CHAPTER 7

RECURRENCES

Recurrence relations define sequences of numbers. They can be obtained

- either by inductive reasoning as we have already seen in examples in Chapter 3,
- or by strategies which divide a size n problem into smaller problems (of size $\leq n-1$), each of which can be solved more easily in general; e.g. strategies of the type 'divide and conquer', solving problems by dichotomy. Several such examples will be studied below.

In the present chapter we will study some methods for explicitly finding the sequences of numbers defined by such relations.

This chapter is mainly devoted to linear recurrences for which there exist classical mathematical theories explaining how and why these recurrences can be explicitly solved. Linear algebra (vector spaces, matrices, eigenvectors and eigenvalues) is one of these theories. Its knowledge is assumed for the proof of Proposition 7.14. For fundamentals about algebra we suggest

C. Norman, *Undergraduate Algebra*, Oxford University Press, Oxford (1986).
Bartel Leenert Van der Waerden, *Algebra*, Frederick Ungar Publishing Company, New York (1970).

We recommend the following handbooks:

Gilles Brassard, Paul Bratley, *Algorithmics: Theory and Practice*, Prentice Hall, London, (1988).
Ronald Graham, Donald Knuth, Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, London (1989).
Donald Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, London (1973).
Chung Laung Liu, *Introduction to Combinatorial Mathematics*, Mc Graw-Hill, New York (1968).

7.1 Introduction: examples, generalities

7.1.1 Examples

The present section consists of a list of examples showing how recurrence relations are obtained, what forms they can take, and gives some ideas on how to solve them.

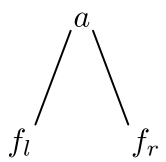
Example 7.1 (Number of binary trees with n nodes) The binary trees studied here can possibly be empty, which is not the case for the trees studied in Chapter 10. Recall (see Example 3.9, 5) that binary trees B labelled by the alphabet $\{a\}$ are recursively defined by

- $\emptyset$ is a binary tree, namely, $\emptyset \in B$ (basis),
- if b_l and b_r are binary trees then (a, b_l, b_r) is also a binary tree (inductive step of the recursive definition).

We can define the number $n(b)$ of nodes of a binary tree b similarly by recurrence

- tree $\emptyset$ has no node, namely $n(\emptyset) = 0$,
- $n((a, b_l, b_r)) = 1 + n(b_l) + n(b_r)$.

We can evaluate the number b_n of binary trees with n nodes as follows. A tree with n nodes can be represented by



where f_l (the left child) is a tree with k nodes and f_r (the right child) is a tree with $n - k - 1$ nodes.

We will evaluate the number b_n of binary trees with n nodes by induction, and we will notice that

- $b_0 = 1$ (the empty tree $\emptyset$ is the only tree with 0 nodes),
- $b_n = \sum_{k=0}^{n-1} b_k \times b_{n-1-k}$: this equality follows from the fact that in order to obtain a tree with n nodes it is necessary (and sufficient) to consider all possible binary trees of the form (a, b_l, b_r) with b_l (resp. b_r) a binary tree with k (resp. $n - k - 1$) nodes; there are b_k possibilities for b_l, and b_{n-k-1} possible choices for b_r, thus $b_k \times b_{n-k-1}$ possible choices for b, and this holds for all possible ks.

Hence, we have the recurrence relation

$$b_n = \sum_{k=0}^{n-1} b_k \times b_{n-1-k} \ . \tag{7.1}$$

We see that in this case, b_n is defined in terms of $b_0, b_1, \ldots, b_{n-1}$: this is one of the most complex cases of recurrence relations that we will encounter.

EXERCISE 7.1 Recall that in a binary tree a node is said to be internal if it has either a non-empty right child, or a non-empty left child or both. Let b_n be the number of binary trees with n internal nodes.

1. Compute b_0, b_1, b_2.
2. Find a recurrence relation giving b_n. $\diamondsuit$

EXERCISE 7.2 Let $\Sigma = \{a_1, a_2, \ldots, a_k\}$ be an alphabet with k elements; recall that the binary trees BIN labelled by Σ are defined inductively by

- $\emptyset$ is a binary tree,
- if $x \in \Sigma$, $b_l \in BIN$, $b_r \in BIN$, then $(x, b_l, b_r) \in BIN$.

The depth $p(b)$ of a binary tree is defined by:

- $p(\emptyset) = 0$,
- $p\big((x, b_l, b_r)\big) = 1 + \sup\{p(b_l), p(b_r)\}$.

Give recurrence relations defining

1. the number u_n of binary trees U_n of depth less than or equal to n (in terms of u_{n-1}). (Computing u_n is not required.)
2. the number v_n of binary trees AB_n of depth exactly n (in terms of v_{n-1} and of the u_is for $i \leq n$). (Computing v_n is not required.) $\diamondsuit$

EXAMPLE 7.2 The Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \ , \tag{7.2}$$

with initial conditions $F_0 = 0$, $F_1 = 1$.

EXERCISE 7.3 n lines are drawn on a plane; they intersect and thus delimit a certain number of bounded regions and of infinite regions. What is the maximum possible number of bounded regions determined? $\diamondsuit$

EXERCISE 7.4 n overlapping circles in the plane are assumed to intersect pairwise in two points, with neither tangential nor triple points. Show that the number of regions thus defined in the plane is determined by the recurrence relation

$$n > 1, \qquad r_n = r_{n-1} + 2(n-1) \ , \tag{7.3}$$

with $r_1 = 2$. $\diamondsuit$

EXAMPLE 7.3 Binary search to determine the maximum in a list of n elements: in order to find the maximum in a length n list, we divide it into two lists of length $n/2$ (assuming n of the form 2^k), we find the maximum of each one of the two lists, then we compare these two maxima. If t_n is the required time for finding the maximum of a length n list, we have

$$t_n = 2t_{n/2} + 1 \quad , \quad t_2 = 1 \ . \tag{7.4}$$

(We assume that the unit of time complexity is the cost of a comparison, hence $t_2 = 1$.)

EXERCISE 7.5 Compute t_n defined in Example 7.3. $\diamondsuit$

EXAMPLE 7.4 Let $f_h(k)$ be the maximum number of leaves on a tree of height h where each node has at most k children. $f_1(k) = 1$, since a tree of height 1 is reduced to a single node being both root and leaf. We immediately check that

$$f_h(k) = kf_{h-1}(k) . \tag{7.5}$$

See also Chapter 10.

EXERCISE 7.6 Compute $s_p = 1 - 2 + \cdots + (2p-3) - (2p-2) + (2p-1) - 2p$, for $p \in \mathbb{N}$. ♢

EXAMPLE 7.5 The following recurrence relations are useful for studying the complexity of Quicksort (see Section 14.1):

$$p_2 = 3, \text{ and for } n > 2, \qquad p_n = n + 1 + p_{n-1} , \tag{7.6}$$

$a_0 = b_0 = a_1 = b_1 = c_0 = c_1 = 0$, and $\forall n \geq 2$,

$$a_n \leq c_n \leq b_n ,$$

$$a_n = n - 1 + 2/n \sum_{k=1}^{n-1} a_k \quad \text{and} \quad b_n = n + 1 + 2/n \sum_{k=1}^{n-1} b_k . \tag{7.7}$$

7.1.2 Generalities, classification

In the preceding section we saw various recurrence relations. They were all of the form

$$u_n = f(n, \{u_0, u_1, \ldots, u_{n-1})\}) , \qquad n \in J \subseteq \mathbb{N} ,$$

with initial conditions enabling us to start the recurrence. Several methods are available for solving them but we must first determine the type of the recurrence relation. To this end, we have three orthogonal classification criteria.

- First, the type of the function f, which can be
  - a *linear combination*, as in the case of relations (7.2), (7.3), (7.4), (7.5), (7.6), (7.7), having constant coefficients ((7.2), (7.3), (7.4), (7.5)), or coefficients depending on n ((7.7)); the recurrence relation is then said to be *linear* ;
  - a *polynomial*, as in the case of relation (7.1); the recurrence relation is then said to be *polynomial* ;
- Second, the set of u_ps needed to compute u_n
  - if we need $u_{n-1}, \ldots, u_{n-k}$ to compute u_n, the recurrence relation is said to be *of degree* k, and then $J \subseteq \{n \,/\, n \geq k\}$. Relation (7.2) is of degree 2, (7.3), (7.5), (7.6) are of degree 1.

