CHAPTER 6

COMBINATORIAL ALGEBRA

In the present chapter we are interested in tools and techniques for counting finite sets and their subsets without enumerating all their elements. Discrete probabilities (see Chapter 12) are also rooted in the study of combinatorics. We start by recalling well-known results about permutations and combinations. We then study some counting techniques for finite sets which enable us to count the number of elements in a union, in a partition and in various combinations of finite sets.

We advise the following further reading:

Ronald Graham, Donald Knuth, Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, London (1989).

Donald Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, London (1973).

6.1 Basics

6.1.1 Generalities

Definition 6.1 *A permutation p of a finite set E is a bijection from E to E. The number of permutations of a set with n elements will be denoted by P_n .*

Identifying E with $\{1,\ldots,n\}$, a permutation p is characterized by a bijection $\{1,\ldots,n\} \longrightarrow \{1,\ldots,n\}$ determining a total ordering on the elements of E, given by the sequence $p(1), p(2), \ldots, p(n)$. Because there are n possible choices for $p(1)$, it follows that there are $(n-1)$ possible choices for $p(2)$, etc. (the word 'etc.' hides a proof by induction), therefore, $P_n = n!$.

Definition 6.2
1. *A k-permutation, $k \leq n$, of a finite set E with cardinality n is a totally ordered subset of E with k elements. A_n^k denotes the number of k-permutations of a set with n elements.*
2. *A k-combination, $k \leq n$, of a finite set E with cardinality n is a subset of E with k elements. $\binom{n}{k}$ denotes the number of k-combinations of a set with n elements.*

Exercise 6.1 Show that $A_n^k = \dfrac{n!}{(n-k)!}$. ♢

Combinations are unordered, whilst k-permutations are ordered; hence each k-combination yields $k!$ k-permutations, and thus $A_n^k = k!\binom{n}{k}$. We have $A_n^n = P_n = n!$, $\quad A_n^0 = 1$, $\quad A_n^1 = n$. We also have $A_n^k = \dfrac{n!}{(n-k)!}$ (see Exercise 6.1), and thus $\binom{n}{k} = \dfrac{n!}{(n-k)!k!} = \binom{n}{n-k}$.

Example 6.3 Let $E = \{a, b, c\}$. Then

- (a, b, c), (b, c, a), (c, a, b), (b, a, c), (c, b, a), (a, c, b) are the permutations of E,
- (a, b), (b, a), (a, c), (c, a), (b, c), (c, b) are the 2-permutations of E and
- $\{a, b\}$, $\{a, c\}$, $\{c, b\}$ are the 2-combinations of E.

Remark 6.4 A k-permutation is characterized by an *injection*

$$i: \{1, \ldots, k\} \longrightarrow E,$$

and a k-combination is characterized by *the image* of an injection $\{1, \ldots, k\} \longrightarrow E$. As $k!$ different injections have the same image, we return to the previously stated result: $A_n^k = k!\binom{n}{k}$.

Exercise 6.2 Show that $\displaystyle\binom{a+b}{p} = \sum_{k=0}^{\inf(p,a)} \binom{a}{k}\binom{b}{p-k}$, with $p \leq a + b$. ♢

Exercise 6.3
1. Show that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
2. Show that $\displaystyle\sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{n}{k}\binom{n}{i}\binom{n}{i+k} = \binom{3n}{n}$. ♢

The k-permutations (resp. the k-combinations) are also called k-permutations without repetition (resp. k-combinations without repetition). Lastly, terms of the form $\binom{n}{p}$ are also called binomial coefficients or coefficients of *Pascal's triangle*.

Proposition 6.5 (Recurrence relations on the $\binom{n}{k}$s) *The binomial coefficients verify the identities*

$$\text{(i)} \qquad \binom{n}{k} = \binom{n}{n-k} \quad \textit{for } 0 \leq k \leq n \; ,$$

$$\text{(ii)} \qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \textit{for } 2 \leq k \; , \; 1 \leq n \; . \tag{6.1}$$

Proof. We have already proved (i). To prove (ii), choose an element $e \in E$, where E has cardinality n, and divide the $\binom{n}{k}$ combinations in two disjoint sets:

- those combinations which do not contain e, of which there are $\binom{n-1}{k}$, and
- those combinations which contain e, of which there are $\binom{n-1}{k-1}$.

