Some results on $\mathcal{C}$-varieties

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Abstract

In an earlier paper, the second author generalized Eilenberg's variety theory by establishing a basic correspondence between certain classes of monoid morphisms and families of regular languages. We extend this theory in several directions. First, we prove a version of Reiterman's theorem concerning the definition of varieties by identities, and illustrate this result by describing the identities associated with languages of the form $(a_1a_2\cdots a_k)^+$, where $a_1,\ldots,a_k$ are distinct letters. Next, we generalize the notions of Mal'cev product, positive varieties, and polynomial closure. Our results not only extend those already known, but permit a unified approach of different cases that previously required separate treatment.

Résumé

Dans un article antérieur, le second auteur avait proposé une extension de la théorie des variétés d'Eilenberg en établissant une correspondance entre certaines classes de morphismes de monoïdes et certaines classes de langages rationnels. Nous complétons cette théorie dans plusieurs directions. Nous commençons par étendre le théorème de Reiterman relatif à la définition des variétés par identités. Nous illustrons ce résultat en décrivant les identités attachées aux langages de la forme $(a_1a_2\cdots a_k)^+$, où $a_1,\ldots,a_k$ sont des lettres distinctes. Ensuite, nous généralisons les notions de produit de Mal'cev, de variétés positives et de fermeture polynomiale. Nos résultats permettent non seulement d'étendre les résultats déjà connus, mais proposent également une approche unifiée pour des cas qui nécessitaient jusqu'ici un traitement séparé.

1 Introduction

Work of Eilenberg and Schützenberger [3, 4] in the 1970's underscored the importance of *varieties* of finite semigroups and monoids (also called *pseudovarieties*) in the study of the behavior of finite automata and the languages they accept. Since that time, a rich research literature on varieties has arisen, treating both the applications to automata theory and the fundamental underlying algebra. (We refer the reader to [8, 2] for an account of recent progress and a comprehensive bibliography.)

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It was recognized very early on, particularly in the work of Brzozowski and Simon [15, 16] on locally testable languages, that in many instances one needs to study the structure of the syntactic semigroup of a language, rather than the syntactic monoid, and for this reason Eilenberg developed two parallel theories of varieties, one for finite semigroups, and the other for finite monoids.

In studying the circuit complexity of regular languages, Barrington, et al. [1] and Straubing [17] came across a curious phenomenon: Membership of a regular language in the circuit complexity class AC^0, as well as some related families, is not determined by the syntactic monoid or semigroup of the language, and therefore these language families do not correspond to varieties in the usual sense. However the same families do admit succinct algebraic characterizations in terms of the *syntactic morphism*, as well as characterizations in generalized first-order logic that closely resemble the descriptions found for varieties in the usual sense.

Straubing [18] generalized the definition of variety so as to include these examples. The elements of these new varieties are not semigroups or monoids, but morphisms from free finitely-generated monoids onto finite monoids, also called *stamps*. Associated with each such variety $\mathbf{V}$ is a category $\mathcal{C}$ of admissible morphisms between free finitely-generated monoids with the property that if $\varphi : A^* \to M$ is in $\mathbf{V}$ and $f : B^* \to A^*$ is in $\mathcal{C}$, then $\varphi \circ f : B^* \to M$ is in $\mathbf{V}$. When $\mathcal{C}$ is the family of all morphisms between finitely generated free monoids, $\mathbf{V}$ includes all the morphisms onto M, and becomes a variety of monoids in the usual sense. One likewise recovers the varieties of finite semigroups by restricting $\mathcal{C}$ to contain non-erasing morphisms. With additional restrictions on $\mathcal{C}$, one recovers the language families from [1].

In the original definition of these "$\mathcal{C}$-varieties", $\mathcal{C}$ was permitted to be any class of morphisms between free finitely-generated monoids as long as it is closed under composition. We have preferred to alter the definition so that $\mathcal{C}$ is required to contain all the length-preserving morphisms between free monoids. This condition is satisfied by all the examples encountered so far in applications, and it smoothes out various technical difficulties.

While [18] established the basic correspondence between $\mathcal{C}$-varieties and families of regular languages, fundamental questions about the underlying algebra were not considered. In the present paper we begin to fill this gap.

We first give in Section 3 a version of Reiterman's theorem [14] concerning the definition of $\mathcal{C}$-varieties by identities. Independently of us, Kunc [5] also developed the equational theory for $\mathcal{C}$-varieties. Kunc worked with the original definition of these varieties, but as a drawback, his identities must be interpreted in a non-standard manner. The new definition of $\mathcal{C}$-varieties allows one a simplified presentation which follows closely the corresponding proof for varieties of finite monoids, as given for instance in [6].

As an illustration, when $\mathcal{C}$ is the class of length-preserving morphisms, we give the identities describing the $\mathcal{C}$-variety generated by the syntactic morphism of a language of the form $(a_1a_2 \cdots a_k)^+$, where $a_1, \ldots, a_k$ are distinct letters. This example actually occurred in a preliminary study of the long-standing open conjecture that there are languages of generalized star-height > 1. This problem is indeed equivalent to the existence of non-trivial identities satisfied by the syntactic morphisms of all languages of generalized star-height ≤ 1.

In Section 4, we study an analogue of the Mal'cev product of $\mathcal{C}$-varieties, and show in particular that if $\mathcal{C}$ is the class of length-multiplying morphisms, the

$\mathcal{C}$-variety of quasi-aperiodic stamps, which occurs in [1], can be decomposed as a Mal'cev product of two simpler varieties. Identities for the Mal'cev product of two varieties are described in Section 6.

We outline in Section 5 the theory of positive $\mathcal{C}$-varieties, and use it in Section 7 to extend to this setting the results of [12] on the polynomial closure of a class of languages. In particular it permits to greatly simplify the presentation of [12], which originally needed two separate definitions for the polynomial closure: one when languages are considered as subsets of a free monoid, and another one when they are considered as subsets of a free semigroup.

2 $\mathcal{C}$-varieties

Let $\mathcal{C}$ be a class of morphisms between finitely generated free monoids that satisfies the following properties:

(1) $\mathcal{C}$ is closed under composition. That is, if A, B and C are finite alphabets, and $f : A^* \to B^*$ and $g : B^* \to C^*$ are elements of $\mathcal{C}$, then $g \circ f$ belongs to $\mathcal{C}$.

(2) $\mathcal{C}$ contains all length-preserving morphisms.

Examples include the classes of all length-preserving morphisms (morphisms for which the image of each letter is a letter), of all length-multiplying morphisms (morphisms such that, for some integer k, the length of the image of a word is k times the length of the word), all non-erasing morphisms (morphisms for which the image of each letter is a nonempty word), all length-decreasing morphisms (morphisms for which the image of each letter is either a letter of the empty word) and all morphisms.

A *stamp* is a morphism from a finitely generated free monoid onto a finite monoid. The *size* of a stamp $\varphi : A^* \to M$ is by definition the size of M. A $\mathcal{C}$*-morphism* from a stamp $\varphi : A^* \to M$ to a stamp $\psi : B^* \to N$ is a pair (f, α), where $f : A^* \to B^*$ is in $\mathcal{C}$, $\alpha : M \to N$ is a monoid morphism, and $\psi \circ f = \alpha \circ \varphi$.

$$\begin{tikzcd} A^* \arrow[r, "f"] \arrow[d, "\varphi"'] & B^* \arrow[d, "\psi"] \\ M \arrow[r, "\alpha"] & N \end{tikzcd}$$

Figure 2.1: A $\mathcal{C}$-morphism.

A $\mathcal{C}$-morphism (f, α) is a $\mathcal{C}$*-projection* if the map $f : A^* \to B^*$ satisfies $f(A) = B$. Note that, in this case, f and α are necessarily onto. Indeed, $f(A^*) = B^*$ and thus f is onto. Furthermore, ψ is onto, and thus $\psi \circ f = \alpha \circ \varphi$ is onto. It follows that α is onto.

A $\mathcal{C}$-morphism (f, α) is a $\mathcal{C}$*-inclusion* if the morphism $\alpha : M \to N$ is injective. In particular, if $\varphi : B^* \to M$ is a stamp, $f : A^* \to B^*$ is an element of $\mathcal{C}$ and

$\iota : \mathrm{Im}(\varphi \circ f) \to M$ is the inclusion morphism, then the pair (f, ι) is a $\mathcal{C}$-inclusion from $\varphi \circ f : A^* \to \mathrm{Im}(\varphi \circ f)$ into φ.

We say that a stamp $\varphi : A^* \to M$ $\mathcal{C}$*-divides* a stamp $\psi : B^* \to N$ and write $\varphi \prec \psi$ if there is a stamp $\theta : C^* \to K$, a $\mathcal{C}$-inclusion $(f, \alpha) : \theta \to \psi$ and a $\mathcal{C}$-projection $(g, \beta) : \theta \to \varphi$.

In [18], a different definition of division was given. There, it was said that φ $\mathcal{C}$-divides ψ if there is a *division diagram*

$$\begin{tikzcd}
A^* \arrow[rr, "f"] \arrow[d, "\varphi"'] & & B^* \arrow[d, "\psi"] \\
M & \mathrm{Im}(\psi \circ f) \subseteq \arrow[l, "\eta"'] & N
\end{tikzcd}$$

Figure 2.2: A division diagram.

with η onto, $f \in \mathcal{C}$ and $\varphi = \eta \circ \psi \circ f$.