– if we need $u_0, \ldots, u_{n-1}$ to compute u_n, the recurrence relation is said to be *complete*, and then $J = \{n \,/\, n \geq 1\}$. For instance, relation (7.1) is complete, and so are relations (7.7).

– if we only need $u_{n/a}$ with a constant, $a \in \mathbb{N}$, $a > 1$, to compute u_n, the recurrence relation is then said to be a *partition recurrence*; in that case $J = \{n \,/\, a \text{ divides } n\}$. For instance, relation (7.4) is of this type. Usually, partition recurrences are obtained by dividing a size n problem into one or more smaller problems, hopefully simpler to solve, e.g. using dichotomic methods, and more generally strategies of the type 'divide and conquer'.

• Third, the fact that function f does or does not have parameters and terms other than multiples of the u_js, $j < n$.

– if f depends only on the u_js, $j < n$, the recurrence relation is said to be *homogeneous*: e.g. recurrences (7.1), (7.2), (7.5).

– if f depends on terms other than the u_js, most often the recurrence relation will be of the form $u_n = f(\{u_p \;/\; p < n\}) + g(n)$, the recurrence is said to be *non-homogeneous* and $g(n)$ is called its *right-hand side*: e.g. recurrences (7.3), (7.4).

Let $v_n = f(\{v_p \;/\; p < n\})$, $n \in J \subseteq \mathbb{N}$, be a recurrence relation. Solving this recurrence consists of finding a sequence $(u_i)_{i\geq 0}$ such that $\forall n \in J$, $u_n = f(\{u_p \;/\; p < n\})$. Among the solutions, we are interested in those satisfying *initial conditions*, given by a set of values $\{a_i \,/\, i \in I\}$, where I is a subset of $\mathbb{N}$. A sequence $(u_i)_{i\geq 0}$ satisfies the initial conditions if $\forall n \in I$, $u_n = a_n$.

Many useful and systematic methods are available for solving linear recurrences of finite degree, whether homogeneous or not. Some methods are available in the case of partition recurrences, though to a lesser extent. Finally, for polynomial or complete recurrences, the solution is more complex, and will involve more elaborate tools such as generating series and differential equations. We will study some examples in Chapter 8 (generating series).

Proposition 7.6 *Let $v_n = f(\{v_p \;/\; p < n\})$ be a recurrence relation; assume that f is a mapping (i.e. its domain is the whole set $\mathbb{N}$);*

• *if the recurrence is complete, and if u_0 is given, the recurrence has at least one solution.*

• *if the recurrence is of degree k, and if we are given the initial values $u_0, \ldots, u_j$, $j < k$, the recurrence has at least one solution.*

• *if the recurrence is a partition recurrence of the form $v_n = f(v_{n/a})$, and if we are given $u_0, \ldots, u_j$, $j < a$, the recurrence has at least one solution.*

Proof. Verify, by induction on n, the property $P(n)$: $\exists u_0, u_1, u_2, \ldots, u_n$, satisfying the given recurrence; assume, for instance, that f is complete, and that u_0 is given. Then,

- u_0 exists since it is given, and u_0 trivially satisfies the recurrence;
- assume $u_0, \ldots, u_{n-1}$ satisfying the recurrence exist, and let $u_n = f(u_0, \ldots, u_{n-1})$; u_n satisfies the recurrence by construction; we have thus shown the existence of $u_0, \ldots, u_n$ satisfying the recurrence, hence the induction hypothesis and the result.

The cases when f is a linear or a partition recurrence are similar. □

REMARK 7.7 A solution may not exist if the initial conditions are too demanding, e.g. initial values $u_0, \ldots, u_{k+p}$, where $u_k, \ldots, u_{k+p}$, are incompatible with the recurrence relation in the case of a linear recurrence of degree k.

EXERCISE 7.7 Find examples of non-solvable recurrences. ♢

In spite of the preceding remark, the problem of the existence of the solutions will in general not occur for the recurrences obtained in cost or complexity evaluations. The problem of uniqueness of solution is more complex.

Proposition 7.8 *Let $v_n = f(\{v_p \ / \ p < n\})$ be a recurrence relation, then its solution u_n is uniquely defined if*

- *f is a recurrence of degree k and $u_0, \ldots, u_{k-1}$ are given,*
- *f is a complete recurrence and u_0 is given,*
- *f is a partition recurrence of the form $v_n = f(v_{n/a})$ and all the u_is such that $i < a$ or $i \geq a$ and a does not divide i are given.*

Proof. The existence of a sequence $(u_n)_{n\geq 0}$ satisfying the given conditions follows from Proposition 7.6.

Let $(u_n)_{n\geq 0}$ and $(u'_n)_{n\geq 0}$ be two solutions of $v_n = f(\{v_p \ / \ p < n\})$, $n \in J$, both satisfying the given initial conditions. Show by induction on n the property $P(n) : u_n = u'_n$ for all n. We will use the second induction principle: we must thus prove that

$$\forall n \in \mathbb{N}, \qquad \big(\forall l < n,\ P(l)\big) \quad \Longrightarrow \quad P(n)\,. \tag{7.8}$$

Since $(u_n)_{n\geq 0}$ and $(u'_n)_{n\geq 0}$ satisfy $v_n = f(\{v_p \ / \ p < n\})$, $\forall n \in J$, it is clear that

$$\forall n \in J, \qquad \big(\forall l < n,\ u_l = u'_l\big) \quad \Longrightarrow \quad u_n = u'_n\,.$$

Moreover, since $(u_n)_{n\geq 0}$ and $(u'_n)_{n\geq 0}$ satisfy the same initial conditions $\{a_i \ / \ i \in I\}$, we have $\forall i \in I,\ u_i = u'_i$. Hence,

$$\forall n \in I \cup J, \qquad \big(\forall l < n,\ u_l = u'_l\big) \quad \Longrightarrow \quad u_n = u'_n\,.$$

It can easily be seen that, in all three considered cases, $I \cup J = \mathbb{N}$, hence (7.8), then $\forall n,\ u_n = u'_n$ and thus the solution is unique. □

REMARK 7.9 Combining the first two clauses of Proposition 7.6 with the corresponding clauses of Proposition 7.8, we obtain necessary and sufficient conditions for the existence and uniqueness of the solutions of recurrence relations of degree k and complete recurrences.

EXERCISE 7.8 Find examples of non-unique solutions for various types of recurrence. ♢

REMARK 7.10 The condition implying uniqueness of the solution of partition recurrences is quite restrictive. Usually, we will thus try to obtain the uniqueness more simply:

- either by restricting the domain and computing the u_ns on a subset of $\mathbb{N}$.
- or by finding only estimates of the solutions, namely, by studying the asymptotic behaviour of the solutions instead of the exact solutions themselves (see Example 9.17).

EXAMPLE 7.11 Consider the partition recurrence $u_n = bu_{n/a} + d(n)$. Assuming u_1, u_n is uniquely defined on

$$S = \{n = a^k \ / \ k \in \mathbb{N}\}\,.$$

We will solve this recurrence by letting $v_k = u_{a^k}$; v_k is then defined by $v_0 = u_1$ and $v_{k+1} = bv_k + d(a^{k+1})$. We reduced the problem to that of solving a linear recurrence of degree 1, which will be treated in the next section.

EXAMPLE 7.12 Let the partition recurrence $u_n = 2u_{n/2}$ (corresponding, for instance, to a binary merge-sort), with $u_1 = 1$. Then u_n is uniquely determined on $S = \{n = 2^k \ / \ k \in \mathbb{N}\}$; but, for $n \notin S$ we can no longer uniquely determine u_n, i.e. for $n = 2^k v$ with v odd we will have $u_n = 2^k u_v$, namely, u_n will be determined up to the coefficient u_v.

