c' = d' f b' b a b b' g c'$$

which is in M_a since $d' f b' b$ and $b b' g c'$ are in C_∞.

It follows that $m' \in M_v$ where $v = \overline{x}_n \dots \overline{x}_1$. Now in G we have $[v] = [w]^{-1} = h^{-1} g$ so that $(h, m', g) \in \operatorname{Mor}(h, g)$ and this is an inverse of (g, m, h). Thus C is regular. ☐

Remark. In the proof of Theorem 8.6, we could also use an analogous construction based on inverses rather than weak inverses, which would be more natural in the context of regular monoids.

In an orthodox monoid M we have $C_\infty(M) = E(M)$ and so the following corollary is immediate. We thus obtain the covering theorem discovered independently by McAlister, Szendrei, and Takizawa for orthodox semigroups [40, 51, 54] and consequently the original covering theorem of McAlister for inverse monoids [38, 39].

Corollary 8.7. (1) *Every orthodox monoid has an E-unitary orthodox cover.*

(2) *Every inverse monoid has an E-unitary inverse cover.*

Finally we specialise Proposition 7.18 to the regular case. We denote the covering map of the cover obtained in Theorem 8.6 by $\alpha_M \colon \widehat{M} \to M$.

Proposition 8.8. *Let M, N be regular monoids and let $\varphi \colon M \to N$ be a morphism. Then there is a morphism $\widehat{\varphi} \colon \widehat{M} \to \widehat{N}$ such that the square*

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{\varphi}} & \widehat{N} \\ {\scriptstyle \alpha_M}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_N} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

is commutative. If φ is surjective and $W(a)\varphi = W(a\varphi)$ for all $a \in M$, then $\widehat{\varphi}$ is surjective.

Proof. If φ is surjective, then by (2) of Lemma 8.4, $(C_\infty(M))\varphi = C_\infty(N)$ and the result is now immediate by Proposition 7.18. □

9. Finite monoids

In finite semigroup theory most of the classification results are stated not "up to isomorphism" but "up to division". For instance, the well-known Krohn-Rhodes theorem states that every finite monoid divides (i.e., is a quotient of a submonoid of) a wreath product of groups and aperiodic monoids. A good account of this point of view is given in Rhodes' survey [49]. This approach amounts to working with relational morphisms rather than morphisms.

In this section, we briefly consider finite versions of our main theorems. First we observe that a finite monoid is always E-dense, and thus the results of Section 7 can be applied directly. In particular, the proof of Theorem 7.13 can be readily adapted to obtain the following structure theorem.

Theorem 9.1. *For a finite monoid M, the following conditions are equivalent:*

(1) *There is a finite group G and a surjective morphism $\varphi \colon M \to G$ with $1\varphi^{-1} = D(M)$,*
(2) *M is D-unitary,*
(3) *The morphism $\eta \colon M \to \pi_1(M)$ is surjective with $1\eta^{-1} = D(M)$,*
(4) *M is isomorphic to C/G where G is a finite group acting freely and transitively on a finite strongly connected category C with $D(C) = \delta(C)$,*
(5) *M is isomorphic to C/G where G is a finite group acting freely and transitively on a finite strongly connected category C whose maximum groupoid quotient is simplicial.*

Although Theorem 7.16 guarantees the existence of an E-dense, D-unitary cover for a finite monoid, the cover need not be finite. However, as mentioned in the introduction, a finite covering theorem is a straightforward consequence of Ash's work [4, 5]. More precisely, by Proposition 3.4 of [4] and Proposition 4.1 of [47], if M is a finite monoid, then there is a finite group G and a surjective relational morphism $\tau\colon M \to G$ such that $D(M) = 1\tau^{-1}$. The next theorem (which has also been obtained by Trotter and Zhonghao Jiang [59]) thus follows from Corollary 5.4 since, when a finite group is used, the category constructed in Lemma 5.3 is finite.