Proposition 2.1 *The two definitions of division are equivalent.*

Proof. Suppose first that φ $\mathcal{C}$-divides ψ according to the definition of this paper. We then have a diagram with two commuting squares (see Figure 2.3 below) such that α is injective and $g(C) = A$. There thus exists $k : A \to C$ such that $g \circ k = 1_A$. The map k extends to a length-preserving morphism (also denoted by k) from A^* into C^*.

$$\begin{tikzcd}
A^* \arrow[r, dashed, bend right, "k"'] \arrow[d, "\varphi"'] & C^* \arrow[l, "g"'] \arrow[r, "h"] \arrow[d, "\theta"] & B^* \arrow[d, "\psi"] \\
M & K \arrow[l, "\beta"'] \arrow[r, "\alpha"] & N
\end{tikzcd}$$

Figure 2.3: φ $\mathcal{C}$-divides ψ.

Setting $f = h \circ k$ and $\eta = \beta \circ \alpha^{-1}$, we obtain the diagram of Figure 2.2. Now since $\mathcal{C}$ contains all length-preserving morphisms and is closed under composition, k and f are in $\mathcal{C}$. Next, observe that

$$\mathrm{Im}(\psi \circ f) = \psi \circ h(k(A^*)) \subseteq \psi \circ h(C^*) = \alpha \circ \theta(C^*) \subseteq \mathrm{Im}(\alpha)$$

so $\beta \circ \alpha^{-1}$ is well defined. Secondly,

$$\eta \circ \psi \circ f = \eta \circ \psi \circ h \circ k = \eta \circ \alpha \circ \theta \circ k = \beta \circ \alpha^{-1} \circ \alpha \circ \theta \circ k = \beta \circ \theta \circ k = \varphi \circ g \circ k = \varphi$$

Thus φ divides ψ in the sense of [18].

Conversely, suppose we have the division diagram of Figure 2.2. We then have a commutative diagram

$$\begin{tikzcd}
A^* \arrow[d, "\varphi"'] & A^* \arrow[l, "1_{A^*}"'] \arrow[r, "f"] \arrow[d, "\psi \circ f"] & B^* \arrow[d, "\psi"] \\
M & \mathrm{Im}(\psi \circ f) \arrow[l, "\eta"'] \arrow[r, "\iota"] & N
\end{tikzcd}$$

where ι is the inclusion morphism. This shows that φ divides ψ in the sense of this paper. $\square$

When the class $\mathcal{C}$ of morphisms is understood, we omit the prefix $\mathcal{C}$- and simply use the terms projection, inclusion and divides.

It follows directly from closure under composition and the definition of division given in [18] that division is transitive. Note that division is *not* antisymmetric, but if $\varphi \prec \psi$ and $\psi \prec \varphi$ then the finite monoids $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\psi)$ are isomorphic.

The *restricted direct product* of two stamps φ_1 and φ_2 is the stamp φ with domain A^* defined by $\varphi(a) = (\varphi_1(a), \varphi_2(a))$. The image of φ is a submonoid of the monoid $M_1 \times M_2$.

$$\begin{tikzcd}
 & & M_2 \\
A^* \arrow[urr, "\varphi_2"] \arrow[r, "\varphi"] \arrow[drr, "\varphi_1"'] & \mathrm{Im}(\varphi) \subseteq M_1 \times M_2 \arrow[ur, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
 & & M_1
\end{tikzcd}$$

Figure 2.4: The restricted direct product of two stamps.

A *$\mathcal{C}$-variety of stamps* is a class of stamps closed under $\mathcal{C}$-division and finite restricted direct products (possibly empty). Equivalently, a $\mathcal{C}$-variety is a class of stamps closed under $\mathcal{C}$-projections, $\mathcal{C}$-inclusions and finite restricted direct products.

When $\mathcal{C}$ is the class of all (resp. length-preserving length-multiplying, non-erasing, length-decreasing) morphisms, we use the term *all-variety* (resp. *lp-variety*, *lm-variety*, *ne-variety*, *de-variety*).

As an example of $\mathcal{C}$-variety, consider the class **MOD** of all stamps φ from a free monoid A^* onto a finite cyclic group such that, for all $a, b \in A$, $\varphi(a) = \varphi(b)$.

Proposition 2.2 *The class* **MOD** *is an lm-variety (and also an lp-variety).*

Proof. Consider the diagram given in Figure 2.1. First assume that (f, α) is a projection and that φ is in **MOD**. Then N is a quotient of M and thus is also a cyclic group. Furthermore, if $b, b' \in B$ then $b = f(a)$ and $b' = f(a')$ for some $a, a' \in A$. Now since φ is in **MOD**, $\varphi(a) = \varphi(a')$. It follows that

$$\psi(b) = \psi(f(a)) = \alpha(\varphi(a)) = \alpha(\varphi(a')) = \psi(f(a')) = \psi(b')$$

showing that ψ belongs to **MOD**.

Next assume that (f, α) is an inclusion and that ψ is in **MOD**. Then C is a subgroup of D and hence is also cyclic. Furthermore, if $a, a' \in A$, then $|f(a)| = |f(a')|$ since f is length-multiplying. It follows that $\psi(f(a)) = \psi(f(a'))$ and $\alpha(\varphi(a)) = \alpha(\varphi(a'))$ since $\psi \circ f = \alpha \circ \varphi$. Since α is injective, it follows that $\varphi(a) = \varphi(a')$.

Finally, let $\varphi_1 : A^* \to M_1$ and $\varphi_2 : A^* \to M_2$ be two stamps of **MOD**. Then their restricted direct product is clearly in **MOD**, since if $a, a' \in A$, then $(\varphi_1(a), \varphi_2(a)) = (\varphi_1(a'), \varphi_2(a'))$. Thus **MOD** is an lm-variety. □

If $\varphi : A^* \to M$ is a stamp, consider the set $\varphi(A)$ as an element of the monoid $\mathcal{P}(M)$ of the subsets of M. This element has a unique idempotent power, which is also a subsemigroup of M, called the *stable subsemigroup* of φ. A stamp is said to be *quasi-aperiodic* if its stable subsemigroup is aperiodic. More generally, given a variety of finite semigroups **V**, a stamp is said to be a *quasi-***V** *stamp* if its stable subsemigroup belongs to **V**. It is stated in [18] that the quasi-**V** stamps form an lm-variety (and also an lp-variety), denoted by **QV**.

3 The Reiterman theorem for $\mathcal{C}$-varieties

3.1 Metric monoids

Recall that a *metric* on a set E is a map $d : E^2 \to \mathbb{R}^+$ satisfying the following properties:

(1) for every $(u, v) \in E^2$, $d(u, v) = d(v, u)$,

(2) for every $(u, v) \in E^2$, $d(u, v) = 0$ if and only if $u = v$

(3) for every $(u, v, w) \in E^3$, $d(u, w) \leq d(u, v) + d(v, w)$

A metric is an *ultrametric* if it satisfies the stronger condition

(3′) for every $(u, v, w) \in E^3$, $d(u, w) \leq \max(d(u, v), d(v, w))$

A *metric monoid* is a monoid M equipped with a metric d, such that (M, d) is a complete metric space and the multiplication of M is uniformly continuous. Morphisms between two metric monoids are required to be uniformly continuous.

In this section, we will treat every finite monoid M as a metric monoid equipped with the discrete metric d defined by

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ 1 & \text{otherwise} \end{cases}$$

Let **V** be a $\mathcal{C}$-variety of stamps. An important example of metric monoid is the free pro-**V** monoid on A, which we now define. A stamp $\varphi : A^* \to M$ *separates* two words u and v of A^* if $\varphi(u) \neq \varphi(v)$. Given two words $u, v \in A^*$, we set

$$r_{\mathbf{V}}(u,v) = \min\bigl\{\mathrm{Card}(M) \bigm| \text{ there is a stamp } \varphi : A^* \to M \text{ of } \mathbf{V} \text{ that separates } u \text{ and } v\}$$

and $d_{\mathbf{V}}(u,v) = 2^{-r_{\mathbf{V}}(u,v)}$, with the usual conventions $\min \emptyset = +\infty$ and $2^{-\infty} = 0$. We first establish some general properties of $d_{\mathbf{V}}$.

Proposition 3.1 *The following properties hold for every $u, v, w \in A^*$*

(1) $d_{\mathbf{V}}(u,v) = d_{\mathbf{V}}(v,u)$

(2) $d_{\mathbf{V}}(uw,vw) \leq d_{\mathbf{V}}(u,v)$ *and* $d_{\mathbf{V}}(wu,wv) \leq d_{\mathbf{V}}(u,v)$

(3) $d_{\mathbf{V}}(u,w) \leq \max\{d_{\mathbf{V}}(u,v), d_{\mathbf{V}}(v,w)\}$

Proof. The first assertion is trivial. A stamp of $\mathbf{V}$ separating uw and vw certainly separates u and v. Therefore $d_{\mathbf{V}}(uw,vw) \leq d_{\mathbf{V}}(u,v)$, and dually, $d_{\mathbf{V}}(wu,wv) \leq d_{\mathbf{V}}(u,v)$.