EXERCISE 7.9 Solve the recurrence relation

$$\forall n \geq 1, \qquad u_n = 3\sum_{k=0}^{n-1} u_k + 1.$$ ♢

7.2 Linear recurrences

7.2.1 Linear homogeneous recurrences with constant coefficients

A linear homogeneous recurrence relation of degree k with constant coefficients is defined by an equation of the form

$$\forall n \geq k\,, \qquad u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}\ . \tag{7.9}$$

Method of the characteristic polynomial

Definition 7.13 *Let the characteristic polynomial of the recurrence* (7.9)*, be defined by*

$$P(r) = r^k - a_1 r^{k-1} - \cdots - a_{k-1} r - a_k$$

and the characteristic equation of the recurrence (7.9)*, be defined by*

$$r^k = a_1 r^{k-1} + \cdots + a_{k-1} r + a_k \ . \tag{7.10}$$

Proposition 7.14
(i) *The set of solutions of equation* (7.9) *is a vector space of dimension* k.
(ii) 1. *If* $P(r)$ *has* k *pairwise distinct roots* $r_1, \ldots, r_k$*, then the* k *sequences* $\{r_i^n \ / \ n \in \mathbb{N}\}$*,* $i = 1, \ldots, k$*, are a basis of the vector space of the solutions of* (7.9)*, and any solution of* (7.9) *is of the form* $u_n = \sum_{i=1}^k \lambda_i r_i^n$*, where the* λ_i*s are determined by the initial values* $u_0, \ldots, u_{k-1}$.
2. *If the roots of* $P(r)$ *are* r_j*, such that for* $j = 1, \ldots, p$*, with* $p < k$*, each* r_j *is a multiple root of multiplicity* m_j*, then the* k *sequences* $\{(r_j^n)_{n\in\mathbb{N}}, (n{r_j}^n)_{n\in\mathbb{N}}, \ldots, (n^{m_j-1} r_j^n)_{n\in\mathbb{N}}, \quad j = 1, \ldots, p\}$ *are a basis of the vector space of solutions of* (7.9)*; any solution of* (7.9) *will be of the form*

$$u_n = \sum_{i=1}^{p} P_j(n) r_j^n \ ,$$

where $P_j(n)$ *is a polynomial of degree* $\leq m_j - 1$.

Note that the first case is an instance of the second case, with $p = k$ and $m_j = 1$.

Proof.
(i) Clearly, if u_n and v_n are solutions of (7.9), any linear combination $w_n = \lambda u_n + \mu v_n$ is also a solution of (7.9); thus the set of solutions forms a vector space. This vector space is of dimension k at most because giving $u_1, \ldots, u_k$ suffices to uniquely determine u_n. Lastly, this vector space is of dimension k, since k linearly independent sequences can be found; for $j = 1, \ldots, k$, consider the sequences $u^1, \ldots, u^k$, where the initialization of u^j is defined by

for $i = 1, \ldots, k$, $$u_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then the k sequences $u^1, \ldots, u^k$ define a set of k linearly independent sequences.

(ii) Assume that r is a root of (7.10), then, clearly, $u_n = r^n$ is a solution of the recurrence relation, with the initial conditions $u_i = r^i$ for $i = 0, \ldots, k-1$.

- thus, in the case when the r_js are pairwise distinct it is simple to verify that the k sequences $u^1, \ldots, u^k$, where u^j is defined by the initial conditions $u_i^j = r_j^i$ for $i = 0, \ldots, k-1$, form a basis of the vector space of the solutions: these are indeed k linearly independent solutions since the determinant

$$\det \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_k \\ \vdots & \vdots & & \vdots \\ r_1^{k-1} & r_2^{k-1} & \ldots & r_k^{k-1} \end{vmatrix} = \prod_{j<i} (r_i - r_j)$$

is non-zero (Vandermonde's determinant). Any solution of (7.9) is thus a linear combination of the solutions $u^1, \ldots, u^k$.

- Assume now that r_j is a root of multiplicity m_j of the characteristic equation $P(r) = r^k - (a_1 r^{k-1} + \cdots + a_{k-1} r + a_k) = 0$, and check that the m_j sequences $(r_j^n)_{n\in\mathbb{N}}, (nr_j^n)_{n\in\mathbb{N}}, \ldots, (n^{m_j-1} r_j^n)_{n\in\mathbb{N}}$ are all solutions of (7.9). For the sequence $(r_j^n)_{n\in\mathbb{N}}$, it is straightforward since r_j is a root of the characteristic polynomial $P(r)$. For the other sequences $(nr_j^n)_{n\in\mathbb{N}}, \ldots, (n^{m_j-1} r_j^n)_{n\in\mathbb{N}}$ we need a result from algebra (Lemma 7.15).

For any $n \geq k$ and $p \geq 0$, let $Q_{n,p}(r)$ be the polynomial

$$\begin{aligned} Q_{n,p}(r) = n^p r^n - \big(& a_1 (n-1)^p r^{n-1} + a_2 (n-2)^p r^{n-2} \\ & + \cdots + a_{k-1}(n-k+1)^p r^{n-k+1} + a_k (n-k)^p r^{n-k}\big). \end{aligned}$$

We prove by induction on n that there exist polynomials $R_{n,p,i}(r)$ such that for $i = 0, \ldots, p$

$$Q_{n,p}(r) = \sum_{i=0}^{p} R_{n,p,i}(r) P^{(i)}(r) \,, \qquad (E)$$

where $P^{(i)}$ is the ith derivative of P.

(B) If $p = 0$, $Q_{n,0}(r) = r^{n-k} P(r)$; we thus let $R_{n,0,0}(r) = r^{n-k}$.

(I) We assume that (E) is true for p and note that

$$\begin{aligned} Q_{n,p+1}(r) &= r Q'_{n,p}(r) \\ &= r \sum_{i=0}^{p} \left(R'_{n,p,i}(r) P^{(i)}(r) + R_{n,p,i}(r) P^{(i+1)}(r) \right), \end{aligned}$$

hence
$$\begin{cases} R_{n,p+1,0}(r) = r R'_{n,p,0}(r) \\ R_{n,p+1,p+1}(r) = r R_{n,p,p}(r) \end{cases}$$

and for $0 < i \leq p$,
$$R_{n,p+1,i}(r) = r\big(R'_{n,p,i}(r) + R_{n,p,i-1}(r)\big).$$

Hence the induction hypothesis and (E).

From (E) and Lemma 7.15 we deduce that $Q_{n,p}(r_j) = 0$ for $p < m_j$; this implies that $(n^p r_j^n)_{n\in\mathbb{N}}$ is a solution of (7.9).

The solutions thus obtained are linearly independent because the determinant

$$\left|\begin{array}{ccccccc} 1 & 0 & 0 & \dots & 0 & 1 & \dots \\ r_1 & r_1 & r_1 & \dots & r_1 & r_2 & \dots \\ r_1^2 & 2r_1^2 & 2^2 r_1^2 & \dots & 2^{m_1-1} r_1^2 & r_2^2 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ r_1^{k-1} & (k-1)r_1^{k-1} & (k-1)^2 r_1^{k-1} & \dots & (k-1)^{m_1-1} r_1^{k-1} & r_2^{k-1} & \dots \end{array}\right.$$

$$\left.\begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \\ r_p & r_p & r_p & \dots & r_p \\ r_p & 2r_p^2 & 2^2 r_p^2 & \dots & 2^{m_p-1} r_p^2 \\ \vdots & \vdots & \vdots & & \vdots \\ r_p^{k-1} & (k-1)r_p^{k-1} & (k-1)^2 r_p^{k-1} & \dots & (k-1)^{m_p-1} r_p^{k-1} \end{array}\right|$$

is equal to $\left(\prod_{1\le j\le p} r_j^{\binom{m_j}{2}}\right)\left(\prod_{1\le j<i\le p} (r_i - r_j)^{m_i m_j}\right)$ and is non-zero (the r_is are non-zero and assume distinct values).

The general solution of (7.9) is again a linear combination of the solutions $(n^{k_j} r_j^n)_{n\in\mathbb{N}}$, $j = 1, \dots, p$, $k_j = 0, \dots, m_j - 1$, which form the basis of the vector space. $\square$

Lemma 7.15 *If r is a root of multiplicity m of $P(x)$, then r is also a root of $P'(x), \dots, P^{(m-1)}(x)$, where $P^{(k)}(x)$ is the kth derivative of P.*

Proof. By induction on k, we prove that, if r is a root of multiplicity m of $P(x)$, then, for $1 \le k \le m-1$, r is also a root of multiplicity $m-k$ of $P^{(k)}(x)$.