These two sets are disjoint, so $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. □

The binomial coefficients can be represented by Pascal's triangle, using the recurrence relation (6.1) for computing the successive $\binom{n}{k}$s.

n \ k	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1
5	. . . .				

Proposition 6.6 (Binomial theorem) *Let A be a ring, $a, b \in A$ such that $ab = ba$, and $n \in \mathbb{N}$. The following identity, called the binomial identity, holds:*

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{p}a^{n-p}b^p + \cdots + \binom{n}{n}b^n \\ &= \sum_{p=0}^{n} \binom{n}{p} a^{n-p}b^p \; .\end{aligned}$$

Proof. By induction on n.
(B) If $n = 0$, $(a+b)^0 = \binom{0}{0} = 1$. If $n = 1$, $a + b = \binom{1}{0}a + \binom{1}{1}b$.
(I) Assume $(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^{n-p}b^p$. Then, taking into account that $(a+b)^{n+1} = (a+b)^n(a+b)$, we have

$$(a+b)^{n+1} = \binom{n}{0}a^{n+1} + \sum_{p=1}^{n}\left(\binom{n}{p-1} + \binom{n}{p}\right)a^{n+1-p}b^p + \binom{n}{n}b^{n+1} \, .$$

Noting that $\binom{n+1}{0} = \binom{n}{0} = \binom{n+1}{n+1} = \binom{n}{n} = 1$, and that $\binom{n}{p-1} + \binom{n}{p} = \binom{n+1}{p}$, we indeed have $(a+b)^{n+1} = \sum_{p=0}^{n+1} \binom{n+1}{p} a^{n+1-p} b^p$. □

Exercise 6.4 Compute the number N of partitions of a set with np elements in n subsets with p elements. ♢

Exercise 6.5 Compute $S = \displaystyle\sum_{q=0}^{p} (-1)^q \binom{n}{q} \binom{n-q}{p-q}$, for $p \leq n$. ♢

Exercise 6.6

1. Show that $\displaystyle\sum_{p=0}^{k} \binom{n+p}{p} = \binom{n+k+1}{k}$, for $k \geq 0$.
2. Show that $\displaystyle\sum_{k=p}^{n} \binom{k}{p} = \binom{p}{p} + \binom{p+1}{p} + \cdots + \binom{n}{p} = \binom{n+1}{p+1}$.

Deduce the value of $\sum_{k=1}^{n} k^p$, for $p = 1, 2, 3$. ♢

Exercise 6.7 Let $P(x)$ be a polynomial of degree less than or equal to n. Show that $\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(x+i) = 0$. ♢

6.1.2 Applications

The notions and results of the preceding section are very basic, but they can nevertheless be applied to counting finite sets, evaluating discrete probabilities or determining complexity and feasibility of algorithms. We illustrate such applications by examples and exercises.

Remark 6.7 Recall (see Example 1.5) that the characteristic function of a subset A of a set E is the function

$$\chi_A : E \longrightarrow \{0, 1\} ,$$

defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, any function $\chi: E \longrightarrow \{0, 1\}$ defines the subset $A = \chi^{-1}(\{1\})$ of E.

We will consider that $\{0, 1\} \subseteq \mathbb{B}$, or $\{0, 1\} \subseteq \mathbb{N}$; the choice will be clear by the context.

EXAMPLE 6.8 We have various methods for computing

$$S_n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

1. By Remark 6.7, there is a one-to-one correspondence between the set of subsets of E and the set of functions $E \longrightarrow \mathbb{B}$. Thus

$$S_n = |\mathcal{P}(\{1, \ldots, n\})| = 2^n,$$

since there are 2^n functions $\{1, \ldots, n\} \longrightarrow \mathbb{B}$.