Let $\varphi : A^* \to M$ be a stamp of $\mathbf{V}$ separating u and w. Then φ separates either u and v, or v and w. It follows that $\min(r_{\mathbf{V}}(u,v), r_{\mathbf{V}}(v,w)) \leq r_{\mathbf{V}}(u,w)$ and hence $d_{\mathbf{V}}(u,w) \leq \max\{d_{\mathbf{V}}(u,v), d_{\mathbf{V}}(v,w)\}$. $\square$

If $\mathbf{V}$ is the $\mathcal{C}$-variety of all stamps onto a finite monoid, we simplify the notation $d_{\mathbf{V}}$ to d.

Proposition 3.2 *The function d is an ultrametric on A^*.*

Proof. Properties (1) and (3) of the definition of an ultrametric follow from Proposition 3.1. Suppose that $d(u,v) = 0$. In particular, the syntactic morphism of the language $\{u\}$ does not separate u from v, showing that $u = v$. Thus by Proposition 3.1, d is an ultrametric. $\square$

In the general case, $d_{\mathbf{V}}$ is not always a metric, because one may have $d_{\mathbf{V}}(u,v) = 0$ even if $u \neq v$. For instance, if $\mathbf{V}$ is a $\mathcal{C}$-variety of stamps onto commutative monoids, $d_{\mathbf{V}}(ab,ba) = 0$, since there is no way to separate ab and ba in a commutative monoid. To work around this inconvenience, we first observe that, by Proposition 3.1, the relation $\sim_{\mathbf{V}}$ defined by

$$u \sim_{\mathbf{V}} v \text{ if and only if } d_{\mathbf{V}}(u,v) = 0$$

is a congruence on A^*. Equivalently, $u \sim_{\mathbf{V}} v$ if and only if, for each stamp $\varphi : A^* \to M$ of $\mathbf{V}$, $\varphi(u) = \varphi(v)$. Let $\pi_{\mathbf{V}}$ be the natural morphism from A^* onto $A^*/{\sim_{\mathbf{V}}}$.

Proposition 3.3 *Every stamp $\varphi : A^* \to M$ of $\mathbf{V}$ factors through $\pi_{\mathbf{V}}$.*

Proof. It is now an immediate consequence of the definition of $\pi_{\mathbf{V}}$. $\square$

Proposition 3.2 can now be generalized as follows.

Proposition 3.4

(1) *The function $d_{\mathbf{V}}$ is an ultrametric on $A^*/{\sim_{\mathbf{V}}}$.*

(2) *The product on $A^*/{\sim_{\mathbf{V}}}$ is uniformly continuous for this metric.*

Proof. (1) follows directly from Proposition 3.1, since $d_{\mathbf{V}}(u, v) = 0$ implies $u \sim_{\mathbf{V}} v$ by definition. We use the same proposition to obtain the relation

$$d_{\mathbf{V}}(uv, u'v') \leq \max\{d_{\mathbf{V}}(uv, uv'), d_{\mathbf{V}}(uv', u'v')\} \leq \max\{d_{\mathbf{V}}(v, v'), d_{\mathbf{V}}(u, u')\}$$

which proves (2). $\square$

3.2 Profinite monoids

The completion of the metric space (A^*, d), denoted by $\widehat{A^*}$, is called the *free profinite monoid* on A. The completion of the metric space $(A^*/{\sim_{\mathbf{V}}}, d_{\mathbf{V}})$, denoted by $\hat{F}_{\mathbf{V}}(A)$, is called the *free pro-*$\mathbf{V}$ *monoid* on A. These topological monoids satisfy the following properties:

Proposition 3.5 *Let* $\mathbf{V}$ *be a* $\mathcal{C}$*-variety of stamps and* A *a finite alphabet.*

(1) *The monoid* $\hat{F}_{\mathbf{V}}(A)$ *is compact.*

(2) *The natural morphism* $\pi_{\mathbf{V}}$ *from* (A^*, d) *onto* $(A^*/{\sim_{\mathbf{V}}}, d_{\mathbf{V}})$ *is uniformly continuous.*

(3) *Every stamp* $\varphi : A^* \to M$ *of* $\mathbf{V}$ *is uniformly continuous for* $d_{\mathbf{V}}$*. Furthermore, there is a uniformly continuous morphism* π *from* $A^*/{\sim_{\mathbf{V}}}$ *onto* M *such that* $\varphi = \pi \circ \pi_{\mathbf{V}}$*.*

(4) *Every morphism of* $\mathcal{C}$ *is uniformly continuous for* $d_{\mathbf{V}}$*.*

Proof. (1) Since $\hat{F}_{\mathbf{V}}(A)$ is complete, it suffices to verify that, for every $n > 0$, A^* is covered by a finite number of open balls of radius $< 2^{-n}$. Consider the congruence $\sim_n$ defined on A^* by

$$u \sim_n v \text{ if and only if } \varphi(u) = \varphi(v) \text{ for every stamp of } \mathbf{V} \text{ of size} \leq n.$$

Since A is finite, there are only finitely many morphisms from A^* onto a monoid of size $\leq n$, and thus $\sim_n$ is a congruence of finite index. Furthermore, $d_{\mathbf{V}}(u, v) < 2^{-n}$ if and only if u and v are $\sim_n$-equivalent. It follows that the $\sim_n$-classes are open balls of radius $< 2^{-n}$ and cover A^*.

(2) Let $\pi_{\mathbf{V}}$ be the natural morphism from A^* onto $A^*/{\sim_{\mathbf{V}}}$. Since $d_{\mathbf{V}}(u, v) \leq d(u, v)$, $\pi_{\mathbf{V}}$ is uniformly continuous.

(3) Let φ be a stamp of $\mathbf{V}$ of size n. If $d_{\mathbf{V}}(u, v) < 2^{-n}$, then in particular $\varphi(u) = \varphi(v)$. Therefore φ is uniformly continuous. Since $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$ implies $\varphi(u) = \varphi(v)$, there is a morphism π from $A^*/{\sim_{\mathbf{V}}}$ onto M such that $\varphi = \pi \circ \pi_{\mathbf{V}}$. This morphism is uniformly continuous for the same reason as φ.

(4) Let $f : A^* \to B^*$ be a morphism of $\mathcal{C}$. If φ is a stamp of $\mathbf{V}$ separating $f(u)$ and $f(v)$, then $\varphi \circ f$ is a stamp of $\mathbf{V}$ separating u and v. It follows that $d_{\mathbf{V}}(f(u), f(v)) \leq d_{\mathbf{V}}(u, v)$, and thus f is uniformly continuous. $\square$

It is a well known fact that a uniformly continuous function from a metric space (E, d) into a metric space (E', d') admits a uniformly continuous extension $\hat{\varphi} : \hat{E} \to \hat{E}'$. Furthermore this extension is unique.

Corollary 3.6

(1) *The morphism $\pi_{\mathbf{V}}$ extends uniquely to a uniformly continuous morphism from $(\widehat{A^*}, d)$ onto $(\hat{F}_{\mathbf{V}}(A), d_{\mathbf{V}})$.*
(2) *Every stamp $\varphi : A^* \to M$ of $\mathbf{V}$ extends uniquely to a uniformly continuous morphism from $\widehat{A^*}$ onto M and induces a unique uniformly continuous morphism $\hat{\pi}$ from $\hat{F}_{\mathbf{V}}(A)$ onto M.*
(3) *Every morphism $f : A^* \to B^*$ of $\mathcal{C}$ induces a unique uniformly continuous morphism from $\hat{F}_{\mathbf{V}}(A)$ into $\hat{F}_{\mathbf{V}}(B)$.*

Proof. The corollary is an immediate consequence of Proposition 3.5 and of the result on extensions of uniformly continuous functions recalled above. $\square$

Note that the distance $d_{\mathbf{V}}$ on $\hat{F}_{\mathbf{V}}(A)$ can be defined directly by setting

$$r_{\mathbf{V}}(u, v) = \min\bigl\{\mathrm{Card}(M) \bigm| \text{ there is a stamp } \varphi : A^* \to M \text{ of } \mathbf{V} \text{ such that } \hat{\varphi}(u) \neq \hat{\varphi}(v)\}$$

and $d_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u,v)}$.

We now state the key property of the free pro-$\mathbf{V}$ monoid.

Theorem 3.7 *Let $\varphi : A^* \to M$ be a stamp. Then φ belongs to $\mathbf{V}$ if and only if there is a uniformly continuous morphism $\hat{\pi}$ from $\hat{F}_{\mathbf{V}}(A)$ onto M such that $\varphi = \hat{\pi} \circ \pi_{\mathbf{V}}$.*

$$\begin{tikzcd}
A^* \arrow[rr, "\pi_{\mathbf{V}}"] \arrow[dr, "\varphi"'] & & \hat{F}_{\mathbf{V}}(A) \arrow[dl, "\hat{\pi}"] \\
& M &
\end{tikzcd}$$

Proof. By Corollary 3.6, φ induces a uniformly continuous morphism $\hat{\varphi}$ from $\hat{F}_{\mathbf{V}}(A)$ onto M such that $\varphi = \hat{\varphi} \circ \pi_{\mathbf{V}}$.