Theorem 9.2. *Every finite monoid has a finite D-unitary cover.*

It would be of interest to have an independent proof of this covering theorem because Proposition 3.4 of [4] is a simple consequence.

If we specialise these results to E-monoids, we recover the results of Birget, Margolis, and Rhodes [7]. In this case, the covering theorem states that every finite E-monoid has a finite E-unitary cover and the structure theorem is the finite version of Corollary 7.15.

Corollary 9.3. *For a finite monoid M, the following conditions are equivalent:*

(1) *there is a finite group G and a surjective morphism $\varphi\colon M \to G$ with $1\varphi^{-1} = E(M)$,*
(2) *M is E-unitary,*
(3) *the morphism $\eta\colon M \to \pi_1(M)$ is surjective and $1\eta^{-1} = E(M)$,*
(4) *M is isomorphic to C/G where G is a finite group acting freely and transitively on a finite strongly connected, locally idempotent category C.*

There is a corresponding result for finite regular monoids similar to Theorem 8.3, which we omit. It was also given in [59] where it is noted that a cover may be chosen to be in the same variety, e-variety or pseudovariety as the monoid to be covered, provided all finite groups are contained in the relevant class.

If idempotents commute, the results of [2, 3] and [36] follow from Theorem 9.2 and Corollary 9.3.

Appendix

In Section 6 we introduced the monoid M with presentation

$$\mathrm{Mon}(a, t_i, t_i^{-1} \mid t_i^{-1} a t_i = a^{p_i}, t_i t_i^{-1} = t_i^{-1} t_i = 1, t_i t_j = t_j t_i \text{ for all } i, j \in \mathbb{N})$$

where p_i is the ith prime. We now examine the properties of M in some detail. First, we observe that the presentation looks very similar to that of an *HNN* extension (see [26, 13]).

Indeed, M is obviously a quotient of the *HNN* extension with presentation

$$\mathrm{Mon}(a, t_i, t_i^{-1} \mid t_i^{-1} a t_i = a^{p_i}, t_i t_i^{-1} = t_i^{-1} t_i = 1, \text{ for all } i \in \mathbb{N})$$

but this fact does not shed much light on the structure of M. To study M we find a normal form for its elements and use this to show that M is reversible and cancellative. Recall that M is *reversible* if, for any $x, y \in M$, there exist $t, t', z, z' \in M$ such that $tx = t'y$ and $xz = yz'$.

It then follows from Theorem 1.24 of [10] that M has a group of quotients. It is well known and easy to see that if a monoid S has a group of quotients G, then G is isomorphic to $\pi_1(S)$ and that if we have a monoid presentation for S, then the same presentation is a group presentation for G. Thus in our case,

$$\mathrm{gp}(a, t_i \mid t_i^{-1} a t_i = a^{p_i}, t_i t_j = t_j t_i \text{ for all } i, j \in \mathbb{N})$$

is a group presentation for $\pi_1(M)$. Finally, we show that M has a maximum group quotient which is not isomorphic to $\pi_1(M)$.

Let F be the free abelian group on $\{t_i \mid i \in \mathbb{N}\}$. Note that if $\langle a \rangle$ denotes the cyclic monoid generated by a, then M is the quotient of the *monoid* free product $\langle a \rangle * F$ by the congruence generated by $\{(t_i^{-1} a t_i, a^{p_i}) \mid i \in \mathbb{N}\}$. If $x \in F$, then $x = \prod_{i \geqslant 1} t_i^{n_i}$ where the n_i are integers and only finitely many of them are non-zero. We say that t_i *occurs in* x if $n_i \neq 0$. We define $e(x)$ to be the integer $\prod_{i \geqslant 1} p_i^{|n_i|}$. The element x is *non-negative* if $n_i \geqslant 0$ for all i and is *non-positive* if $n_i \leqslant 0$ for all i. Let

$$C = \{x \in F \mid x \text{ is non-negative}\},$$
$$D = \{x \in F \mid x \text{ is non-positive}\}.$$

and note that both C and D are submonoids of F. Let

$$\mathscr{N} = \{(c, a^n, d) \in C \times \langle a \rangle \times D \mid \text{if } t_i \text{ occurs in both } c \text{ and } d, \text{ then } p_i \not| \, n\}.$$