2. Note that $\binom{n}{k}$ represents the number of subsets with k elements of $\{1, \ldots, n\}$. We introduce the notation $+$ for the disjoint union: if A and B are disjoint, i.e. if $A \cap B = \emptyset$, then $A \cup B$ is denoted by $A + B$, and this notation is justified by the fact that $|A + B| = |A| + |B|$. Note, finally, that if P_k denotes the set of k-element subsets of $\{1, \ldots, n\}$, $P_0, \ldots, P_n$ form a partition of the set $\mathcal{P}(\{1, \ldots, n\})$ of subsets of $\{1, \ldots, n\}$. Since $\binom{n}{k} = |P_k|$ and

$$2^n = |\mathcal{P}(\{1, \ldots, n\})| = |P_0 + P_1 + \cdots + P_n| = |P_0| + |P_1| + \cdots + |P_n|,$$

we deduce $2^n = \binom{n}{0} + \cdots + \binom{n}{n}$.

3. Check that $S_n = 2S_{n-1}$ for $n \geq 1$. We apply the recurrence relation (6.1) on the $\binom{n}{k}$s: then

$$\begin{aligned}
S_n &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{k} + \cdots + \binom{n}{n} \\
&= \binom{n}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{k} + \cdots + \binom{n-1}{n-1} \\
&\quad + \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{k-1} + \cdots + \binom{n-1}{n-2} + \binom{n}{n} \\
&= 2S_{n-1} \qquad (\text{since } \binom{n}{0} = \binom{n-1}{0} = 1 = \binom{n-1}{n-1} = \binom{n}{n}).
\end{aligned}$$

Hence, noting that $S_0 = 1$ and multiplying the equalities $S_n = 2S_{n-1}$ yields $S_n = 2^n$.

4. Finally, we can apply the binomial identity, Proposition 6.6, with $a = b = 1$, and deduce $S_n = (1 + 1)^n$.

EXERCISE 6.8 Given twenty-seven white cubes, we stack them to build a cube three times larger. The outside of the big cube is painted in red, then the big cube is pulled down and the pieces are given to a blind person who is asked to rebuild it. Compute $p = n_f/n$, where n_f is the number of ways to rebuild a red cube (number of favourable cases) and n is the total number of ways to rebuild a cube (number of possible cases)? (p is the probability that the rebuilt cube is red.) $\diamondsuit$

Exercise 6.9 How many five-card hands, chosen from a deck of thirty-two cards (four suits), are there:

1. containing a four-of-a-kind (four cards of equal face values)?
2. containing a three-of-a-kind (three cards of equal face values) and nothing else?
3. containing a pair (two cards of equal face values) and nothing else? $\diamondsuit$

Exercise 6.10 Compute the number of strings of sixteen bits containing eight bits equal to 1. $\diamondsuit$

Example 6.9 This example shows how to use combinatorial algebra to prove the (in)tractability of some algorithms by evaluating their complexity *a priori*. We consider the problem of a travelling salesman who wishes to visit n pairwise connected cities $\{1, \ldots, n\}$ (i.e. forming a complete graph with n nodes). The distance between cities i and j is denoted by c_{ij}. He starts and ends his tour in city 1, visits each city exactly once, and wants to drive as few miles as possible. See also Chapter 10.

The simplest algorithm is to enumerate all the cycles starting at node 1 and to compute the length of each cycle; then choosing the shortest possible cycle will do the job. For each cycle consisting of n cities, the computation of its length needs $n-1$ additions, and since there are $(n-1)!$ cycles starting at node 1, the total cost of such an algorithm is $(n-1) \times (n-1)!$ additions. For a tour of fifty cities, we have $49 \times 49!$ (circa $3 \cdot 10^{64}$) additions (see Chapter 9 on asymptotic behaviours for the order of magnitude of $n!$). A computer performing 10^9 additions per second will need 10^{47} years to complete the computation of the optimal path. This cost is prohibitive for the sales of the travelling salesman. Practically, this algorithm will thus be excluded, and we must consider heuristic methods which will involve some cycles only. We will no longer find the shortest path but only the shortest path among the class of considered paths, the asset being that this path will be obtained after a reasonable amount of time.

6.2 Applications: counting techniques for finite sets

6.2.1 Fundamentals

Here we recall results which can be found in a slightly different form in Chapter 4. We generalize the notion of characteristic function as follows. Let $f\colon E \to \{0, 1, 2\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in A_1, \\ 2 & \text{if } x \in A_2 \text{ and } x \notin A_1, \\ 0 & \text{if } x \notin A_2. \end{cases}$$

Exercise 6.11 E is a finite set with n elements.