Conversely, suppose there is a uniformly continuous morphism $\hat{\pi}$ from $\hat{F}_{\mathbf{V}}(A)$ onto M such that $\varphi = \hat{\pi} \circ \pi_{\mathbf{V}}$. The set

$$D = \{(u, v) \in \hat{F}_{\mathbf{V}}(A) \times \hat{F}_{\mathbf{V}}(A) \mid \hat{\pi}(u) = \hat{\pi}(v)\}$$

is the inverse image under $\hat{\pi}$ of the diagonal of $M \times M$, and since M is discrete and $\hat{\pi}$ is continuous, it is a clopen subset of $\hat{F}_{\mathbf{V}}(A) \times \hat{F}_{\mathbf{V}}(A)$. Let $\mathcal{F}$ be the class of all morphisms of the form $\hat{\alpha} : \hat{F}_{\mathbf{V}}(A) \to M_\alpha$, where $\alpha : A^* \to M_\alpha$ is a stamp of $\mathbf{V}$. For each $\alpha \in \mathcal{F}$, let

$$C_\alpha = \{(u, v) \in \hat{F}_{\mathbf{V}}(A) \times \hat{F}_{\mathbf{V}}(A) \mid \hat{\alpha}(u) \neq \hat{\alpha}(v)\}$$

Each C_α is open by continuity of $\hat{\alpha}$. Furthermore, if (u, v) does not belong to any C_α, then $\hat{\alpha}(u) = \hat{\alpha}(v)$ for each stamp of $\mathbf{V}$, which gives $d_{\mathbf{V}}(u, v) = 0$, $u = v$ and $\hat{\pi}(u) = \hat{\pi}(v)$, and thus $(u, v) \in D$. It follows that the family $D \cup (C_\alpha)_{\alpha \in \mathcal{F}}$ is a covering of $\hat{F}_{\mathbf{V}}(A) \times \hat{F}_{\mathbf{V}}(A)$ by open sets, and since $\hat{F}_{\mathbf{V}}(A)$ is compact, it admits a finite subcovering, say $D \cup (C_\alpha)_{\alpha \in F}$, where F is a finite set.

Let $u, v \in A^*$ and $\alpha \in F$. If $\alpha(u) = \alpha(v)$, then $\hat{\alpha} \circ \pi_{\mathbf{V}}(u) = \hat{\alpha} \circ \pi_{\mathbf{V}}(v)$ and thus $(\pi_{\mathbf{V}}(u), \pi_{\mathbf{V}}(v)) \notin C_\alpha$. It follows that if $\alpha(u) = \alpha(v)$ for each $\alpha \in F$, then $(\pi_{\mathbf{V}}(u), \pi_{\mathbf{V}}(v)) \in D$, which implies that $\hat{\pi} \circ \pi_{\mathbf{V}}(u) = \hat{\pi} \circ \pi_{\mathbf{V}}(v)$, that is $\varphi(u) = \varphi(v)$. Consequently φ is a projection of a substamp of the stamp $\prod_{\alpha \in F} \alpha$ and thus belongs to $\mathbf{V}$. $\square$

Let A be a finite alphabet and let $\varphi : A^* \to M$ be a stamp. By Corollary 3.6, φ extends to a morphism from $\widehat{A^*}$ onto M, also denoted by φ. Furthermore, any $\mathcal{C}$-morphism $f : A^* \to B^*$ extends to a morphism from $\widehat{A^*}$ onto $\widehat{B^*}$, also denoted by f.

3.3 Identities

We now extend the notion of identity as follows. Let $u, v \in \widehat{A^*}$. A stamp $\varphi : B^* \to M$ *satisfies the identity* $u = v$ if, for every $\mathcal{C}$-morphism $f : A^* \to B^*$, $\varphi \circ f(u) = \varphi \circ f(v)$. A variety $\mathbf{V}$ satisfies a given identity if every stamp of $\mathbf{V}$ satisfies this identity. We also say in this case that the given identity is an identity of $\mathbf{V}$.

We now show that identities are stable under the morphisms of $\mathcal{C}$.

Proposition 3.8 *Let* $\mathbf{V}$ *be a* $\mathcal{C}$*-variety and let* $u = v$ *be an identity of* $\mathbf{V}$*, with* $u, v \in \widehat{A^*}$*. If* $f : A^* \to B^*$ *is a morphism of* $\mathcal{C}$*, then* $f(u) = f(v)$ *is also an identity of* $\mathbf{V}$*.*

Proof. Let $\varphi : C^* \to M$ be a stamp of $\mathbf{V}$ and let $g : B^* \to C^*$ be a $\mathcal{C}$-morphism. Then, $g \circ f : A^* \to C^*$ is also a $\mathcal{C}$-morphism, and thus $\varphi \circ g \circ f(u) = \varphi \circ g \circ f(v)$. It follows that $f(u) = f(v)$ is an identity of $\mathbf{V}$. $\square$

We now show that identities of $\mathbf{V}$ are closely related to free pro-$\mathbf{V}$ monoids.

Proposition 3.9 *Let* A *be a finite alphabet. Given two elements* u *and* v *of* $\widehat{A^*}$*,* $u = v$ *is an identity of* $\mathbf{V}$ *if and only if* $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$*.*

Proof. If $u = v$ is an identity of $\mathbf{V}$, then u and v cannot be separated by any stamp of $\mathbf{V}$. Thus $d_{\mathbf{V}}(u, v) = 0$, $u \sim_{\mathbf{V}} v$ and $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$. Conversely, suppose that $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$, and let $\varphi : B^* \to M$ be a stamp of $\mathbf{V}$. If $f : A^* \to B^*$ is a $\mathcal{C}$-morphism, then $\varphi \circ f$ is in $\mathbf{V}$ and by the definition of $d_{\mathbf{V}}$, $\varphi \circ f(u) = \varphi \circ f(v)$. It follows that $u = v$ is an identity of $\mathbf{V}$. $\square$

Corollary 3.10 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be two* $\mathcal{C}$*-varieties of stamps satisfying the same identities on the alphabet* A*. Then* $\hat{F}_{\mathbf{V}}(A)$ *and* $\hat{F}_{\mathbf{W}}(A)$ *are isomorphic.*

We are now ready to state the generalization of Reiterman's theorem. Given a set E of identities, we denote by $[\![E]\!]$ the class of stamps satisfying all the identities of E.

Theorem 3.11 *A class of stamps is a* $\mathcal{C}$*-variety if and only if it can be defined by a set of identities.*

Proof. The fact that every class of stamps defined by a set of identities is a variety can be proved easily.

Let $\mathbf{V}$ be a $\mathcal{C}$-variety. Let E be the class of all identities which are satisfied by every stamp of $\mathbf{V}$ and let $\mathbf{W} = [\![E]\!]$. Clearly $\mathbf{V} \subseteq \mathbf{W}$. Let $\varphi : A^* \to M$ be a stamp of $\mathbf{W}$. This stamp can be extended to a uniformly continuous morphism from $\widehat{A^*}$ onto M. Let $u, v \in \widehat{A^*}$. By Proposition 3.9, if $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$, then $u = v$ is an identity of $\mathbf{V}$ and thus is satisfied by φ. In particular, $\pi_{\mathbf{V}}(u) = \pi_{\mathbf{V}}(v)$ implies $\varphi(u) = \varphi(v)$ and thus there exists a morphism $\hat{\pi} : \hat{F}_{\mathbf{V}}(A) \to M$ such that $\varphi = \hat{\pi} \circ \pi_{\mathbf{V}}$. We claim that $\hat{\pi}$ is uniformly continuous. Since $\hat{F}_{\mathbf{V}}(A)$ is compact by Proposition 3.5, it suffices to verify that $\hat{\pi}$ is continuous. Let F be a subset of the discrete monoid M. We first observe that $\hat{\pi}^{-1}(F) = \pi_{\mathbf{V}}(\varphi^{-1}(F))$. Since φ is continuous, $\varphi^{-1}(F)$ is closed. Now, $\widehat{A^*}$ is compact, $\pi_{\mathbf{V}}$ is continuous, and $\hat{F}_{\mathbf{V}}(A)$ is Hausdorff. It follows that $\pi_{\mathbf{V}}(\varphi^{-1}(F))$ is closed, proving the claim. It now follows from Theorem 3.7 that φ is in $\mathbf{V}$. Thus $\mathbf{V} = \mathbf{W}$. □

We now give some examples of identities.

(1) **MOD** is defined by the single identity $a^{\omega-1}b = 1$, both as an lm-variety and an lp-variety. It is worth taking a moment to prove this, noting that the identity implies, in both cases, the identities $a^\omega = 1$ and $a = b$. Without the condition that $\mathcal{C}$ contains length-preserving morphisms we cannot arbitrarily substitute one letter for another and deduce $a^\omega = 1$ from $a^{\omega-1}b$. In the context of *all*-varieties and *ne*-varieties, the same identities imply that the semigroup is trivial.

(2) The lm-variety of stamps onto aperiodic monoids is defined by the familiar identity $x^\omega = x^{\omega+1}$. But there is no finite basis for the identities of the lp-variety of stamps onto aperiodics.