(B) r is a root of multiplicity m of $P(x)$ implies that $P(x) = (x-r)^m Q(x)$, with $Q(x)$ a polynomial such that r is not a root of $Q(x)$. Since $P'(x) = m(x-r)^{m-1}Q(x) + (x-r)^m Q'(x) = (x-r)^{m-1}\big(mQ(x) + (x-r)Q'(x)\big)$, r is a root of multiplicity $m-1$ of $P'(x)$.
(I) The inductive step is similar: for any $k < m-1$, we can check that if r is a root of multiplicity $m-k$ of $P^{(k)}(x)$, then r is a root of multiplicity $m-k-1$ of $P^{(k+1)}(x)$. $\square$

Exercise 7.10 Find the solutions of the recurrence relation

$$u_n = 5u_{n-1} - 8u_{n-2} + 4u_{n-3}, \quad n \ge 3\,,$$

with the initial conditions $u_0 = 0$, $u_1 = 1$, $u_2 = 2$. ♢

EXERCISE 7.11 Let $\Sigma = \{a, b\}$ and let B be the subset of Σ^* defined inductively by

(B) $a \in B$, $b \in B$,
(I) $w \in B \quad \Longrightarrow (abw \in B, baw \in B)$.

1. Write six elements of B.
2. Prove: $w \in B \Longrightarrow |w|$ odd ($|w|$ denotes the length of w). Is the converse true?
3. Let $u_n = card\{w \;/\; w \in B \text{ and } |w| = n\}$. Compute u_1 and u_2. Find a recurrence relation for u_n and solve it. ♢

EXERCISE 7.12 Let $\Sigma = \{a, b, c, d\}$ and let

$$\begin{aligned} L &= \{w \in \Sigma^* \,/\, ab \text{ is not a factor of } w\} \\ &= \{w \in \Sigma^* \,/\, \nexists w_1, w_2 \in \Sigma^* \text{ with } w = w_1abw_2\} \,. \end{aligned}$$

$\forall n \geq 0$, let

$$L_n = L \cap \Sigma^n,$$

$$u_n = |L_n| \qquad (\text{ the cardinality of } L_n) \,.$$

Recall that Σ^n consists of the length n words on the alphabet Σ.

1. Compute $L_0, L_1, L_2, u_0, u_1, u_2$.
2. Find a necessary and sufficient condition for a word $w = xw'$, with $x \in \Sigma$, to be in L_n.
3. Express L_n in terms of L_{n-1} and L_{n-2}; deduce that the recurrence defining u_n is given by $u_n = 4u_{n-1} - u_{n-2}$.
4. Compute u_n.
5. Also compute $v_n = |L'_n|$ with $L'_n = L' \cap \Sigma^n$ and

$$L' = \{w \in \Sigma^* \,/\, w \text{ has neither } ab \text{ nor } ac \text{ as factor}\} \,.$$ ♢

EXERCISE 7.13 Let

$$u_n = au_{n-1} + bu_{n-2} \tag{7.11}$$

be a linear homogeneous recurrence of degree 2 with constant coefficients whose associated polynomial $r^2 - ar - b$ has two conjugate complex roots ce^{-it} and ce^{it}. Show that the general solution of (7.11) is of the form $u_n = \lambda c^n \cos(nt) + \mu c^n \sin(nt)$. ♢

EXERCISE 7.14 Solve the recurrence relations

1. $\forall n \geq 2$, $u_n = u_{n-1} - 2u_{n-2}$.
2. $\forall n \geq 2$, $u_n = u_{n-1} - 2u_{n-2} + 4$. ♢

EXERCISE 7.15 Solve the following recurrences:

1. $\forall n \geq 2$, $2u_n = 3u_{n-1} - u_{n-2}$.
2. $\forall n \geq 2$, $u_n = 4u_{n-1} - 4u_{n-2}$. ♢

Matrix method

Another method for solving linear homogeneous recurrences with constant coefficients is available: the matrix method. Consider again the recurrence relation (7.9) (page 127)

$$\forall n \geq k\,, \qquad u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k}\,. \tag{7.9}$$

It can be rewritten as

$$\begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_{k-1} & a_k \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \times \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_{n-k} \end{bmatrix} = M \times \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_{n-k} \end{bmatrix}.$$

We deduce, by multiplication,

$$\begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix} = M^{n-k} \times \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \end{bmatrix},$$

whence the following computation method: assume that M has k distinct eigenvalues, let $r_1, \dots, r_k$ be the eigenvalues of M and let N be the associated eigenvectors matrix such that

$$M = N \times \begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & r_k \end{bmatrix} \times N^{-1}\,,$$

then

$$\begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix} = N \times \begin{bmatrix} r_1^{n-k} & 0 & \dots & 0 \\ 0 & r_2^{n-k} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & r_k^{n-k} \end{bmatrix} \times N^{-1} \times \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \end{bmatrix},$$

namely, in order to compute the general solution of (7.9), it is enough to diagonalize M as follows:

- Find the eigenvalues of

$$M = \begin{bmatrix} a_1 & a_2 & \dots & a_{k-1} & a_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} ;$$

this can be done by writing that the following determinant is equal to zero

$$\det \begin{vmatrix} a_1 - r & a_2 & \dots & a_{k-1} & a_k \\ 1 & -r & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -r \end{vmatrix} ,$$

which is precisely equal to $P(r)$ up to the plus or minus sign.

- Determine the eigenvector's matrix from M to its diagonal form.

The matrix method also applies for solving simultaneous linear homogeneous recurrences of degree 1 with constant coefficients; consider, for instance, the recurrence relations

$$\begin{aligned} u_n &= a_1 u_{n-1} + a_2 v_{n-1} + a_3 w_{n-1} \ , \\ v_n &= b_1 u_{n-1} + b_2 v_{n-1} + b_3 w_{n-1} \ , \\ w_n &= c_1 u_{n-1} + c_2 v_{n-1} + c_3 w_{n-1} \ . \end{aligned}$$

They can be written in the form

$$\begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \times \begin{bmatrix} u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{bmatrix} = M \times \begin{bmatrix} u_{n-1} \\ v_{n-1} \\ w_{n-1} \end{bmatrix} .$$

We deduce, by multiplication,

$$\begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} = M^{n-1} \times \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} ,$$

which is solved as previously.

Exercise 7.16 Solve the recurrences

1. $$\forall n \geq 1, \qquad \begin{cases} u_n &= 4u_{n-1} + 2v_{n-1}, \\ v_n &= -3u_{n-1} - v_{n-1}, \end{cases}$$

with $u_0 = a,\ v_0 = b$.

2. $$\forall n \geq 1, \qquad \begin{cases} u_n &= u_{n-1}^4 v_{n-1}^2 , \\ v_n &= \dfrac{1}{u_{n-1}^3 v_{n-1}} , \end{cases}$$

with $u_0 = a > 0,\ v_0 = b > 0$. ♢

Exercise 7.17 Let $\Sigma = \{0,1\}$. Interpret a word f in Σ^* as the binary representation of an integer. For instance, 101 represents 5, 1101 represents 13. Let $L_i = \{f \in 1\cdot\Sigma^* \,/\, f \equiv i[3]\}$, $i = 0,1,2$, otherwise stated,

$$L_i = \{f \in \Sigma^* \,/\, f \text{ starts with 1 and } f \equiv i \text{ modulo } 3\}, \quad i = 0,1,2\,.$$

For all $n \geq 1$, let:

$$u_n = |L_0 \cap \Sigma^n|$$
$$v_n = |L_1 \cap \Sigma^n|$$
$$w_n = |L_2 \cap \Sigma^n|$$

1. Compute u_1, v_1, w_1 and u_2, v_2, w_2.
2. If $f \in L_i$ in which set is the word $f0$? In which set is the word $f1$? Deduce that $u_{n+1} = u_n + v_n$, $v_{n+1} = u_n + w_n$, $w_{n+1} = w_n + v_n$.
3. Compute u_n, v_n, w_n. ♢

Method of the generating series

Finally, the third method for solving linear recurrences consists of generating series. Consider a sequence u_n defined by the recurrence relation (7.9), and define the series $u(z) = \sum_{n\geq 0} u_n z^n$, called the generating series associated with the sequence u_n. We will compute the generating series, and deduce the u_ns.

