Congruence Preservation

Irène Guessarian
IRIF

Joint work with
André Arnold & Patrick Cégielski & Serge Grigorieff
LABRI & LACL & IRIF

November 9, 2020

Outline

Outline

Congruences

A **CONGRUENCE** on algebra $\mathcal{A} = \langle A, \star \rangle$ algebra with operations $\star$. is an equivalence relation $\sim$ on A which is compatible with the operations, i.e.,

$$a \sim a' \text{ and } b \sim b' \quad \Longrightarrow \quad a \star b \sim a' \star b'$$

$f : A \to A$ is **Congruence Preserving (CP)** iff for every congruence $\sim$,

$$a \sim b \Longrightarrow f(a) \sim f(b)$$

Congruence preserving functions

More generally, $\mathcal{A} = \langle A, \star \rangle$ algebra

Definition

$f : A^n \to A$ is congruence preserving iff, for any congruence $\sim$ on A:

$\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$\bigwedge_{i=1}^{i=n} x_i \sim y_i \quad \Longrightarrow \quad f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n)$

Example: "Polynomial functions" = functions expressed by terms with variables $x_1, \ldots, x_n$ and with constants in A.

Anything else ?

Outline

Congruence preservation on $\langle \mathbb{N}, + \rangle$

Non polynomial congruence preserving functions $\mathbb{N} \to \mathbb{N}$

$$f(x) = \lfloor e^{1/a} a^x x! \rfloor \quad \text{for } a \in \mathbb{N} \setminus \{0, 1\}$$

third kind Bessel function g

$$g(x) = \frac{\Gamma(1/2)}{2 \times 4^x \times x!} \int_1^\infty e^{-t/2} (t^2 - 1)^x dt$$

Idem for $f(x) = \lceil e^{1/a} a^x x! \rceil$

What about other algebras ??

Outline

On other algebras

Abbrev: CP= congruence preserving

On $\langle \mathbb{N}, \times \rangle$: much simpler characterization

Theorem

$f : \mathbb{N} \longrightarrow \mathbb{N}$;

$$f \ \ CP \ on \ \langle \mathbb{N}, \times \rangle$$
$$\Updownarrow$$
$$f(x) = f(1) \times x^k, \ with \ k \in \mathbb{N}$$

Congruence preservation on a non commutative algebras

Theorem

On the algebra of words with concatenation, $\mathcal{S} = \langle \Sigma^*, \cdot \rangle$,
$| \Sigma | \geq 3$,
f *CP* $\iff$ $f : x \mapsto w_0 x w_1 x w_2 \cdots x w_k$,
$k \in \mathbb{N}$, $w_0, w_1, \ldots, w_k \in \Sigma^*$.

An algebra is affine complete iff:

for all f**:** f **CP** $\iff$ f **polynomial.**

$\mathcal{S}$, $\langle \mathbb{N}, \times \rangle$ are affine complete.
$\langle \mathbb{N}, + \rangle$ is not affine complete

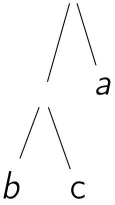
Outline

Algebras of binary trees with labelled leaves

Σ alphabet having at least 3 letters.

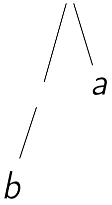
$\mathcal{T}_{\mathcal{C}}$: complete trees generated by Σ, $\star$

example



$\mathcal{T}$: non complete trees generated by $\emptyset$, Σ, $\star$

example



Tree algebras are affine complete

Theorem

If $f : \mathcal{T}^n \to \mathcal{T}$ is CP then there exists a polynomial P such that $f = P$.

Proof uses only 2 basic types of congruences (to have same skeleton, graftings)

What if Σ has less that 3 letters ???