(3) It is shown in [10] that the languages of generalized star-height ≤ 1 form an lp-variety of languages, to which corresponds an lp-variety of stamps. Thus the long-standing open conjecture that there are languages of generalized star-height > 1 is equivalent to the existence of non-trivial identities for this variety. Thus an important problem is to try to find a nontrivial identity for this lp-variety (assuming one exists!)

3.4 An example

Let A_k denote the k-letter alphabet $\{a_1, \ldots, a_k\}$ and let $\mathbf{V}_k$ be the lp-variety of stamps generated by the syntactic morphism of the language $(a_1 a_2 \cdots a_k)^+$ of A_k^*. The goal of this section is to give the identities describing $\mathbf{V}_k$. The proof requires some arguments from combinatorics on words.

Let A be a finite alphabet, let $k \geq 2$ and let $0 \leq r < k$. Given a word w, let $S_{r,k}(w)$ be the set of letters of w occurring in positions that are congruent to r modulo k. That is, if $w = c_0 \cdots c_{n-1}$, with each c_i in A, then

$$S_{r,k}(w) = \{c_r, c_{r+k}, c_{r+2k}, \ldots\}$$

Let us say that w is *k-redundant* if the sets $S_{r,k}$, for $0 \leq r < k$, are *not* pairwise distinct.

Example 3.1 Let $A = \{a, b, c, d, r\}$ and $w = abracadabra$. Then $S_{0,3} = \{a, d, r\}$, $S_{1,3} = \{a, b, c\}$, $S_{2,3} = \{a, b, r\}$. Thus w is not redundant.

Define an equivalence relation $\sim_k$ on A^* by setting $u \sim_k v$ if and only if either u and v are both k-redundant, or if $|u| \equiv |v| \bmod k$ and $S_{r,k}(u) = S_{r,k}(v)$ for $0 \le r < k$.

Lemma 3.12 *For each $k \ge 2$, $\sim_k$ is a congruence on A^**

Proof. It is clear that $\sim_k$ is a congruence relation, and that it has finite index. To prove that it is a congruence, we need to verify that if $u \sim_k v$ and $a \in A$, then $ua \sim_k va$ and $au \sim_k av$. This is obviously the case if u and v are both k-redundant. If not, then the desired equivalences follow at once from the following facts:

(a) $S_{r,k}(ua) = \begin{cases} S_{r,k}(u) \cup \{a\} \text{ if } |u| \equiv r-1 \bmod k \\ S_{r,k}(u) \text{ otherwise} \end{cases}$

(b) $S_{r,k}(au) = \begin{cases} S_{0,k}(u) \cup \{a\} \text{ if } r = 1 \\ S_{r-1,k}(u) \text{ otherwise} \end{cases}$

Thus the equivalence classes of ua and au depend only on the equivalence class of u. $\square$

Observe that the congruence class consisting of the k-redundant words is the zero element of the quotient monoid $A^*/{\sim_k}$.

We are now ready to state the main result of this subsection:

Theorem 3.13 *Let $\varphi : A^* \to M$ be a stamp. The following conditions are equivalent:*

(1) *φ factors through $\sim_k$,*
(2) *φ belongs to $\mathbf{V}_k$,*
(3) *φ satisfies the following identities, where, for $0 \le i \le k$, $u_i = x_1 \cdots x_i$.*
  (a) *for $0 \le i \le k-2$, $yxu_ix = xu_ix = xu_ixy$,*
  (b) *$xu_{k-1}y = yu_{k-1}x$,*
  (c) *$u_k^2 = u_k$.*

Proof. (1) implies (2). It suffices to show that the projection morphism from A^* onto $A^*/{\sim_k}$ belongs to $\mathbf{V}_k$. This morphism factors through the restricted direct product of the syntactic morphisms of the classes of $\sim_k$, so it suffices to show that each of these classes belongs to the lp-variety $\mathcal{V}_k$ of languages generated by $(a_1a_2\cdots a_k)^+$. Consider first a nonzero congruence class. It is either the set consisting of the empty word alone, or else of the form

$$(B_1 \cdots B_k)^* B_1 \cdots B_i,$$

where $1 \le i \le k$ and the B_j are pairwise disjoint subsets of A. This language is the intersection of the languages

$$(A_1 \cdots A_k)^* A_1 \cdots A_i,$$

over all partitions $\{A_1, \ldots, A_k\}$ of A such that $B_j \subseteq A_j$ for $1 \le j \le k$. Each of these is in turn the inverse image of the language of A_k^*

$$L = (a_1a_2\cdots a_k)^* a_1 \cdots a_i$$

under the length-preserving morphism that maps each A_j to the letter a_j. Finally,

$$L = (a_1 a_2 \cdots a_k)^+ (a_{i+1} \cdots a_k)^{-1}$$

which is in $\mathcal{V}_k(A^*)$, so each congruence class is in $\mathcal{V}$.

The language $\{1\}$ of A^* is the inverse image of the language $\{1\}$ of A_k^* under any length-preserving morphism. Further, we have

$$\{1\} = (a_1 a_2 \cdots a_k)^+ (a_1 \cdots a_k)^{-1} \setminus (a_1 a_2 \cdots a_k)^+$$

and thus $\{1\} \in \mathcal{V}(A^*)$.

Finally, since the zero congruence class is the complement of the union of all the other classes, it too is in $\mathcal{V}(A^*)$.

(2) implies (3). It is easy to see that the syntactic morphism of $(a_1 a_2 \cdots a_k)^+$ satisfies the given identities.

(3) implies (1). We describe a distinguished representative for each nonzero congruence class. For this, we suppose that the alphabet A is linearly ordered. Given pairwise disjoint subsets $(S_{r,k})_{0 \le r < k}$ of A and $0 \le m < k$, we construct a word $w = a_1 \cdots a_t$ using the following algorithm. The i^{th} step of the algorithm determines the letter a_i. If, after i steps, all the letters of $\bigcup_{1 \le j \le i} S_{j,k}$ have been used, and $i \equiv m \bmod k$, then the algorithm terminates, and we set $w = a_1 \cdots a_i$. Otherwise we set a_{i+1} to be the least unused letter of $S_{(i+1) \bmod k, k}$ if such an unusued letter exists, or the greatest letter of $S_{(i+1) \bmod k, k}$, under the ordering on A, if all letters of this set have been used.

For example, suppose $k = 3$, $A = \{a, b, c, d, e, f\}$, $S_{0,3} = \{a, c, d\}$, $S_{1,3} = \{b, e\}$, $S_{2,3} = \{f\}$ and $m = 2$. We write these three sets as the columns of a table, then fill out the shorter columns by repeating the last element:

$$\begin{array}{ccc} a & b & f \\ c & e & f \\ d & e & f \end{array}$$

We then read the normal form from the table, proceeding row by row until the final row, stopping when the length of the word is congruent to m mod k. In this instance, the word obtained is $abfcefde$.

Obviously, each word w in A^* is either congruent to 0, or congruent to a unique word in this normal form. We show that w can be transformed, in a sequence of steps, to this unique normal form, and that each step preserves the value under φ. To describe the transformations, we again write a word, row by row, in the form of a table with k columns. Since the length of the word might not be exactly divisible by k, the last row of the table might be shorter than the other rows, and thus some of the rightmost columns might have one fewer letter than the other columns.

The identity (b) implies that we can permute the letters of any column without changing the value, under φ, of the associated word. We'll call this a *permutation transformation*. The identity (c) implies that we can replace any row (with the exception of the last row, if it is incomplete) by two copies of the same row, and, similarly, replace two identical adjacent rows by a single copy, without changing the value under φ. These are called *idempotent transformations*. Note that both types of transformations preserve the $\sim_k$-class as well.

By using permutation transformations we can always transform a word into one in which the columns of the associated table are sorted according to the order of A. If the word represented is not in normal form, and the columns are sorted, then either some column contains two occurrences of a letter that is not the greatest letter of the column, or some column contains an unnecessary repetition of its greatest letter.

In the first instance, the table contains rows of the form xax', yby' and zcz', where $x, x', y, y', z, z' \in A^*$, $a, b \in A$, $|x| = |y| = |z|$, $|x'| = |y'|$, and zcz' is the last row of the table. We can permute the columns so that the row yay' is replaced by yby', and zaz' becomes the last row of the table. We can then duplicate the row yby'. We then use another sequence of permutation transformations to obtain rows xax', yby' and zcz' as the last row. We then eliminate the duplicated row and sort the columns. As a result we arrive at a word with one less occurrence of a in the column in question. We do this for each repeated occurrence of a letter that is not the greatest in its column.

For the second instance, of a column in a sorted table contains an unnecessary repetition of its last letter, then every letter in the last complete row of the table must appear in an earlier row of the table, and thus the last two complete rows of the table are identical. We then apply an idempotent transformation to eliminate the duplicated row.

Now suppose $u \sim_k v$. If u and v are both k-redundant, then they can each be transformed, by a series of permutation transformations, to words in which some row contains a repeated letter. The identities (a) imply that any such word is the (necessarily unique) zero element of M, and thus $\varphi(u) = \varphi(v)$. If u and v are not k-redundant, then the above argument shows that there is a word w in normal form such that $\varphi(u) = \varphi(w) = \varphi(v)$. This completes the proof. $\square$

It might be helpful to see an example of the reduction process described above.