7.2.2 Non-homogeneous linear recurrences with constant coefficients

Non-homogeneous linear recurrences are also called recurrences with a right-hand side. They assume the general form

$$\forall n \geq k\,, \qquad u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + b(n)\ . \tag{7.12}$$

Proposition 7.16 *The solutions of the equation* (7.12) *form an affine space of dimension k, whose associated vector space is the set of solutions of the associated homogeneous recurrence* (7.9)*.*

Proof. Let v_n be a particular solution of (7.12); all other solutions v'_n are of the form $v'_n = v_n + w_n$, where w_n is a solution of the recurrence relation (7.9) (page 127)

$$w_n = a_1 w_{n-1} + \cdots + a_k w_{n-k}\ . \tag{7.9}$$

Indeed, if we have

$$v_n = a_1 v_{n-1} + \cdots + a_k v_{n-k} + b(n)$$

and

$$v'_n = a_1 v'_{n-1} + \cdots + a_k v'_{n-k} + b(n) .$$

We deduce by subtraction that $w_n = v'_n - v_n$ satisfies

$$w_n = a_1 w_{n-1} + \cdots + a_k w_{n-k} .$$

Conversely, it is clear that if w_n is a solution of (7.9) and v_n is a particular solution of (7.12), $u_n = w_n + v_n$ is also a solution of (7.12). □

Therefore, to obtain the general solution of (7.12), it 'suffices' to find a particular solution v_n, then to solve the associated homogeneous recurrence (7.9), and finally to add the general solution w_n of (7.9) to a particular solution v_n of (7.12). The problem is that no systematic method for finding a particular solution v_n exists except for a special type of function $b(n)$, which we will now study. First study a simple example.

Example 7.17 Consider the recurrence relation

$$n \geq 2, \quad u_n = 2u_{n-1} + 1/n , \tag{7.13}$$

with $u_1 = 1$. We easily verify that $v_n = \sum_{i=2}^{n} 2^{n-i}/i$ is a particular solution of (7.13). Moreover, the associated homogeneous recurrence can be written $w_n = 2w_{n-1}$, and its characteristic polynomial has the root $r = 2$; thus its general solution is of the form $\lambda 2^n$, and the general solution of (7.13) is of the form $u_n = \lambda 2^n + v_n$; taking into account that $u_1 = 1$ and $v_1 = 0$, we obtain $\lambda = 1/2$, hence $u_n = 2^{n-1} + \sum_{i=2}^{n} 2^{n-i}/i$.

The generating series method also applies, see Section 8.2.

Method of the characteristic polynomial

This method can be applied when the 'right-hand side' $b(n)$ of the non-homogeneous recurrence (7.12) is of the form $\sum_{i=1}^{l} b_i^n P_i(n)$, where, for $1 \leq i \leq l$, $P_i(n)$ is a polynomial in n, $b_i \neq 0$, and the b_is are distinct numbers.

Note that when the right-hand side of a linear recurrence relation is of the form $\sum_{i=1}^{l} c_i(n)$, the solution of

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + \sum_{i=1}^{l} c_i(n)$$

is the sum of the solutions of the l recurrences

$$u_n^{(i)} = a_1 u_{n-1}^{(i)} + \cdots + a_k u_{n-k}^{(i)} + c_i(n) .$$

In practice, in order to solve a recurrence whose right-hand side is $\sum_{i=1}^{l} b_i^n P_i(n)$ we will apply

- either the general method given in Proposition 7.18 below,
- or solve the l recurrences with right-hand sides $b_i^n P_i(n)$ separately.

Proposition 7.18 *Let the recurrence relation be defined by*

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + \sum_{i=1}^{l} b_i^n P_i(n) , \tag{7.14}$$

where all the b_is are distinct, and $P_i(n)$ is a polynomial in n of degree d_i. Then u_n is the solution of a linear homogeneous recurrence relation of degree $q = k + \sum_{i=1}^{l}(1 + d_i)$,

$$u_n = c_1 u_{n-1} + \cdots + c_q u_{n-q} ,$$

whose characteristic polynomial $r^q - c_1 r^{q-1} - \cdots - c_q$ is

$$\left(r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k\right) \prod_{i=1}^{l} (r - b_i)^{d_i+1} . \tag{7.15}$$

Any solution of (7.14) *is thus of the form $u_n = \sum_{j=1}^{p} Q_j(n) r_j^n$, where r_j is a root of multiplicity m_j of* (7.15) *and $Q_j(n)$ is a polynomial of degree $m_j - 1$.*

Preliminary remark. In order to simplify the notations in the proof, we assume here that the polynomial zero is of degree -1. Indeed, if $Z(n)$ is the polynomial zero

$$\begin{aligned} u_n &= a_1 u_{n-1} + \cdots + a_k u_{n-k} \\ &= a_1 u_{n-1} + \cdots + a_k u_{n-k} + \sum_{i=1}^{l} b_i^n Z(n) \end{aligned}$$

and we have

$$r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k = \left(r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k\right) \prod_{i=1}^{l} (r - b_i)^0 .$$

Proof. We prove the proposition by induction on $\delta = \sum_{i=1}^{l}(1 + d_i)$.

Basis. In the preliminary remark we saw that the result holds for $\delta = 0$.

Inductive step. In order to prove that if it holds for δ then it also holds for $\delta + 1$, we note that $\delta > 0$ implies that $d_i \geq 0$ for at least one i, and we prove the following property.

Lemma 7.19 *Let $P(n)$ be a polynomial of degree $d \geq 0$, and let $P_i(n)$ be a polynomial of degree $d_i \geq 0$, for $i = 1, \ldots, l$. Let $b, b_1, \ldots, b_l$ be non-zero constants with $b \neq b_i$, for all $i = 1, \ldots, l$. The solution of*

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + b^n P(n) + \sum_{i=1}^{l} b_i^n P_i(n) \tag{7.16}$$

is also the solution of

$$u_n = c_1 u_{n-1} + \cdots + c_{k+1} u_{n-k-1} + b^n Q(n) + \sum_{i=1}^{l} b_i^n Q_i(n) ,$$

with
- $r^{k+1} - c_1 r^k - c_2 r^{k-1} - \cdots - c_{k+1} = (r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k)(r - b)$,
- *the degree of $Q(n)$ is $d - 1$,*
- *the degree of $Q_i(n)$ is d_i .*

Proof. For $n \geq k + 1$,

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} + b^n P(n) + \sum_{i=1}^{l} b_i^n P_i(n) ,$$

$$u_{n-1} = a_1 u_{n-2} + \cdots + a_k u_{n-k-1} + b^{n-1} P(n-1) + \sum_{i=1}^{l} b_i^{n-1} P_i(n-1) ;$$

multiplying the second equality by b and subtracting it from the first equality, we have

$$\begin{aligned} u_n - b u_{n-1} = a_1(u_{n-1} - b u_{n-2}) + \cdots + a_k(u_{n-k} - b u_{n-k-1}) \\ + b^n Q(n) + \sum_{i=1}^{l} b_i^n Q_i(n) , \end{aligned} \tag{7.17}$$

with
$$\begin{aligned} Q(n) &= P(n) - P(n-1) , \\ Q_i(n) &= P_i(n) - \frac{b}{b_i} P_i(n-1) . \end{aligned}$$

The recurrence (7.17) can be rewritten as

$$u_n = c_1 u_{n-1} + \cdots + c_{k+1} u_{n-k-1} + b^n Q(n) + \sum_{i=1}^{l} b_i^n Q_i(n) ,$$

with

$$\begin{aligned} c_1 &= a_1 + b \\ c_2 &= a_2 - a_1 b \\ &\vdots \\ c_k &= a_k - a_{k-1} b \\ c_{k+1} &= -a_k b \ , \end{aligned}$$

hence $r^{k+1} - c_1 r^k - c_2 r^{k-1} - \cdots - c_k r - c_{k+1} = r^{k+1} - a_1 r^k - a_2 r^{k-1} - \cdots - a_k r - b r^k + a_1 b r^{k-1} + \cdots + a_k b = (r^k - a_1 r^{k-1} - a_2 r^{k-2} - \cdots - a_k)(r - b)$.

It remains to prove: if $P(n)$ is a polynomial of degree $d \geq 0$, $Q(n) = P(n) - cP(n-1)$ is a polynomial of degree $d-1$ if $c = 1$, of degree d otherwise.