Theorem 1. *Every element of M can be expressed uniquely in the form $ca^n d$ for some $(c, a^n, d) \in \mathscr{N}$.*

Proof. If $m \in M$, then $m = a^{n_1} x_1 a^{n_2} x_2 \ldots x_k a^{n_k}$ for some non-negative integers $n_1, \ldots, n_k$ and elements $x_1, \ldots, x_k$ of F. Each x_j can be written as $c_j d_j$ for some $c_j \in C$, $d_j \in D$ having no letters in common. Now repeatedly using $t_i^{-1} a = a^{p_i} t_i^{-1}$ and $a t_i = t_i a^{p_i}$ we obtain $m = c_1 \ldots c_k a^n d_1 \ldots d_k$ for some $n \geqslant 0$. Putting $c = c_1 \ldots c_k$, $d = d_1 \ldots d_k$, we have $c \in C$, $d \in D$ and $m = ca^n d$. For a non-negative integer $r = p_i s$ we have

$$t_i a^r t_i^{-1} = (t_i a^{p_i} t_i^{-1})^s = a^s \tag{1}$$

so that if p_i is a factor of n and t_i occurs in both c and d, then repeated use of (1) gives an expression for m in the desired form.

To prove uniqueness we construct an injective morphism φ^* from M into $T(\mathscr{N})$, the monoid of transformations of the set $\mathscr{N}$.

For each t_i, define $t_i\varphi$ by the rule that

$$(c, a^n, d)(t_i\varphi) = \begin{cases} (c, a^n, dt_i) & \text{if } t_i \text{ occurs in } d, \\ (ct_i, a^{np_i}, d) & \text{otherwise,} \end{cases}$$

and $t_i^{-1}\varphi$ by the rule that

$$(c, a^n, d)(t_i^{-1}\varphi) = \begin{cases} (ct_i^{-1}, a^{\ell_i}, d) & \text{if } t_i \text{ occurs in } c \text{ and } n = \ell_i p_i, \\ (c, a^n, dt_i^{-1}) & \text{otherwise.} \end{cases}$$

It is clear that $t_i\varphi$ and $t_i^{-1}\varphi$ are well-defined mappings from $\mathscr{N}$ into itself and that

$$(t_i\varphi)(t_i^{-1}\varphi) = I_{T(\mathscr{N})} = (t_i^{-1}\varphi)(t_i\varphi) \quad \text{and} \quad (t_i\varphi)(t_j\varphi) = (t_j\varphi)(t_i\varphi)$$

for all $i, j \in \mathbb{N}$. Hence φ extends to a morphism $\varphi\colon F \to T(\mathscr{N})$.

Next we define $a\varphi \in T(\mathscr{N})$ by the rule that

$$(c, a^n, d)(a\varphi) = (c, a^{n+e(d)}, d).$$

If t_i occurs in d, then $p_i | e(d)$ so that if t_i also occurs in c, then p_i is not a factor of n and hence not a factor of $n + e(d)$. Thus $(c, a^{n+e(d)}, d) \in \mathscr{N}$ so that $a\varphi \in T(\mathscr{N})$ and we obtain a morphism $\varphi\colon \langle a \rangle \to T(\mathscr{N})$.

The universal property of free products now ensures that we have a morphism

$$\varphi\colon \langle a \rangle * F \to T(\mathscr{N})$$

given by

$$(a^{n_1} x_1 a^{n_2} x_2 \dots x_k a^{n_k})\varphi = (a^{n_1}\varphi)(x_1\varphi) \dots (x_k\varphi)(a^{n_k}\varphi)$$

where $x_i \in F$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots k$.