Example 3.2 Let $k = 3$ and let $w = abecdeabfab$. This word is not 3-redundant. The tabular representation is

$$\begin{array}{ccc} a & b & e \\ c & d & e \\ a & b & f \\ a & b & \end{array}$$

Using permutation transformations we sort the columns

$$\begin{array}{ccc} a & b & e \\ a & b & e \\ a & b & f \\ c & d & \end{array}$$

Using an idempotent transformation, we eliminate the repeated row:

$$\begin{array}{ccc} a & b & e \\ a & b & f \\ c & d & \end{array}$$

The following steps eliminate the repeated a in the first column:

$$
\begin{array}{ccc} a & b & e \\ c & b & f \\ a & d & \end{array}
\qquad
\begin{array}{ccc} a & b & e \\ c & b & f \\ c & b & f \\ a & d & \end{array}
\qquad
\begin{array}{ccc} c & b & e \\ a & b & f \\ a & b & f \\ c & d & \end{array}
\qquad
\begin{array}{ccc} c & b & e \\ a & b & f \\ c & d & \end{array}
\qquad
\begin{array}{ccc} a & b & e \\ c & b & f \\ c & d & \end{array}
$$

We proceed similarly to remove the repeated b in the second column, and eventually arrive at:

$$
\begin{array}{ccc} a & b & e \\ c & d & f \\ c & d & \end{array}
$$

which gives the normal form $abecdfcd$.

4 Mal'cev products

In this section, we extend to $\mathcal{C}$-varieties of stamps the classical notion of Mal'cev product of varieties.

Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be varieties of finite semigroups and let M and N be finite monoids. Recall that a relational morphism $\tau : M \to N$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-*relational morphism* if, for every subsemigroup T of N in $\mathbf{V}_2$, the semigroup $\tau^{-1}(T)$ belongs to $\mathbf{V}_1$.

Let $\mathbf{W}$ be a $\mathcal{C}$-variety of stamps. Denote by $(\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W}$ the class of all stamps $\varphi : A^* \to M$ for which there exists a stamp $\psi : A^* \to N$ of $\mathbf{W}$ such that the relational morphism $\tau = \psi \circ \varphi^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism.

$$
\begin{tikzcd}
& A^* \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
M \arrow[rr, "\psi \circ \varphi^{-1}"] & & N
\end{tikzcd}
$$

If $\mathbf{V}_1 = \mathbf{V}$ and $\mathbf{V}_2$ is the trivial variety of semigroups, the notation simplifies to $\mathbf{V} \circledM \mathbf{W}$ (this is the *Mal'cev product* of $\mathbf{V}$ and $\mathbf{W}$).

Proposition 4.1 *The class $(\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W}$ is a $\mathcal{C}$-variety.*

Proof. Let $\mu : A^* \to M$ and $\nu : B^* \to N$ be two stamps and let (f, α) be a $\mathcal{C}$-morphism from μ to ν. By definition, $\nu \circ f = \alpha \circ \mu$.

First assume that $\mu \in (\mathbf{V}_1, \mathbf{V}_2) \circledM \mathbf{W}$ and that (f, α) is a projection. Then f and α are onto and there is a stamp $\kappa : A^* \to K$ of $\mathbf{W}$ such that $\kappa \circ \mu^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism. Since $f(A) = B$, there is a map $g : B \to A$ such that $f \circ g = 1_B$. Let us extend g into a length-preserving morphism from B^* into A^*. Let $\gamma = \kappa \circ g$ and let $R = \mathrm{Im}(\gamma)$. By construction R is a submonoid of K. Let us denote by $\iota : R \to K$ the inclusion map. Then $\gamma = \iota^{-1} \circ \kappa \circ g$ and the pair (g, ι) is an inclusion from γ into κ. In particular, the stamp γ is in $\mathbf{W}$. The situation is summarized in the diagram below:

$$\begin{tikzcd}
K & A^* \arrow[l, "\kappa"'] \arrow[r, "\mu"] \arrow[d, "f"] & M \arrow[d, "\alpha"] \\
R \arrow[u, "\iota"] & B^* \arrow[l, "\gamma"] \arrow[r, "\nu"'] \arrow[u, bend left, "g"] & N
\end{tikzcd}$$

Let $\tau = \gamma \circ \nu^{-1}$. We claim that τ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism. First observe that

$$\begin{aligned}
\tau^{-1} = \nu \circ \gamma^{-1} = \nu \circ g^{-1} \circ \kappa^{-1} \circ \iota = \nu \circ f \circ g \circ g^{-1} \circ \kappa^{-1} \circ \iota \\
\subseteq \nu \circ f \circ \kappa^{-1} \circ \iota = \alpha \circ \mu \circ \kappa^{-1} \circ \iota
\end{aligned}$$

Now, let T be a subsemigroup of R in $\mathbf{V}_2$. Then $\tau^{-1}(T) = \alpha \circ \mu \circ \kappa^{-1} \circ \iota(R)$. Since ι is injective, $\iota(R)$ is in $\mathbf{V}_2$. Furthermore, since $\kappa \circ \mu^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism, the semigroup $T' = \mu \circ \kappa^{-1} \circ \iota(R)$ is in $\mathbf{V}_1$. It follows that $\alpha(T')$ is also in $\mathbf{V}_1$, proving the claim. Therefore ν belongs to $(\mathbf{V}_1, \mathbf{V}_2) \textcircled{M} \mathbf{W}$.

Suppose now that $\nu \in (\mathbf{V}_1, \mathbf{V}_2) \textcircled{M} \mathbf{W}$ and that (f, α) is an inclusion. Then α is injective and there is a stamp $\kappa : B^* \to K$ of $\mathbf{W}$ such that $\kappa \circ \nu^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism. The situation is summarized in the diagram below

$$\begin{tikzcd}
A^* \arrow[r, "f"] \arrow[d, "\mu"'] & B^* \arrow[d, "\nu"] \arrow[dr, "\kappa"] & \\
M \arrow[r, "\alpha"] & N & K
\end{tikzcd}$$

Let $\gamma = \kappa \circ f$. Since $f \in \mathcal{C}$, γ is a stamp of $\mathbf{W}$. Let $\tau = \gamma \circ \mu^{-1}$. We claim that τ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism. Let T be a subsemigroup of K in $\mathbf{V}_2$ and let $T' = \nu \circ \kappa^{-1}(T)$. Since $\kappa \circ \nu^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism, T' is in $\mathbf{V}_1$. Furthermore, since α is injective, $\alpha^{-1} \circ \alpha$ is the identity on M. Therefore, using the relation $\alpha \circ \mu = \nu \circ f$, we obtain

$$\begin{aligned}
\tau^{-1}(T) = \alpha^{-1} \circ \alpha \circ \gamma^{-1} \circ \kappa^{-1}(T) = \alpha^{-1} \circ \nu \circ f \circ f^{-1} \circ \kappa^{-1}(T) \\
\subseteq \alpha^{-1} \circ \nu \circ \kappa^{-1}(T) = \alpha^{-1}(T')
\end{aligned}$$

Now $\alpha^{-1}(T')$ is in $\mathbf{V}_1$, proving the claim. Therefore μ belongs to $(\mathbf{V}_1, \mathbf{V}_2) \textcircled{M} \mathbf{W}$.

Finally, let $\mu_1 : A^* \to M_1$ and $\mu_2 : A^* \to M_2$ be two stamps of $(\mathbf{V}_1, \mathbf{V}_2) \textcircled{M} \mathbf{W}$. By definition, there exist two stamps of $\mathbf{W}$, $\kappa_1 : A^* \to K_1$ and $\kappa_2 : A^* \to K_2$ such that $\kappa_1 \circ \mu_1^{-1}$ and $\kappa_2 \circ \mu_2^{-1}$ are $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphisms. Let $\kappa : A^* \to K$ be the restricted direct product of κ_1 and κ_2 and $\mu : A^* \to M$ be the restricted direct product of μ_1 and μ_2. We claim that $\kappa \circ \mu^{-1}$ is a $(\mathbf{V}_1, \mathbf{V}_2)$-relational morphism from M to K. Let π_1 and π_2 be the projections from

$K_1 \times K_2$ onto K_1 and K_2, respectively. Let T be a subsemigroup of K in $\mathbf{V}_2$. Then

$$\mu \circ \kappa^{-1}(T) = \mu \circ \kappa_1^{-1}(\pi_1(T)) \cap \mu \circ \kappa_2^{-1}(\pi_2(T))$$

It follows that $\mu \circ \kappa^{-1}(T)$ is in $\mathbf{V}_1$, proving the claim. Therefore μ belongs to $(\mathbf{V}_1, \mathbf{V}_2) \textcircled{M} \mathbf{W}$. $\square$

Recall that a variety of finite semigroups $\mathbf{V}$ is *monoidal* if $S \in \mathbf{V}$ implies $S^1 \in \mathbf{V}$.

Proposition 4.2 *If* $\mathbf{V}$ *is a monoidal variety of finite semigroups, then* $\mathbf{QV} = \mathbf{V} \textcircled{M} \mathbf{MOD}$.