- If $d = 0$, $P(n)$ is a constant $k \neq 0$, and $Q(n) = k(1-c)$ is 0 if and only if $c = 1$.
- Otherwise, $P(n) = kn^d + k'n^{d-1} + R(n)$ with $k \neq 0$, and $R(n)$ of degree less than or equal to $d-2$. Then,

$$Q(n) = kn^d - ck(n-1)^d + k'n^{d-1} - ck'(n-1)^{d-1} + R(n) - cR(n-1).$$

Expanding $(n-1)^d$ and $(n-1)^{d-1}$ by the binomial theorem and grouping together the terms of degree less than or equal to $d-2$ with R, we obtain

$$Q(n) = kn^d - ck(n^d - dn^{d-1}) + k'n^{d-1} - ck'n^{d-1} + R'(n) \ ,$$

which we simplify in $n^d k(1-c) + n^{d-1}\big(k'(1-c) + ckd\big) + R'(n)$.

  - If $c = 1$, $Q(n) = n^{d-1}kd + R'(n)$, and as k and d are non-zero, Q is of degree $d-1$.
  - If $c \neq 1$, Q is of degree d. □

Using Proposition 7.18 we can solve non-homogeneous recurrences of the form (7.14) in the same way as homogeneous recurrences.

EXERCISE 7.18 Let $\Sigma = \{a, b, c, d\}$. Let $L \subseteq \Sigma^*$ be the language containing no word of the form $ubvaw$, with u, v, w in Σ^* (namely, the letter a never occurs **after** the letter b).

Compute by induction on n the number u_n of length n words in the language L. ◇

EXAMPLE 7.20 (cf. Exercise 7.4) Let the recurrence: $u_n = 2u_{n-1}+1$, for $n > 1$, with $u_1 = 1$. This recurrence is obtained by evaluating the time complexity of the recursive algorithm for the tower of Hanoi puzzle*; u_n also represents the maximal length of a preorder traversal in a binary tree of depth $\leq n$ (see Exercise 3.23).

It is a recurrence of the form (7.14) with $l = 1$, $b_1 = 1$ and $P_1(n) = 1$. The characteristic equation is $(r-2)(r-1) = 0$ and the general solution of the recurrence is $u_n = \lambda 1^n + \mu 2^n$. We have $u_1 = \lambda + 2\mu = 1$; we need a second condition to find λ and μ; we will use the recurrence and deduce $u_2 = 3$; hence

$$\lambda + 2\mu = 1$$
$$\lambda + 4\mu = 3$$

and thus $\lambda = -1$, $\mu = 1$, i.e. $u_n = 2^n - 1$ for $n \geq 1$, which was also obtainable by a direct summation.

EXERCISE 7.19 Solve the recurrence $u_n = u_{n-1} + 2u_{n-2} + (-1)^n$, $n \geq 2$, with $u_0 = u_1 = 1$ (studied in Section 8.2.2, by the generating series method). ♢

EXERCISE 7.20 Let the recurrence $u_{n+2} = 3u_{n+1} - 2u_n + 4n$, for all $n \geq 0$ with the initial conditions $u_0 = u_1 = 0$. Compute the general term u_n. ♢

Method of undetermined coefficients

To solve the recurrence (7.14) a slightly different method can be used, consisting of

1. First finding a particular solution of the non-homogeneous recurrence, a particular solution that must be of the form $\sum_{i=1}^{l} b_i^n Q_i(n)$ with
   - $deg(Q_i) = deg(P_i)$ if b_i is not a root of the characteristic polynomial of the homogeneous recurrence associated with (7.14),
   - $deg(Q_i) = deg(P_i) + m_i$ if b_i is root of multiplicity m_i of the characteristic polynomial of the homogeneous recurrence associated with (7.14).

   The particular solution v_n is found by the method of *undetermined coefficients*, namely: in (7.14) substitute for v_n a term $\sum_{i=1}^{l} b_i^n Q_i(n)$, where the coefficients of the Q_is are unknown, and determine these coefficients by identifications.

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* Recall that the tower of Hanoi puzzle consists of transferring n disks initially stacked in decreasing order on a first peg to a second peg, initially empty, moving only one disk at a time, and never moving a larger disk on top of a smaller one: on any peg the disks will at any time be stacked in decreasing order throughout the whole transfer. To this end a third peg, initially empty, is available; the idea is to transfer the $n-1$ smaller disks from peg 1 to peg 3, in time u_{n-1}, then to transfer the biggest disk from peg 1 to peg 2, where it will be at the right place, and to finally transfer the $n-1$ smaller disks from peg 3 to peg 2, in time u_{n-1}. All the disks will then be transferred, in time $u_n = 2u_{n-1} + 1$ for $n > 1$. Clearly when there is a single disk, a single manipulation will do the job, and thus $u_1 = 1$.

2. Then finding the general solution u_n in the form $u_n = v_n + w_n$, where w_n is the general solution of the associated homogeneous recurrence, and the coefficients of w_n are determined by the initial conditions.

Example 7.21
1. Consider again the preceding example : $u_n = 2u_{n-1} + 1$, with $u_1 = 1$. First look for a particular solution $v_n = \lambda 1^n$, substituting the solution v_n in the recurrence gives $\lambda = 2\lambda + 1$ and $\lambda = -1$. Then look for the general solution in the form: $u_n = \mu 2^n - 1$, and for $n = 1$ that gives $2\mu - 1 = 1$, and thus $\mu = 1$.

2. Let the recurrence: $u_n = 2u_{n-1} + n + 2^n$, with the initial condition $u_1 = 0$. It is of the form (7.14) with $b_1 = 1$, $P_1(n) = n$, $b_2 = 2$, $P_2(n) = 1$; moreover, the characteristic polynomial of the associated homogeneous recurrence is $r = 2$. We must thus find a particular solution of the form $v_n = an + b + (cn + d)2^n$. Plugging in the recurrence relation we obtain

$$an + b + (cn + d)2^n = 2a(n - 1) + 2b + (c(n - 1) + d)2^n + n + 2^n,$$

i.e. simplifying

$$0 = (a + 1)n + b - 2a + 2^n(1 - c) \ .$$

This equality being true for all n we deduce $a = -1$, $b = -2$, $c = 1$ and we obtain a particular solution $v_n = -2 - n + n2^n$ for the equation $u_n = 2u_{n-1} + n + 2^n$. We then try to find the general solution in the form $u_n = v_n + \lambda 2^n$, since the general solution of the homogeneous recurrence is $w_n = \lambda 2^n$; taking into account the initial conditions $u_1 = 0$ we obtain $u_1 = -2 - 1 + 1 + 2\lambda = 0$, hence $\lambda = 1$. The solution of our recurrence is thus $u_n = -2 - n + n2^n + 2^n$.

We will note that the initial conditions are of no use to find the particular solution v_n.

7.2.3 Linear recurrences with parameters

There is no general method, except for some special cases; we will study one such case, the *linear recurrences of degree 1 with parameters.* Consider the recurrence relation

$$a(n)u_n = b(n)u_{n-1} + c(n).$$

We can, without loss of generality, assume that $a(n) = 1$ (divide by $a(n)$ which is always $\neq 0$, otherwise the recurrence relation would not define u_n), and we are thus back to

$$u_n = b(n)u_{n-1} + c(n), \quad n > 0.$$

Let $f(n) = \prod_{i=1}^{n} 1/b(i)$, and let the sequence $(v_n)_{n\geq 0}$ be defined by $v_n = f(n)u_n$, for $n > 0$, and $v_0 = u_0$. We verify that, $\forall n > 0$, v_n satisfies the recurrence relation $v_n = v_{n-1} + f(n)c(n)$, which can be solved by summation (the so called *summation factors* method); hence $v_n = v_0 + \sum_{k=1}^{n} f(k)c(k)$, and thus

$$u_n = (1/f(n))v_n = \left(\prod_{i=1}^{n} b(i)\right)\left(u_0 + \sum_{k=1}^{n} f(k)c(k)\right).$$

The summation factors method can also be applied with profit to some linear recurrences with constant coefficients.