We now show that $(t_i^{-1}\varphi)(a\varphi)(t_i\varphi) = a^{p_i}\varphi$. Let $(c, a^n, d) \in \mathscr{N}$ and suppose first that t_i occurs in c and $p_i | n$. Let $n = p_i \ell_i$ and note that t_i does not occur in d. Then

$$\begin{aligned} (c, a^n, d)(t_i^{-1}\varphi)(a\varphi)(t_i\varphi) &= (ct_i^{-1}, a^{\ell_i}, d)(a\varphi)(t_i\varphi) = (ct_i^{-1}, a^{\ell_i + e(d)}, d)(t_i\varphi) \\ &= (c, a^{p_i(\ell_i + e(d))}, d) = (c, a^{n + e(d)p_i}, d) \\ &= (c, a^n, d)(a\varphi)^{p_i}. \end{aligned}$$

Now suppose that p_i is not a factor of n or that t_i does not occur in c. Then

$$\begin{aligned}(c, a^n, d)(t_i^{-1}\varphi)(a\varphi)(t_i\varphi) &= (c, a^n, dt_i^{-1})(a\varphi)(t_i\varphi) = (c, a^{n+e(d)p_i}, dt_i^{-1})(t_i\varphi) \\ &= (c, a^{n+e(d)p_i}, d) = (c, a^n, d)(a\varphi)^{p_i}.\end{aligned}$$

Thus $(t_i^{-1}\varphi)(a\varphi)(t_i\varphi) = a^{p_i}\varphi$ as claimed.

Consequently, φ induces a morphism $\varphi^*\colon M \to T(\mathscr{N})$. Moreover, for any $(c, a^n, d) \in \mathscr{N}$, we have

$$(ca^n d)\varphi^* = (c\varphi)(a^n\varphi)(d\varphi).$$

To see that φ^* is injective, let $(c, a^n, d) \in \mathscr{N}$ and let $m = ca^n d$. Then

$$(1,1,1)(m\varphi^*) = (1,1,1)(c\varphi)(a^n\varphi)(d\varphi) = (c, a^n, d)$$

so that for $m, m' \in M$, the functions $m\varphi^*$ and $m'\varphi^*$ agree on $(1,1,1)$ if and only if $m = m'$. Thus φ^* is injective. $\square$

From now on we say that an expression $ca^n d$ is in *normal form* if $(c, a^n, d) \in \mathscr{N}$. The next two propositions together show that M has a group of quotients.

Proposition 2. *The left and right cancellation laws hold in M.*

Proof. Let $ca^n d, c_1a^{n_1}d_1, c_2a^{n_2}d_2$ be elements of M in normal form and suppose that

$$(ca^n d)(c_1a^{n_1}d_1) = (ca^n d)(c_2a^{n_2}d_2).$$

Since t_i has an inverse in M and $t_it_j = t_jt_i$ for all $i, j \in \mathbb{N}$, we obtain

$$a^n dc_1a^{n_1}d_1' = a^n dc_2a^{n_2}d_2'$$

where $d_1', d_2' \in D$ have no letters in common and $d_1 = d_1'd', d_2 = d_2'd'$ for some $d' \in D$.

We claim that $a^n dc_1a^{n_1}d_1'$ has normal form $c_3a^r d_3d_1'$ where $c_3 \in C$ and $d_3 \in D$ have no letters in common, $dc_1 = c_3d_3$ and $r = e(c_3)n + e(d_3)n_1$. That $a^n dc_1a^{n_1}d_1' = c_3a^r d_3d_1'$ is an easy consequence of the defining relations of M.

To see that $c_3a^r d_3d_1'$ is in normal form, suppose that t_i occurs in both c_3 and d_3d_1'. Then t_i does not occur in d_3 and so must occur in d_1' and hence in d_1. Further, $dc_1 = c_3d_3$ and t_i occurs as a positive power in c_3d_3 so t_i must occur in c_1. Now $c_1a^{n_1}d_1$ is in normal form and hence p_i is not a factor of n_1. Also, p_i is not a factor of $e(d_3)$ since t_i does not occur in d_3. On the other hand, $p_i|e(c_3)$ and so p_i cannot be a factor of r. Thus $c_3a^r d_3d_1'$ is in normal form as claimed.