Proof. Let $\alpha : A^* \to M$ be a stamp of $\mathbf{V} \textcircled{M} \mathbf{MOD}$. By definition, there is a stamp $\beta : A^* \to G$ of $\mathbf{MOD}$, such that $\tau = \beta \circ \alpha^{-1} : M \to G$ is a $\mathbf{V}$-relational morphism.

$$\begin{tikzcd} & A^* \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M \arrow[rr, "\tau"] & & G \end{tikzcd}$$

Let n be an integer such that $\alpha(A^n)$ is the stable subsemigroup of α. Then, in particular, $\alpha(A^n) = \alpha(A^{n|G|})$. On the other hand, since β is a stamp of $\mathbf{MOD}$, $\beta(A^{n|G|}) = (\beta(A)^{|G|})^n = \{1\}$, where 1 is the identity of G. It follows that $\alpha(A^n)$ is a subsemigroup of the semigroup $\tau^{-1}(1)$. But since τ is a $\mathbf{V}$-relational morphism, $\tau^{-1}(1)$ is in $\mathbf{V}$, and so is $\alpha(A^n)$. Thus α is in $\mathbf{QV}$.

Let $\alpha : A^* \to M$ be a stamp of $\mathbf{QV}$ and let $S = \alpha(A^n)$ be the stable subsemigroup of α. Since α is in $\mathbf{QV}$, S is a semigroup of $\mathbf{V}$. Define a morphism $\beta : A^* \to \mathbb{Z}/n\mathbb{Z}$ by setting $\beta(a) = 1$ for all $a \in A$. By construction, β is in $\mathbf{MOD}$ and $\alpha \circ \beta^{-1}(0) = \alpha(A^n)^* = \{1\} \cup S$. It follows that $\beta \circ \alpha^{-1}$ is a $\mathbf{V}$-relational morphism and thus $M \in \mathbf{V} \textcircled{M} \mathbf{MOD}$. $\square$

Corollary 4.3 *The following equality holds:* $\mathbf{QA} = \mathbf{A} \textcircled{M} \mathbf{MOD}$.

5 The ordered case

5.1 Ordered monoids and ordered stamps

Just as Eilenberg's theory of varieties has been extended to the ordered case [7], the theory developed in the present paper can be extended to include C-varieties of ordered stamps. This extension is relatively straightforward, and we only give here the main definitions and results, together with some of the changes required in the proofs of the main results.

A relation $\leq$ on a monoid M is *stable* if, for every $x, y, z \in M$, $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. An *ordered monoid* is a monoid equipped with a stable partial order relation.

A *congruence* on an ordered monoid $(M, \leq)$ is a stable quasi-order which is coarser than $\leq$. In particular, the order relation $\leq$ is itself a congruence. If $\preceq$

is a congruence on $(M, \leq)$, then the equivalence relation $\sim$ associated with $\preceq$ is a monoid congruence on M. Furthermore, there is a well-defined stable order on the quotient set $M/{\sim}$, given by $[s] \leq [t]$ if and only if $s \preceq t$. Thus $(M/{\sim}, \leq)$ is an ordered monoid, also denoted by $M/{\preceq}$.

The product of a family $(M_i)_{i \in I}$ of ordered monoids is the ordered monoid defined on the set $\prod_{i \in I} M_i$. The multiplication and the order relation are defined componentwise.

A *morphism* from an ordered monoid $(M, \leq)$ into an ordered monoid $(N, \leq)$ is a monoid morphism $\varphi : M \to N$ such that $s_1 \leq s_2$ implies $\varphi(s_1) \leq \varphi(s_2)$. Ordered submonoids and quotients are defined in the usual way. Complete definitions can be found in [13].

An *order ideal* I of an ordered monoid $(M, \leq)$ is a subset of M such that if $x \in I$ and $y \leq x$ then $y \in I$.

An *ordered stamp* is an onto morphism from a finitely generated free monoid onto a finite ordered monoid. A $\mathcal{C}$-morphism from a stamp $\varphi : A^* \to (M, \leq)$ to a stamp $\psi : B^* \to (N, \leq)$ is a pair (f, α), where $f : A^* \to B^*$ is in $\mathcal{C}$, $\alpha : M \to N$ is a morphism of ordered monoids, and $\psi \circ f = \alpha \circ \varphi$.

The notions of $\mathcal{C}$-inclusion, $\mathcal{C}$-projection, $\mathcal{C}$-division, $\mathcal{C}$-varieties of ordered stamps and Mal'cev products can now be readily extended to the ordered case.

5.2 Languages

A language L of A^* is recognized by an ordered monoid $(M, \leq)$ if and only if there exist an order ideal I of M and a monoid morphism η from A^* into M such that $L = \eta^{-1}(I)$.

Let A^* be a free monoid. Given a language L of A^* we define the *syntactic congruence* $\sim_L$ and the *syntactic preorder* $\leq_L$ as follows:

(1) $u \sim_L v$ if and only if for all $x, y \in A^*$, $xvy \in L \Leftrightarrow xuy \in L$,

(2) $u \leq_L v$ if and only if for all $x, y \in A^*, xvy \in L \Rightarrow xuy \in L$.

The monoid $A^*/{\sim_L}$ is called the *syntactic monoid* of L, and is denoted by $M(L)$. The monoid $A^*/{\sim_L}$, ordered with the stable order relation induced by $\leq_L$ is called the *ordered syntactic monoid* of L. The syntactic (ordered) monoid of a rational language is finite.

A set of languages closed under finite intersection and finite union is called a *positive boolean algebra*. Thus a positive boolean algebra always contains the empty language and the full language A^* since $\emptyset = \bigcup_{i \in \emptyset} L_i$ and $A^* = \bigcap_{i \in \emptyset} L_i$. A positive boolean algebra closed under complementation is a *boolean algebra*.

A *class of recognizable languages* is a correspondence $\mathcal{V}$ which associates with each alphabet A a set $\mathcal{V}(A^*)$ of recognizable languages of A^*.

A *positive $\mathcal{C}$-variety of languages* is a class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $\mathcal{V}(A^*)$ is a positive boolean algebra,

(2) if $\varphi\colon A^* \to B^*$ is a morphism of $\mathcal{C}$, $L \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A^*)$,

(3) if $L \in \mathcal{V}(A^*)$ and if $a \in A$, then $a^{-1}L$ and La^{-1} are in $\mathcal{V}(A^*)$.

A *$\mathcal{C}$-variety of languages* is a positive $\mathcal{C}$-variety of languages closed under complement.

If $\mathbf{V}$ is a $\mathcal{C}$-variety of ordered stamps, we denote by $\mathcal{V}(A^*)$ the set of languages whose ordered syntactic morphisms are in $\mathbf{V}$. Then $\mathcal{V}$ is a positive $\mathcal{C}$-variety of languages. Conversely, one can associate with any positive $\mathcal{C}$-variety of languages $\mathcal{V}$ the $\mathcal{C}$-variety of stamps $\mathbf{V}$ generated by the ordered syntactic morphisms of the languages of $\mathcal{V}$.

The following result extends simultaneously the results of the first author [7] and of the second author [18].

Theorem 5.1 *The correspondences $\mathbf{V} \to \mathcal{V}$ and $\mathbf{V} \to \mathcal{V}$ define bijections between the $\mathcal{C}$-varieties of ordered stamps and the positive $\mathcal{C}$-varieties of languages.*

The proof is a straightforward generalization of the two aforementioned results and is therefore omitted.

6 Identities of Mal'cev products

In [11], Pin and Weil gave a set of identities defining the Mal'cev product of two varieties of finite semigroups. This result can be adapted as follows.

Theorem 6.1 *Let $\mathbf{W}$ be a $\mathcal{C}$-variety of stamps and let $\mathbf{V} = [\![E]\!]$ be a variety of ordered semigroups. Then $\mathbf{V} ⓜ \mathbf{W}$ is defined by the identities of the form $\hat{\sigma}(x) \leq \hat{\sigma}(y)$, where $x \leq y$ is an identity of E with $x, y \in \widehat{B^*}$ for some finite alphabet B and $\sigma : B^+ \to A^+$ is a semigroup morphism such that, for all $b, b' \in B$, $\mathbf{W}$ satisfies the identity $\sigma(b) = \sigma(b') = \sigma(b^2)$.*

We now apply this result to the variety $\mathbf{QA} = \mathbf{A} ⓜ \mathbf{MOD}$. The variety of semigroups $\mathbf{A}$ is defined by the single identity $x^\omega = x^{\omega+1}$. On the other hand, the free pro-$\mathbf{MOD}$ monoid is a group and thus, if an identity of the form $u = u^2$ holds in $\mathbf{MOD}$, then the identity $u = 1$ also holds in $\mathbf{MOD}$. Therefore, we obtain

Proposition 6.2 *The lm-variety (resp. lp-variety) $\mathbf{QA}$ is defined by the set of identities of the from $u^\omega = u^{\omega+1}$, where $u = 1$ is an identity of $\mathbf{MOD}$.*

One can take for instance $u = a^{\omega-1}b$, or $u = ba^{\omega-2}cab^{\omega-2}b$, etc. If $\mathbf{QA}$ is considered as an lm-variety, this result can be improved as follows (see [5]):

Proposition 6.3 *The lm-variety $\mathbf{QA}$ is defined by the single identity*

$$(x^{\omega-1}y)^\omega = (x^{\omega-1}y)^{\omega+1}$$

Proof. It follows from Proposition 6.2 that the identity $(x^{\omega-1}y)^\omega = (x^{\omega-1}y)^{\omega+1}$ is satisfied by $\mathbf{QA}$.