1. What is the number N_1 of pairs (A_1, A_2) such that

$$A_1 \subseteq E \ , \ A_2 \subseteq E \ , \quad \text{and } A_1 \subseteq A_2 \ ? \tag{6.2}$$

2. Let N_2 be the number of triples (A_1, A_2, A_3) verifying

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq E. \tag{6.3}$$

Compute N_2. $\diamondsuit$

The operations on the subsets A of E will then correspond to operations on the corresponding characteristic functions, and to the operations on $\mathbb{B}$ through which the operations on the characteristic functions are defined (see Section 4.1.3). An operation on $\mathbb{B}$ is described by its truth table. For example, the unary operation corresponding to negation (or complement) is $\neg x = 1 - x$ with the truth table:

x	0	1
$\neg x$	1	0

The binary operations corresponding to the disjunction $x \vee y$ and the conjunction $x \wedge y$ are described on $\mathbb{B}$ by the tables:

$\wedge$	x	0	1
	y		
	0	0	0
	1	0	1

$\vee$	x	0	1
	y		
	0	0	1
	1	1	1

and they can be characterized on $\mathbb{N}$ by (see Example 4.6)

$$x \wedge y = xy \quad \text{and} \quad x \vee y = x + y - xy \ (= x + y \bmod 2).$$

Lemma 6.10 *Let A and B be two subsets of E, and let $\alpha = \chi_A$ (resp. $\beta = \chi_B$) be the characteristic function of A (resp. B). Then $\alpha \wedge \beta = \alpha\beta$ (resp. $\alpha \vee \beta = \alpha + \beta - \alpha\beta$) is the characteristic function of $A \cap B$ (resp. $A \cup B$). $\neg\alpha = 1 - \alpha$ is the characteristic function of $\bar{A}$.*

Lemma 6.11 $\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$. *That is, the complement of a union is the intersection of the complements.*

Proof. See De Morgan's laws in Chapter 1 or Proposition 4.3 in Chapter 4. $\square$

Lemma 6.12 *$|A| = \sum_{e\in E} \chi_A(e)$ for any subset A of E.*

Proof. Here, χ_A is considered to be a function with values in $\{0,1\} \subseteq \mathbb{N}$. Since $e \in A \Longleftrightarrow \chi_A(e) = 1$,

$$\sum_{e\in E} \chi_A(e) = \sum_{e\in A} 1 = |A|. \qquad \square$$

6.2.2 Inclusion–exclusion principle and applications

We will apply the preceding techniques in order to compute the cardinality of sets (unions of subsets of E, number of surjections, injections, etc.).

Proposition 6.13 *Let $A_i \subseteq E$ be subsets of E, for $i = \{1, \ldots, m\}$. Then, we have Sylvester's identity*

$$\begin{aligned}
|A_1 \cup \cdots \cup A_m| &= |A_1| + \cdots + |A_m| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\
&\quad + \cdots + (-1)^{p-1} \sum_{i_1<\cdots<i_p} |A_{i_1} \cap \cdots \cap A_{i_p}| + \cdots \\
&\quad + (-1)^{m-1} |A_1 \cap \cdots \cap A_m| \\
&= \sum_{p=1}^{m} (-1)^{p-1} \sum_{i_1<\cdots<i_p} |A_{i_1} \cap \cdots \cap A_{i_p}|.
\end{aligned}$$

Proof. First method: We will apply the three lemmas stated at the end of the preceding section. Let $A = A_1 \cup \cdots \cup A_m$, $\chi_A = 1 - \chi_{\overline{A}}$; also let $\chi_{A_i} = \alpha_i$. Then $\chi_{\overline{A_i}} = 1 - \alpha_i$ and, since $\overline{A} = \overline{A_1} \cap \cdots \cap \overline{A_m}$, we have

$$\begin{aligned}
\chi_{\overline{A}} &= \prod_{i=1}^{m} \chi_{\overline{A_i}} = \prod_{i=1}^{m} (1 - \alpha_i) \\
&= 1 - (\alpha_1 + \cdots + \alpha_m) + \sum_{i<j} \alpha_i \alpha_j - \sum_{i<j<k} \alpha_i \alpha_j \alpha_k + \cdots .
\end{aligned}$$