EXAMPLE 7.22 Let $u_n = u_{n-1} + 2(n-1)$, for $n > 0$, with $u_0 = 2$. We obtain $u_n = 2 + 2(0 + 1 + 2 + \cdots + (n-2) + (n-1)) = 2 + n(n-1)$.

7.3 Other recurrence relations

Non-linear recurrences are less easy to solve. However, some techniques can be applied, e.g. substitutions or image transformations.

7.3.1 Partition recurrences and substitutions

Recall that a partition recurrence is often obtained in a 'divide and conquer'-type algorithm, where a problem of size n is split into b subproblems of size n/a; if u_n represents the cost of the solution of the size n problem, and $c(n)$ the cost of the creation and the utilization of the b subproblems of size a/n, we have the recurrence relation, for $n > 1$,

$$u_n = bu_{n/a} + c(n). \tag{7.18}$$

In order to solve this recurrence exactly, the domain is restricted to integers of the form a^k, and we apply the substitution $v_k = u_{a^k}$. We have $u_{a^k} = bu_{a^{k-1}} + c(a^k)$, thus $v_k = bv_{k-1} + c(a^k)$; hence a linear recurrence of degree 1, and we obtain (see Section 7.2.3)

$$\begin{aligned} v_k &= b^k \left(v_0 + \sum_{j=1}^{k} c(a^j)/b^j\right) \\ &= v_0 b^k + \sum_{j=0}^{k-1} b^{k-j-1} c(a^{j+1}) \\ &= v_0 b^k + \sum_{j'=1}^{k} b^{j'-1} c(a^{k-j'+1}) \,. \end{aligned}$$

Hence, as $n = a^k$, and thus $k = \log_a n$,

$$u_n = v_0 b^{\log_a n} + \sum_{j=1}^{\log_a n} b^{j-1} c(n/a^{j-1}) \ . \tag{7.19}$$

Equality (7.19) will also allow us to evaluate the order of magnitude of u_n (see Proposition 9.18).

EXERCISE 7.21 Solve the recurrence $u_n = 4u_{n/2} + n^2$, for $n = 2^k$. $\diamondsuit$

Note that a sequence u_n can be considered as a mapping $u: \mathbb{N} \longrightarrow \mathbb{C}$; the substitution technique that we have just seen consists of transforming the domain $\mathbb{N}$, namely, of composing u with a mapping $f: \mathbb{N} \longrightarrow \mathbb{N}$ and considering the recurrence $v = u \circ f$. In the preceding example we had $f(k) = 2^k$. It can also be fruitful to transform the image of u, namely, to compose u with $g: \mathbb{C} \longrightarrow \mathbb{C}$, and to consider the recurrence $w = g \circ u$. We will give two such examples in the next section.

7.3.2 Image transformations

EXAMPLE 7.23 Consider the recurrence $u_n = u_{n-1} - u_n u_{n-1}$, with $u_1 = 1$. Assume $u_n \neq 0$ $\forall n$, and divide by $u_n u_{n-1}$; We obtain $\dfrac{1}{u_{n-1}} = \dfrac{1}{u_n} - 1$ hence $\dfrac{1}{u_n} = \dfrac{1}{u_{n-1}} + 1$. Let $v_n = \dfrac{1}{u_n}$; this boils down to letting $v = g \circ u$ with $g(x) = \dfrac{1}{x}$. We then have $v_1 = 1$ and $v_n = v_{n-1} + 1$, hence $v_n = n$ and $u_n = 1/n$.

EXAMPLE 7.24 Let the recurrence $u_n = n(u_{n/2})^2$, with $u_1 = 6$. Assume that $n = 2^k$ and let $v_k = u_{2^k}$. We deduce $v_k = 2^k v_{k-1}^2$ and $v_0 = 6$. Now let $w_k = \log_2 v_k$; we have $w_k = k + 2w_{k-1}$ and $w_0 = \log_2 6$. The characteristic polynomial is $(r-2)(r-1)^2 = 0$, and the general solution is of the form $w_k = a2^k + b + ck$. We easily obtain, using the equality $w_k = k + 2w_{k-1}$, that $b = -2$ and $c = -1$; we obtain $a = 3 + \log_2 3$ by noting that $w_0 = 1 + \log_2 3 = a - 2$. Hence , finally, $w_k = (3 + \log_2 3)2^k - k - 2$. Then, taking into account that $v = 2^{w_k}$ and that $u_n = v_{\log_2 n}$, we obtain $u_n = 2^{3n-2} 3^n / n$.

EXERCISE 7.22 Solve the following recurrence relations:

1. $\forall n \geq 2, \quad u_n = \dfrac{2}{\dfrac{1}{u_{n-1}} + \dfrac{1}{u_{n-2}}}$, with $u_0 = a$ and $u_1 = b$.
2. $\forall n \geq 2, \quad u_n = \sqrt{u_{n-1} u_{n-2}}$, with $u_0 = 1$ and $u_1 = 2$.
3. $\forall n \geq 2, \quad u_n = \dfrac{a_n}{b_n} = \dfrac{a_{n-1} + a_{n-2}}{b_{n-1} + b_{n-2}}$, with $u_0 = \dfrac{a_0}{b_0}$ and $u_1 = \dfrac{a_1}{b_1}$. $\diamondsuit$

7.3.3 Complete recurrences

They can be solved either by using generating series tools (see Chapter 8 on generating series), or by forming linear combinations of suitably chosen instances of the recurrence relation (see Section 14.1.3 for an example).

7.4 Complements and examples

7.4.1 Operations on sequences

Z A sequence is a mapping $u: \mathbb{N} \longrightarrow \mathbb{C}$, conventionally extended into a mapping $u: \mathbb{Z} \longrightarrow \mathbb{C}$ by letting $u_n = 0$ if $n < 0$ (or $n < k_0 \in \mathbb{Z}$ if for technical reasons the sequences must start with $k_0 \neq 0$). This convention is fundamental in order for the results of the present section to hold. We can define various operations on the sequences allowing ease of manipulation (addition, difference, etc.). We give a short summary.

- E, the *predecessor* operation, is defined by

$$(Eu)_n = u_{n-1} \quad \text{and} \quad (Eu)_0 = 0\ .$$

- The *product* operation is defined by

$$(uv)_n = u_n v_n \quad \text{and} \quad (uv)_0 = u_0 v_0\ .$$

- Δ, the *difference* operation, is defined by:

$$(\Delta u)_n = u_n - u_{n-1} = \big((Id - E)u\big)_n \text{ and } (\Delta u)_0 = \big((Id - E)u\big)_0 = u_0\ ,$$

thus $\Delta = Id - E$. Δ is linear, namely, $\Delta(\lambda u + \mu v) = \lambda \Delta u + \mu \Delta v$, $\forall \lambda, \mu \in \mathbb{C}$. Moreover, $\Delta(u \star v) = (\Delta u) \star v + (Eu) \star (\Delta v)$.

- The $\star$ *product* operation, denoted by $\star$, different from the usual products and in general non-associative, is defined by

$$(u \star v)_n = u_n v_n - u_{n-1} v_{n-1} = \big(\Delta(uv)\big)_n \text{ and } (u \star v)_0 = u_0 v_0\ .$$

- Σ, the *summation* operation, is defined by

$$(\Sigma u)_n = \Sigma_{i=0}^{n} u_i\ .$$

We can verify the equalities

$$\Delta\Sigma = \Sigma\Delta = Id$$

$$\Sigma[(\Delta u) \star v] = u \star v - \Sigma[(Eu) \star (\Delta v)] \quad (\textit{summation by parts}).$$

EXAMPLE 7.25 Let k be fixed, and let $u_n = \binom{n}{k}$, then: $(\Delta u)_n = \binom{n-1}{k-1}$; similarly let $v_n = A_n^k$, i.e. $(v)_n = (A_n^k)_{n\in\mathbb{N}}$,

- for $k > 1$, we have $(\Delta v)_n = (\Delta A_n^k)_n = A_n^k - A_{n-1}^k = kA_{n-1}^{k-1}$;
- $\Delta A_n^1 = 1$, $\Delta A_n^0 = 0$;
- for $k < 0$, let $k = -l$ with $l > 0$, then letting

$$w_n = B_n^k = n(n+1)\cdots(n-k-1);$$

we have:

$$(\Delta w)_n = (\Delta B^k)_n = -kB_n^{k-1}.$$

EXERCISE 7.23 Let k be fixed, $k > 1$, compute Δu where

$$u_n = \frac{1}{n(n+1)\cdots(n+k-1)}\,.$$

$\diamondsuit$

EXAMPLE 7.26 Evaluation of the order of magnitude of the sum ΣH of the harmonic numbers H_n. Recall that

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad H_0 = 0.$$

We have $(\Delta H)_n = \dfrac{1}{n}$, for $n > 0$.