Conversely, suppose $\varphi : A^* \to M$ satisfies this identity. If φ is *not* in $\mathbf{QA}$, then the stable subsemigroup $\varphi(A^k)$ contains a nontrivial group element g. Let $u, v \in A^k$ be such that $\varphi(v) = g$ and $\varphi(u)$ is the identity e of the group generated by g. Since

$$\varphi(v^{\omega-1}u)^\omega = e = \varphi(u)$$

and

$$\varphi(v^{\omega-1}u)^{\omega+1} = g^{-1}$$

the identity is not satisfied, a contradiction. $\square$

Observe that in the second part of the proof, we needed to be able to choose u and v to be words of equal length for any desired length. So this argument does not show $\varphi \in \mathbf{QA}$ if we interpret the identity over $\mathcal{C}_{lp}$. In fact there is no finite basis for the pseudoidentities of $\mathbf{QA}$ as an lp-variety. This is shown by the family of languages

$$L_r = ((a_1^+ \cdots a_r^+)^2)^*$$

over the alphabet $A_r = \{a_1, \ldots, a_r\}$. It is easy to see that the syntactic morphism of L_r is not in $\mathbf{QA}$, but the image of B^* where B is a strict subset of A_r is aperiodic.

7 Polynomial closure

In this section, we extend to $\mathcal{C}$-varieties the main results of [12].

The *polynomial closure* of a class of languages $\mathcal{L}$ of A^* is the set of languages that are finite unions of languages of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the a_i's are letters and the L_i's are elements of $\mathcal{L}$.

By extension, if $\mathcal{V}$ is a positive $\mathcal{C}$-variety of languages, we denote by Pol $\mathcal{V}$ the class of languages such that, for every alphabet A, Pol $\mathcal{V}(A^*)$ is the polynomial closure of $\mathcal{V}(A^*)$.

Let, for $0 \leq i \leq n$, L_i be recognizable languages of A^*, let $\eta_i : A^* \to M(L_i)$ be their syntactic ordered stamps and let

$$\eta : A^* \to \mathrm{Im}(\eta) \subseteq M(L_0) \times M(L_1) \times \cdots \times M(L_n)$$

be the ordered stamp defined by

$$\eta(u) = (\eta_0(u), \eta_1(u), \ldots, \eta_n(u))$$

Let a_1, a_2, $\ldots$, a_n be letters of A and let $L = L_0 a_1 L_1 \cdots a_n L_n$. Let $\mu : A^* \to M(L)$ be the syntactic ordered stamp of L. The properties of the relational morphism

$$\tau = \eta \circ \mu^{-1} : M(L) \to M(L_0) \times M(L_1) \times \cdots \times M(L_n)$$

were intensively studied in the literature. We cite below the most recent of these results [9]. Recall that an ordered semigroup $(S, \leq)$ belongs to the variety of finite ordered semigroups $\mathbf{LJ}^+$ if and only if, for every s in S and every idempotent e in S, $ese \leq e$.

Proposition 7.1 *The relational morphism $\tau : M(L) \to M(L_1) \times M(L_2) \times \cdots \times M(L_n)$ is a $(\mathbf{LJ}^+, \mathbf{LJ}^+)$-relational morphism.*

The algebraic characterization of the polynomial closure was given in [12] for varieties of languages and in [9] for positive varieties. It can be further extended as follows.

Theorem 7.2 *Let $\mathbf{V}$ be a $\mathcal{C}$-variety of ordered stamps and let $\mathcal{V}$ be the associated positive $\mathcal{C}$-variety of languages. Then Pol $\mathcal{V}$ is a positive $\mathcal{C}$-variety and the associated $\mathcal{C}$-variety of ordered stamps is the Mal'cev product $(\mathbf{LJ}^+, \mathbf{LJ}^+) \circledast \mathbf{V}$.*

It is interesting to observe that, in [12], two different definitions of the polynomial closure were used. One for classes of languages of A^* — the one given above — and another one for classes of languages of A^+: the *polynomial closure* of a class of languages $\mathcal{L}$ of A^+ is the set of languages of A^+ that are finite unions of languages of the form $u_0L_1u_1 \cdots L_nu_n$, where $n \geq 0$, the u_i's are words of A^* and the L_i's are elements of $\mathcal{L}$. If $n = 0$, one requires of course that u_0 is not the empty word.

This definition can be recovered in the framework of varieties of stamps. The first step consists in interpreting the notion of variety of finite semigroups.

Let us introduce a notation: if $(S, \leq)$ is an ordered semigroup, denote by $(S^I, \leq)$ the ordered monoid defined as follows: $S^I = S \cup \{I\}$, where I is a new identity, and the order on S^I is the order on S, together with the trivial relation $I \leq I$.

Let $\mathbf{V}$ be a variety of finite ordered semigroups. We now associate to $\mathbf{V}$ the *ne*-variety of ordered stamps $\mathbf{V}'$ consisting of all morphisms $\varphi : A^* \to M$ such that the semigroup $\varphi(A^+)$ is in $\mathbf{V}$. In particular, if $(S, \leq) \in \mathbf{V}$ and $\varphi : A^+ \to S$ is an onto morphism, the stamp $\varphi^I : A^* \to S^I$ defined by setting $\varphi^I(1) = I$, is in $\mathbf{V}'$.

We now compare $\mathcal{V}$ to $\mathcal{V}'$, the positive *ne*-varieties of languages corresponding to $\mathbf{V}$ and $\mathbf{V}'$, respectively.

Lemma 7.3 *Every language of $\mathcal{V}(A^+)$ is a language of $\mathcal{V}'(A^*)$. Conversely, if L is a language of $\mathcal{V}'(A^*)$, then $L \cap A^+$ is a language of $\mathcal{V}(A^+)$.*

Proof. Let L be a language of $\mathcal{V}(A^+)$ and let $\varphi : A^+ \to (S, \leq)$ be its ordered syntactic morphism. Then $(S, \leq) \in \mathbf{V}$ and thus $\varphi^I \in \mathbf{V}'$. Let J be an order ideal of S such that $L = \varphi^{-1}(J)$. Then J is also an order ideal of S^I and thus L is a language of $\mathcal{V}'(A^*)$.

Conversely, if $L \in \mathcal{V}'(A^*)$, its ordered syntactic morphism $\varphi : A^* \to (M, \leq)$ belongs to $\mathbf{V}'$ and thus $\varphi(A^+) \in \mathbf{V}$. It follows that $L \cap A^+ \in \mathcal{V}(A^+)$. $\square$

Using $\mathcal{V}'$ in the place of $\mathcal{V}$ avoids the need for two separate definitions for the polynomial closure and Theorem 7.2 is sufficient in all the cases. More precisely:

Proposition 7.4 *For each alphabet A, Pol $\mathcal{V}'(A^*)$ is the set of languages that are finite union of languages of the form $u_0L_1u_1 \cdots L_nu_n$, where $n \geq 0$, the u_i's are words of A^* and the L_i's are elements of $\mathcal{L}$.*

Proof. First, every language of the form $L_0a_1L_1 \cdots a_nL_n$, with $a_0, \ldots, a_n \in A$, is also of the form $u_0L_1u_1 \cdots L_nu_n$.

We claim that $\mathcal{V}'(A^*)$ contains the languages $\{1\}$ and A^+. Indeed, the trivial semigroup 0 belongs to $\mathbf{V}$. Therefore the stamp $\varphi^I : A^* \to S^I = \{1, 0\}$ defined by $\varphi^I(1) = 1$ and $\varphi^I(a) = 0$ for each $a \in A$ is in $\mathbf{V}'$. This proves the claim, since $\{1\} = (\varphi^I)^{-1}(1)$ and $A^+ = (\varphi^I)^{-1}(0)$.

Consider now a language L which is a union of languages of the form

$$u_0L_1u_1 \cdots L_nu_n$$

where the u_i's are words of A^* and the L_i's are elements of $\mathcal{L}$. Observing that if $u = a_1 \cdots a_k$, then $\{u\} = \{1\}a_1\{1\}a_2\{1\} \cdots \{1\}a_k\{1\}$, we may assume that each word u_i is either a letter or the empty word. Next, the formula

$$KK' = K(\{1\} \cap K') \cup \bigcup_{a \in A} Ka(a^{-1}K')$$

shows that the u_i's equal to the empty word can be eliminated, since if $L \in \mathcal{L}$, then $a^{-1}L \in \mathcal{L}$. Thus L belongs to Pol $\mathcal{V}'(A^*)$. $\square$

Similar subtleties occurring in [9] for the definition of the Schützenberger product of n semigroups also disappear within the framework of $\mathcal{C}$-varieties.

8 Conclusion

In this paper, we generalized to $\mathcal{C}$-varieties a number of algebraic results of varieties of monoids, but there is still a lot to do. In particular, it is possible to extend the theory of wreath products to $\mathcal{C}$-varieties. Due to space limitation, it was not possible to include this generalization in the present paper and it will be the topic of a subsequent article.

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