We will then use

$$\begin{aligned}
\chi_A &= 1 - \chi_{\overline{A}} = \alpha_1 + \cdots + \alpha_m - \sum_{i<j} \alpha_i \alpha_j + \sum_{i<j<k} \alpha_i \alpha_j \alpha_k - \cdots \\
&= \sum_{p=1}^{n} (-1)^{p-1} \sum_{i_1<i_2<\cdots<i_p} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_p} ,
\end{aligned}$$

and we will apply Lemma 6.12, which tells us that $|A| = \sum_{e\in E} \chi_A(e)$. Thus

$$|A| = \sum_{e\in E} \chi_A(e) = \sum_{e\in E}\sum_{p=1}^{m}(-1)^{p-1} \sum_{i_1<\cdots<i_p} \alpha_{i_1}(e)\cdots\alpha_{i_p}(e)$$
$$= \sum_{p=1}^{m}(-1)^{p-1} \sum_{i_1<\cdots<i_p}\sum_{e\in E} \alpha_{i_1}(e)\cdots\alpha_{i_p}(e)$$
$$= \sum_{p=1}^{m}(-1)^{p-1} \sum_{i_1<\cdots<i_p} |A_{i_1}\cap\cdots\cap A_{i_p}|$$

(by noting that $\alpha_{i_1}\cdots\alpha_{i_p} = \chi_{A_{i_1}\cap\cdots\cap A_{i_p}}$).

Second method: By induction on m.

(B) Straightforward for $m = 1$ and $m = 2$.

(I) Assume

$$|A_1\cup\cdots\cup A_m| = \sum_{p=1}^{m}(-1)^{p-1} \sum_{1\le i_1<\cdots<i_p\le m} |A_{i_1}\cap\cdots\cap A_{i_p}|$$
$$= |A_1| + \cdots + |A_m| + \sum_{p=2}^{m}(-1)^{p-1} \sum_{1\le i_1<\cdots<i_p\le m} |A_{i_1}\cap\cdots\cap A_{i_p}| \,.$$

and compute, for $m \ge 2$, $|A_1\cup\cdots\cup A_m\cup A_{m+1}|$. It follows that

$$|A_1\cup\cdots\cup A_m\cup A_{m+1}| = |A_1\cup\cdots\cup A_m| + |A_{m+1}| - |(A_1\cup\cdots\cup A_m)\cap A_{m+1}| \,.$$

Moreover, letting $A'_i = A_i \cap A_{m+1}$, we have

$$(A_{i_1}\cup\cdots\cup A_{i_p})\cap A_{m+1} = A'_{i_1}\cup\cdots\cup A'_{i_p}$$

and

$$(A_{i_1}\cap\cdots\cap A_{i_p})\cap A_{m+1} = A'_{i_1}\cap\cdots\cap A'_{i_p} \;;$$

hence, by the induction hypothesis

$$|(A_1\cup\cdots\cup A_m)\cap A_{m+1}| = |A'_1\cup\cdots\cup A'_m|$$
$$= |A'_1| + \cdots + |A'_m| + \sum_{p=2}^{m}(-1)^{p-1} \sum_{1\le i_1<\cdots<i_p\le m} |A'_{i_1}\cap\cdots\cap A'_{i_p}|$$
$$= |A'_1| + \cdots + |A'_m| + \sum_{p=2}^{m}(-1)^{p-1} \sum_{1\le i_1<\cdots<i_p\le m} |A_{i_1}\cap\cdots\cap A_{i_p}\cap A_{m+1}|$$

and

$$\begin{aligned}|A_1 \cup \cdots \cup A_m \cup A_{m+1}| = {} & |A_1| + \cdots + |A_m| + |A_{m+1}| - |A'_1| - \cdots - |A'_m| \\ & + \sum_{p=2}^{m} (-1)^{p-1} \sum_{1 \le i_1 < \cdots < i_p \le m} |A_{i_1} \cap \cdots \cap A_{i_p}| \\ & - \sum_{p=2}^{m} (-1)^{p-1} \sum_{1 \le i_1 < \cdots < i_p \le m} |A_{i_1} \cap \cdots \cap A_{i_p} \cap A_{m+1}| \\ = {} & \sum_{p=1}^{m+1} (-1)^{p-1} \sum_{1 \le i_1 < \cdots < i_p \le m+1} |A_{i_1} \cap \cdots \cap A_{i_p}| \,. \qquad \square\end{aligned}$$