Similarly, letting $dc_2 = c_4d_4$ where $c_4 \in C$ and $d_4 \in D$ have no letters in common, we have that $a^n dc_2a^{n_2}d_2'$ has normal form $c_4a^s d_4d_1'$ where $s = e(c_4)n + e(d_4)n_2$. Comparing the two normal forms gives $c_3 = c_4$, $d_3d_1' = d_4d_2'$ and $r = s$ whence $e(d_3)n_1 = e(d_4)n_2$.

Clearly, if $d_3 = d_4$, then $n_1 = n_2$, $c_1 = c_2$ and $d_1 = d_2$, so that $c_1a^{n_1}d_1 = c_2a^{n_2}d_2$.

Let $d_3 = d_3'd''$ and $d_4 = d_4'd''$ where $d_3', d_4', d'' \in D$ and d_3', d_4' have no letters in common. Then $e(d_h) = e(d_h')e(d'')$ for $h = 3, 4$ so that $e(d_3')n_1 = e(d_4')n_2$. Moreover, $d_4'd_2' = d_3'd_1'$ since $d_4d_2' = d_3d_1'$. In addition, since $dc_1 = c_3d_3$ and c_3, d_3 have no letters in common, we must have $d = \bar{d}d''$ for some $\bar{d} \in D$. It follows that $\bar{d}c_1 = c_3d_3'$ and $\bar{d}c_2 = c_4d_4'$.

Suppose that t_i occurs in d_3'. Then t_i occurs in d_2' since $d_4'd_2' = d_3'd_1'$. Also t_i occurs in $\bar{d}$, since $\bar{d}c_1 = c_3d_3'$ and c_3 and d_3' have no letters in common. Now d_3' and d_4' have no letters in common and $\bar{d}c_2 = c_4d_4'$, so t_i occurs in c_2. As $c_2a^{n_2}d_2'$ is in normal form, we see that p_i is not a factor of n_2. Also $p_i \nmid e(d_4')$ and so p_i is not a factor of $e(d_3')n_1$ contradicting the assumption that t_i occurs in d_3'. Thus $d_3' = 1$ and similarly, $d_4' = 1$ so that $d_3 = d_4$ and $n_1 = n_2$. Consequently, $c_1a^{n_1}d_1 = c_2a^{n_2}d_2$ and left cancellation holds in M.

A similar argument with the roles of C and D interchanged shows that M is right cancellative. □

Next we show that M is reversible.

Proposition 3. *The monoid M is left and right reversible.*

Proof. Let $c_1a^{n_1}d_1, c_2a^{n_2}d_2$ be elements of M in normal form. Put $d_1' = c_1^{-1}$ and $d_2' = c_2^{-1}$ so that $d_1', d_2' \in D$. Let $d_3, d_4 \in D$ be such that $d_3d_1 = d_4d_2$ and let k_1, k_2 be non-negative integers such that $k_1 + e(d_3)n_1 = k_2 + e(d_4)n_2$. Then

$$\begin{aligned}(a^{k_1}d_3d_1')(c_1a^{n_1}d_1) &= a^{k_1}d_3a^{n_1}d_1 = a^{k_1+e(d_3)n_1}d_3d_1 = a^{k_2}d_4a^{n_2}d_2\\ &= (a^{k_2}d_4d_2')(c_2a^{n_2}d_2)\end{aligned}$$

so that M is right reversible. A similar argument shows that M is also left reversible. □

As we remarked earlier, it now follows that $\pi_1(M)$ is the group of quotients of M and that

$$\mathrm{gp}(a, t_i \mid t_i^{-1}at_i = a^{p_i}, t_it_j = t_jt_i \text{ for all } i, j \in \mathbb{N})$$

is a group presentation for $\pi_1(M)$. It follows from Theorem 1 that $\langle a \rangle$ is embedded in M and hence in $\pi_1(M)$. Thus in $\pi_1(M)$, $at_1 = t_1a^2 \neq t_1a$ so that $\pi_1(M)$ is not abelian.