Noting that $(\Delta u)_n = 1$ for all n, a summation by parts with $u_n = n+1$ and $v_n = H_n$ gives

$$\Sigma H = \Sigma\big((\Delta u)\star H\big) = u\star H - \Sigma(Eu)\star\Delta(H)\,,$$

hence

$$(\Sigma H)_n = (n+1)H_n - (\Sigma\frac{n}{n})_n = (n+1)H_n - n\,.$$

But $H_n \sim \log n$. This comes from the following remark: the function $1/x$ being decreasing, we have the squeeze

$$\frac{1}{n} > \int_n^{n+1}\frac{dx}{x} > \frac{1}{n+1}\,, \qquad \forall n > 0;$$

thus considering the sequence of intervals $[1,2[$, $[2,3[$,..., $[n,n+1[$, we have

$$\int_1^{n+1}\frac{dx}{x} < H_n < \int_1^n\frac{dx}{x} + 1,$$

and thus

$$\log(n+1) < H_n < \log(n) + 1.$$

The situation is illustrated by Figure 7.1. Therefore $(\Sigma H)_n \sim n\log n$.

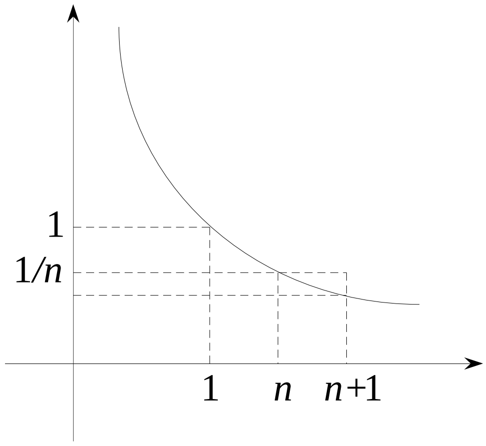

Figure 7.1

7.4.2 Applications: counting, Stirling numbers

EXAMPLE 7.27 (Number of surjections of A in B) Let A, B be such that $|A| = a$, $|B| = b$, $a \geq b$, and let S_a^b be the number of surjections $f: A \longrightarrow B$. In Proposition 6.14 we computed S_a^b by considering $E_i = \{f: A \longrightarrow B \ / \ i \notin f(A)\}$ and by noting that f is non-surjective if and only if $f \in \cup_{i \in B} E_i$, which gave

$$\begin{aligned} S_a^b &= b^a - \left(\sum_{p=1}^{b} (-1)^{p+1} \binom{b}{p} (b-p)^a \right) \\ &= b^a - \big(b(b-1)^a + \cdots + (-1)^{p+1} \binom{b}{p} (b-p)^a + \cdots + (-1)^b b \big) \\ &= \sum_{p=0}^{b} (-1)^p \binom{b}{p} (b-p)^a . \end{aligned}$$

We can also compute S_a^b by exhibiting a recurrence relation satisfied by S_a^b. Let $A = \{1, \ldots, a\}$ and $B = \{1, \ldots, b\}$. Let $f_{|A-\{a\}}$ be the restriction of f to $A - \{a\}$. If f is surjective, then one of the two following conditions is realized:

- Either $f' = f_{|A-\{a\}}$ is surjective, and then f' is a surjection from $\{1, \ldots, a-1\}$ onto B and there are b possibilities for choosing the image of a, thus there are bS_{a-1}^b possible choices for f in this case.
- Or $f' = f_{|A-\{a\}}$ is not surjective. Let $f(a) = j$; since f is surjective, $f'': (A - \{a\}) \longrightarrow (B - \{j\})$, also is surjective, and we thus have S_{a-1}^{b-1} possible choices for f''. Moreover there are b possible choices for the element j which we took out, hence bS_{a-1}^{b-1} possible choices for f in that case.

We finally obtain the recurrence relation

$$S_a^b = bS_{a-1}^b + bS_{a-1}^{b-1}, \quad 0 < b \leq a, \tag{7.20}$$

with the initial condition $S_1^1 = 1$.

EXAMPLE 7.28 (Number of partitions of A in b classes) Letting $|A| = a \geq b > 0$, we would like to determine the number P_a^b of partitions of A of the form $A = A_1 + \cdots + A_b$, namely, the A_is are b subsets of A, pairwise disjoint. To this end we will use S_a^b that we just computed.

Recall the following theorem first.

Theorem 7.29 *Let $f\colon A \longrightarrow B$ be a surjection, and let $\equiv_f$ be the equivalence relation associated with f which is defined by $x \equiv_f y$ if and only if $f(x) = f(y)$. f can be factored into $f = i \circ p$, where $p\colon A \longrightarrow A/\equiv_f$ is the canonical projection and $i\colon A/\equiv_f \longrightarrow B$ is an isomorphism.*

Corollary 7.30 $S_a^b = b!P_a^b$.

Proof. To each surjection $f\colon A \longrightarrow B$ corresponds a partition of A in b classes of equivalence according to the preceding theorem.

Conversely, if $A = A_1 + \cdots + A_b$ is a partition of A, $f(a) = i \iff a \in A_i$ defines a surjection $A \longrightarrow B = \{1, \ldots, b\}$. Let p be an arbitrary permutation of B, then $p \circ f\colon A \longrightarrow B$ is also a surjection corresponding to the same partition of A. Thus each partition determines $b!$ surjections. □

We can also also determine a recurrence relation satisfied by the P_a^bs. Let P be a partition of A in b classes, and let $\alpha \in A$.

- Either $\{\alpha\}$ (the singleton α) is a class of P on its own, and then the other classes of P partition $A - \{\alpha\}$ in $b-1$ classes; there are thus P_{a-1}^{b-1} possible choices for P in that case.
- Or $\{\alpha\}$ is not a class, but belongs to one of the b classes of P, and in that case P partitions $A - \{\alpha\}$ in b classes, and there are b possible choices for putting α in one of the classes, hence bP_{a-1}^b possible choices for P in that case. Finally, we obtain

$$P_a^b = bP_{a-1}^b + P_{a-1}^{b-1}, \qquad 0 < b \leq a\ . \tag{7.21}$$

Terminology: The P_a^b are called *Stirling numbers* of the second kind. Stirling numbers of the first kind satisfy the recurrence relation

$$K_a^b = (a-1)K_{a-1}^b + K_{a-1}^{b-1}, \qquad 0 < b \leq a\ . \tag{7.22}$$

The combinatorial interpretation of K_a^b is the following: K_a^b is the number of permutations of a letters containing b cycles. A *cycle* in a permutation p of a letters is a subset of $n < a$ letters $\{\alpha_1, \ldots, \alpha_n\}$, such that $p(\alpha_i) = \alpha_{i+1}$, for $i < n$ and $p(\alpha_n) = \alpha_1$. By a case analysis similar to the preceding ones we obtain: let $\alpha \in A$, and let p be a permutation with b cycles;

- either α is invariant by p, namely, $\{\alpha\}$ is a cycle, and there are K_{a-1}^{b-1} such possibilities
- or α is not invariant, there are then K_{a-1}^b possible choices of permutations of $a-1$ letters with b cycles, and $a-1$ possible choices for inserting α in one of the b cycles: if the b cycles have $a_1, a_2, \ldots, a_b$ letters, respectively, we have $a_1 + a_2 + \cdots + a_b = a - 1$ possibilities, hence $(a-1)K_{a-1}^b$ possibilities altogether.

Stirling numbers have numerous applications in combinatorics: two simple examples are the expansion of the binomial coefficients in powers and, conversely, the expression of powers in terms of the binomial coefficients; we have

$$x(x-1)\cdots(x-n+1) = \sum_{k=1}^{n} (-1)^{n-k} K_n^k x^k,$$

$$x^n = \sum_{k=1}^{n} P_n^k \binom{x}{k} k!\,.$$