Exercise 6.12 $|E| = n$, $A \cap B = \emptyset$, $|A| = n_1$, $|B| = n_2$. Compute the number N of subsets with p elements, with $p \geq 2$, and with

1. exactly one element from A and one element from B,
2. at least one element from A and one element from B. ♢

Exercise 6.13 Let $\{a, b, c, d\}$ be a four-letter alphabet. What are:

1. the number of strings of length n over this alphabet?
2. the number of strings of length n in which each of the letters a, b, c, d occurs at least once? ♢

Exercise 6.14 Among the permutations of $\{a, b, c, d, e, f\}$, how many contain neither '*ac*' nor '*bde*' ? ♢

Exercise 6.15 What is the number u_n of binary strings with n bits containing neither 010 nor 11. ♢

Proposition 6.14 *Let A and B be two sets with cardinality $|A| = m$ and $|B| = n$.*

1. *The number of mappings from A to B is n^m.*
2. *The number of injections (or one-to-one mappings) from A to B is A_n^m, if $m \leq n$.*
3. *The number of surjections (or onto mappings) from A to B is*

$$S_n^m = \begin{cases} 0 & \text{if } m < n, \\ n! & \text{if } m = n, \\ \sum_{p=0}^{n} (-1)^p \binom{n}{p} (n-p)^m & \text{if } m > n. \end{cases}$$

Proof.

1. Indeed, there are n possible choices for the image of each element $a_1, \ldots, a_m$ in A, thus n^m choices altogether (see Proposition 1.9 (iv)). As an exercise, the reader is invited to give a formal proof by induction on m.

2. See Remark 6.4.
3. It is an application of the preceding proposition. The first two cases are straightforward, and so the only case requiring a proof is the case when $m > n$. We first compute the number of mappings from A to B. We then determine, using the preceding proposition, the number of non-surjective mappings from A to B. We finally deduce by difference the number of surjections from A to B, since, clearly, the set of mappings from A to B is the disjoint union of surjections on the one hand, and of mappings that are not surjections on the other hand.

Let $N = \{f\colon A \longrightarrow B \;/\; f \text{ non-surjective}\}$; f is non-surjective if and only if $\exists b_i \in B$ such that $b_i \notin f(A)$. Thus let,

$$A_i = \{f\colon A \longrightarrow B \;/\; b_i \notin f(A)\}, \quad i = 1, \ldots, n\,.$$

We will have

$$\begin{aligned} N &= \{f\colon A \longrightarrow B \;/\; f \text{ non-surjective}\} \\ &= \{f\colon A \longrightarrow B \;/\; f(A) \neq B\} = A_1 \cup \cdots \cup A_n\,. \end{aligned}$$

By the preceding proposition, we thus have

$$|N| = \sum_{p=1}^{n} (-1)^{p-1} \Bigg(\sum_{i_1 < \cdots < i_p} |A_{i_1} \cap \cdots \cap A_{i_p}| \Bigg)$$

and it suffices to compute $|A_{i_1} \cap \cdots \cap A_{i_p}|$. Note now that

$$\begin{aligned} A_{i_1} \cap \cdots \cap A_{i_p} &= \{f\colon A \longrightarrow B \;/\; b_{i_1} \notin f(A), \ldots, b_{i_p} \notin f(A)\} \\ &= \{f\colon A \longrightarrow B - \{b_{i_1}, \ldots, b_{i_p}\}\}\ , \end{aligned}$$

and thus $A_{i_1} \cap \cdots \cap A_{i_p}$ is the set of mappings from A, a set with m elements, to $B - \{b_{i_1}, \ldots, b_{i_p}\}$, a set with $n-p$ elements. There are $(n-p)^m$ such mappings by Proposition 6.14, 1. Since, moreover, there are $\binom{n}{p}$ possible choices of $b_{i_1}, \ldots, b_{i_p}$ in $\{b_1, \ldots, b_n\}$, we deduce $|N| = \sum_{p-1}^{n} (-1)^{p-1} \binom{n}{p} (n-p)^m$. Finally, noting that $n^m = \binom{n}{0}(n-0)^m$, we have

$$S_n^m = n^m - |N| = \sum_{p=0}^{n} (-1)^p \binom{n}{p} (n-p)^m.$$

Another way of computing S_n^m is given in Example 7.27. □

EXERCISE 6.16 A function f from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$ is said to be increasing if $x < y$ implies $f(x) < f(y)$.