We now consider which groups can be quotient groups of M.

Proposition 4. *Let $\theta\colon M \to G$ be a surjective morphism onto a group G. Then G is abelian and countably generated.*

Proof. Let $a\theta = x$ and $t_i\theta = y_i$ for all $i \in \mathbb{N}$. Then $y_i^{-1}xy_i = x^{p_i}$ and $y_iy_j = y_jy_i$ for all $i, j \in \mathbb{N}$. Since θ is surjective, $x^{-1} = m\theta$ for some element m of M. Let m have normal form ca^nd. Then $x^{-1} = gx^nh$ where $g = c\theta$ and $h = d\theta$. Now $ac = ca^{n(c)}$ for some positive integer $n(c)$ since $c \in C$. Thus $xg = gx^{n(c)}$ so that $1 = xgx^n h = gx^{n(c)+n}h$ and $x^{n(c)+n} = g^{-1}h^{-1}$. Put $s = n(c) + n$ and $z = g^{-1}h^{-1}$ so that $x^s = z$. Now $xy_1 = y_1 x^2$ so that $zy_1 = x^s y_1 = y_1 x^{2s} = y_1 z^2$. Since z is a member of the subgroup generated by the elements y_i $(i \in \mathbb{N})$ and this subgroup is abelian, we have $z = 1$, that is, $x^s = 1$. Hence x has finite order since $s \neq 0$.

In G, however, $y_i^{-1} x y_i = x^{p_i}$ for all $i \in \mathbb{N}$, so that for all primes p, the elements x and x^p have the same order. It follows that $x = 1$.

Thus G is generated by $\{y_i \mid i \in \mathbb{N}\}$ and so G is abelian. □

Corollary 5. *The maximum group quotient of M is F, the free abelian group on the set $\{t_i \mid i \in \mathbb{N}\}$.*

Proof. We can define a morphism θ from M onto F by putting $a\theta = 1$, $t_i\theta = t_i$ and $t_i^{-1}\theta = t_i^{-1}$ for all $i \in \mathbb{N}$. It is clear from the proposition that every group quotient of M is a factor group of F and the corollary follows. □

Since $\pi_1(M)$ is not abelian it cannot be a quotient of M and so certainly not the maximum quotient.

References

[1] J. Almeida, J.-E. Pin, and P. Weil, *Semigroups whose idempotents form a subsemigroup*, Math. Proc. Camb. Phil. Soc. **111** (1992), 241–253.

[2] C.J. Ash, *Finite idempotent-commuting semigroups*, Semigroups and their applications (S. Goberstein and P. Higgins eds.), Reidel, Dordrecht, Holland, 1987, pp. 13–23.

[3] C.J. Ash, *Finite semigroups with commuting idempotents*, J. Austral. Math. Soc. (Series A) **43** (1987), 81–90.

[4] C.J. Ash, *Inevitable sequences and a proof of the type II conjecture*, Proceedings of the Monash Conference on Semigroup Theory, World Scientific, Singapore, 1991, pp. 31–42.

[5] C.J. Ash, *Inevitable Graphs: A proof of the type II conjecture and some related decision procedures*, Int. J. Alg. and Comp. **1** (1991), 127–146.

[6] M. Barr and C. Wells, *Category theory for computing science*, Prentice Hall, New York, 1990.

[7] J. C. Birget, S. Margolis, and J. Rhodes, *Semigroups whose idempotents form a subsemigroup*, Bull. Austral. Math. Soc. **41** (1990), 161–184.

[8] F. Catino and M. M. Miccoli, *On semidirect product of semigroup*, Note Mat. **9** (1989), 189–194.

[9] C. C. Chang and H. J. Keisler, *Model theory*, North Holland, Amsterdam, 1977.

[10] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. I, American Math. Soc., Providence, R.I., 1961.

[11] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. II, American Math. Soc., Providence, R.I., 1967.

[12] P. M. Cohn, *Universal algebra*, Harper and Row, New York, 1965.

[13] D. V. Dekov, *HNN extensions of semigroups*, Semigroup Forum, **49** (1994), 83–87.

[14] P. Dubreil, *Contribution à la théorie des demigroupes*, Mém. Acad. Sci. Inst. France **63** (1941), no. 3, 52pp.

[15] D. Easdown, *A new proof that regular biordered sets come from regular semigroups*, Proc. Roy. Soc. Edinb. **96A** (1984), 109–116.

[16] R. Feigenbaum, *Congruences on regular semigroups*, Semigroup Forum **17** (1979), 373–377.

[17] D. G. FitzGerald, *On inverses of products of idempotents in regular semigroups*, J. Austral. Math. Soc. **13** (1972), 335–337.

[18] J. Fountain, *E-unitary dense covers of E-dense monoids*, Bull. London Math. Soc. **22** (1990), 353–358.

[19] J. Fountain, *$\langle E \rangle$-dense monoids*, Monash conference on semigroup theory (T. E. Hall, P. R. Jones, and J. C. Meakin, eds.), World Scientific, Singapore, 1991, pp. 68–76.

[20] R. Goldblatt, *Topoi: the categorical analysis of logic*, North Holland, Amsterdam, 1984.

[21] G. M. S. Gomes, *A characterization of the group congruences on a semigroup*, Semigroup Forum **46** (1993), 48–53.

[22] T. E. Hall and W. D. Munn, *The hypercore of a semigroup*, Proc. Edinburgh Math. Soc. **28** (1985), 107–112.

[23] S. Hanumantha Rao and P. Lakshmi, *Group congruences on eventually regular semigroups*, J. Austral. Math. Soc. **45** (1988), 320–325.

[24] W. S. Hatcher, *Foundations of mathematics*, W.B.Saunders, Philadelphia, 1968.

[25] P. J. Higgins, *Categories and groupoids*, Van Nostrand, London, 1971.

[26] J. M. Howie, *Embedding theorems for semigroups*, Quarterly Journal of Mathematics **14** (1963), 254–258.

[27] J. M. Howie, *Fundamentals of semigroup theory*, Oxford University Press, Oxford, 1995.

[28] G. Lallement and M. Petrich, *Décomposition I-matricielles d'un demigroupe*, J. Math. Pures Appl. **45** (1966), 67–117.

[29] D. R. LaTorre, *Group congruences on regular semigroups*, Semigroup Forum **24** (1982), 327–340.

[30] F. W. Lawvere, *The category of categories as a foundation for mathematics*, Proceedings of the conference on categorical algebra (La Jolla 1965), Springer-Verlag, Berlin-Heidelberg-New York, 1966, pp. 1–20.

[31] F. W. Levi, *On semigroups*, Bull. Calcutta Math. Soc. **36** (1944), 141–146.

[32] F. W. Levi, *On semigroups II*, Bull. Calcutta Math. Soc. **38** (1946), 123–124.

[33] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[34] S. W. Margolis and J.-E. Pin, *Graphs, inverse semigroups and languages*, Proceedings of the 1984 Marquette conference on semigroups (K. Byleen, P. R. Jones, and F. J. Pastijn, eds.), Marquette University, Milwaukee, 1985, pp. 85–112.

[35] S. W. Margolis and J.-E. Pin, *Inverse semigroups and extensions of groups by semilattices*, J. Algebra **110** (1987), 277–297.

[36] S. W. Margolis and J.-E. Pin, *Inverse semigroups and varieties of finite semigroups*, J. Algebra **110** (1987), 306–323.