1. What is the number of increasing functions (in terms of n and m)?
2. What is the number of increasing functions such that

$$\exists x\colon f(x) = k + 1 \quad \text{for } m = 2k + 1 \text{ and } k > 1\,?$$

3. What is the number of increasing functions such that, for a fixed k,

$$|\{a \,/\, f(a) < k\}| = |\{a \,/\, f(a) > k\}|\,?$$

4. What is the number of injective functions such that

$$|\{a \,/\, f(a) < k\}| = |\{a \,/\, f(a) > k\}|\,? \qquad \diamondsuit$$

6.3 Counting sequences and partitions

We now give some other counting formulas that will be of use in probability theory.

Definition 6.15 *A k-permutation with repetition allowed of a set E with n elements is an ordered sequence with k elements from E in which each element may occur arbitrarily often.*

Example 6.16 Let $E = \{a, b, c\}$. Then aa, ab, ba, bb are 2-permutations with repetition of E. Two k-permutations can differ by the ordering of their elements, by their elements or by both.

Proposition 6.17 *Let E be a set with cardinality n and $k \in \mathbb{N}$. (It is not assumed that $k \leq n$.) A k-permutation with repetition of E is defined by a mapping from $\{1, \ldots, k\}$ to E. There are thus n^k such k-permutations.*

Definition 6.18 *A k-combination with repetition of a set E with n elements is an unordered set with k elements of E in which each element can occur arbitrarily often.*

A set whose elements can occur arbitrarily often is called a *multiset*. The difference between a k-combination with repetition and a k-permutation with repetition is the following: a k-combination with repetition is an *unordered* multiset of elements, possibly repeated, whilst a k-permutation is an *ordered* sequence. For instance the 3-permutations aba and baa correspond to the same 3-combination: $\{a, a, b\}$. Two k-combinations with repetition can differ by their elements, by the number of repetitions or by both. A k-combination with repetition of elements of $E = \{1, \ldots, n\}$ will contain n_1 occurrences of i_1, …, n_p occurrences of i_p, with

- $\forall j, \quad 1 \leq i_j \leq n$,
- $n_1 + \cdots + n_p = k$.

Proposition 6.19 *Let E be a set of cardinality n, and let $k \in \mathbb{N}$. A k-combination with repetition of elements of E is defined by a mapping f from E to $\{0, 1, \dots, k\}$ such that $\sum_{i=1}^{n} f(e_i) = k$. An element e_i of E occurs in the k-combination j_i times if and only if $f(e_i) = j_i$.*

Equivalently, a k-combination with repetition is defined by a solution of the equation $j_1 + \cdots + j_n = k$, with $j_i \in \mathbb{N}, \quad \forall i \in \{1, \dots, n\}$.

Proposition 6.20 *The number of k-combinations with repetition of a set E with n elements is $\binom{n+k-1}{n-1}$.*

Proof. There is a one-to-one correspondence between k-combinations with repetition and the sequences $j_1, \dots, j_n$ such that $j_1 + \cdots + j_n = k$. We can represent such a sequence by the string of length $n + k - 1$ over the alphabet $\{0, 1\}$ given by $0^{j_1} 1 0^{j_2} 1 \dots 0^{j_{n-1}} 1 0^{j_n}$. (The rth sequence of 0s represents the number of repetitions of e_r, and the 1s act as separators.)

Such a k-combination is thus determined by a string of length $k + n - 1$ over the alphabet $\{0, 1\}$ consisting of exactly $n - 1$ occurrences of 1. It thus suffices to determine the number of such strings by characterizing them by the positions where the 1s occur. There are $\binom{n+k-1}{n-1}$ possible choices for fitting $n - 1$ 1s in a string of length $n + k - 1$. □

Remark 6.21
1. $\binom{n+k-1}{n-1}$ is also the number of monomials of degree k on n variables.
2. $\binom{n+k-1}{n-1}$ is also the number of monotone mappings from $\{1, 2, \dots, k\}$ to $\{1, 2, \dots, n\}$.