[37] F. E. Masat, *Right group and group congruences on a regular semigroup*, Duke. Math. J. **40** (1973), 393–402.
[38] D. B. McAlister, *Groups, semilattices and inverse semigroups*, Trans. Amer. Math. Soc. **192** (1974), 227–244.
[39] D. B. McAlister, *Groups, semilattices and inverse semigroups II*, Trans. Amer. Math. Soc. **196** (1974), 351–370.
[40] D. B. McAlister, *On a problem of M.P.Schützenberger*, Proc. Edinb. Math. Soc. **23** (1980), 243–247.
[41] H. Mitsch, *Subdirect products of E-inversive semigroups*, J. Austral. Math. Soc. **48** (1990), 66–78.
[42] K. S. S. Nambooripad, *Structure of regular semigroups I*, Mem. Amer. Math. Soc. (1979), no. 224.
[43] M. Petrich, *Sur certaines classes de demigroupes I*, Bull. Cl. Sci. Acad. Roy. Belgique **49** (1967), 785–798.
[44] M. Petrich, *Sur certaines classes de demigroupes II*, Bull. Cl. Sci. Acad. Roy. Belgique **49** (1967), 888–900.
[45] M. Petrich, *Inverse semigroups*, Wiley, New York, 1984.
[46] J.-E. Pin, *Variétés de langages formels*, Masson, Paris, 1984. English translation: *Varieties of formal languages*, North Oxford, London, and Plenum, New York, 1986.
[47] J.-E. Pin, *On a conjecture of Rhodes*, Semigroup Forum **39** (1989), 1–15.
[48] J.-E. Pin, *Finite semigroups as categories, ordered semigroups or compact semigroups*, Semigroup theory and its applications (K. H. Hofmann and M. W. Mislove, eds.), Cambridge U. P., London Mathematical Society Lecture Note Series **231**, (1996), 107–121.
[49] J. Rhodes, *The relationship of Al Clifford's work to the current theory of semigroups*, Semigroup theory and its applications (K. H. Hofmann and M. W. Mislove, eds.), Cambridge U. P., London Mathematical Society Lecture Note Series **231**, (1996), 43–51.
[50] Ĕ. G. Shutov, *Embeddings of semigroups into simple and complete semigroups. (Russian)*, Mat. Sb. (N.S.) **62(104)** (1963), 496–511.
[51] M. B. Szendrei, *On a pull-back diagram for orthodox semigroups*, Semigroup Forum **20** (1980), 1–10, corr. **25** (1982), 311–324.
[52] M. B. Szendrei, *A generalization of McAlister's P-theorem for E-unitary regular semigroups*, Acta Sci. Math. **51** (1987), 229–249.
[53] K. Takizawa, *E-unitary $\mathscr{R}$-unipotent semigroups*, Bull. Tokyo Gakugei Univ., Series IV **30** (1978), 21–33.
[54] K. Takizawa, *Orthodox semigroups and E-unitary regular semigroups*, Bull. Tokyo Gakugei Univ., Series IV **31** (1979), 41–43.
[55] G. Thierrin, *Demigroupes inversés et rectangulaires*, Bull. Cl. Sci. Acad. Roy. Belgique **41** (1955), 83–92.
[56] B. Tilson, *Categories as algebra: an essential ingredient in the theory of monoids*, J. Pure Appl. Alg. **48** (1987), 83–198.
[57] P. G. Trotter, *Congruence extensions in regular semigroups*, J. Algebra **137** (1991), 166–179.
[58] P. G. Trotter, *Covers for regular semigroups and an application to complexity*, J. Pure Appl. Alg. **105** (1995), 319–328.

[59] P. G. Trotter and Zhonghao Jiang, *Covers for regular semigroups*, Southeast Asian Bull. Math. **18** (1994), 157–161.

[60] Zhonghao Jiang, *E-unitary inversive covers for E-inversive semigroups whose idempotents form a subsemigroup*, Southeast Asian Bull. Math. **18** (1994), 59–64.