Finally, we will study the partitions of a set with n elements in k disjoint sets $A_1, \dots, A_k$ such that $|A_i| = n_i$ and $\sum_{i=1}^{k} n_i = n$, and this will lead us to the definition of multinomial coefficients.

Theorem 6.22 *The number of partitions $\binom{n}{n_1, \dots, n_k}$ of a set E with n elements in k classes $A_1, \dots, A_k$ each having n_i elements, with $\sum_{i=1}^{k} n_i = n$, is $\dfrac{n!}{n_1! \cdots n_k!}$.*

Proof. By induction on k.
(B) If $k = 2$, then choosing a n_1-element set A_1 defines a partition of E in two disjoint sets A_1, A_2 with $|A_2| = n_2 = n - n_1$, hence

$$\binom{n}{n_1, n_2} = \binom{n}{n_1} = \frac{n!}{n_1! n_2!}.$$

(I) Assume $\binom{n}{n_1, \dots, n_k} = \dfrac{n!}{n_1! \cdots n_k!}$, and let $A_1, \dots, A_{k+1}$ be a partition of E in $k + 1$ subsets. Let $B_k = A_k \cup A_{k+1}$. The number of partitions $A_1, \dots, A_{k-1}, B_k$

of E is $\binom{n}{n_1,\ldots,m_k}$, where $m_k = n_k + n_{k+1}$, i.e. $\dfrac{n!}{n_1!\cdots n_{k-1}!(n_k+n_{k+1})!}$. Moreover, the number of partitions of B_k in $A_k \cup A_{k+1}$ is $\dfrac{(n_k+n_{k+1})!}{n_k!n_{k+1}!}$. Hence, by multiplication,

$$\binom{n}{n_1,\ldots,n_k,n_{k+1}} = \frac{n!}{n_1!\cdots n_k!n_{k+1}!}. \qquad \square$$

The $\binom{n}{n_1,\ldots,n_k}$s are also called the *multinomial coefficients*. We have the following multinomial identity.

Proposition 6.23

$$(X_1+\cdots+X_k)^n = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1,\ldots,n_k} X_1^{n_1}\cdots X_k^{n_k}.$$

Proof. See Exercise 6.17. $\square$

Exercise 6.17 We are given n letters not assumed to be pairwise distinct: q_1 letters a_1, ..., and q_p letters a_p, with $q_1+q_2+\cdots+q_p = n$.

1. How many different strings of length n can be written using those n letters?
   (a) Deduce a representation of the formal polynomial $(X_1+X_2+\cdots+X_p)^n$.
   (b) Deduce an expression of the multinomial coefficients in terms of the binomial coefficients.
2. Deduce that $(k!)!$ is divisible by $k!^{(k-1)!}$.
3. Compute the number of strings of length 13 that can be written with $q_1 = 5$ letters a_1, and $q_2 = 8$ letters a_2. $\diamondsuit$

Exercise 6.18 Compute $\sum_{p=0}^{n} p^2\binom{2n}{2p}$, for $n \geq 2$.
Hint: Let $g(x) = \sum_{p=1}^{n} p^2\binom{2n}{2p}x^{2p-2}$ and $f(x) = (1+x)^{2n}+(1-x)^{2n}$, and find a relation among $g(x)$, $f'(x)$, and $f''(x)$. $\diamondsuit$

Exercise 6.19 For $n \in \mathbb{N}$ and $p \in \mathbb{N}-\{0\}$, denote by $F(n,p)$ the number of p-tuples $(x_1,\ldots,x_p) \in \mathbb{N}^p$ such that $x_1+\cdots+x_p = n$. Compute $F(n,p)$.

Method 1
   (a) Show that $F(n,p+1) = \sum_{k=0}^{n} F(k,p)$.
   (b) Show that $\binom{n+p}{p} = \sum_{k=0}^{n}\binom{k+p-1}{p-1}$, for $n \geq 0$, and $p \geq 1$.
   (c) Compute $F(n,p)$.

Method 2
   (a) Show that $F(n,p+1) = F(n,p) + F(n-1,p+1)$.
   (b) Show that $F(n,p) = \binom{n+p-1}{n}$. $\diamondsuit$