Both ways rational functions

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Transductions

input and output alphabets

Definition

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input: c a n a d a

output: c c a n n a d d a

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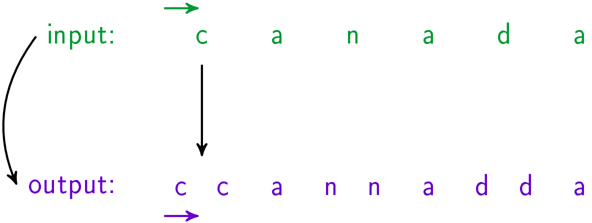

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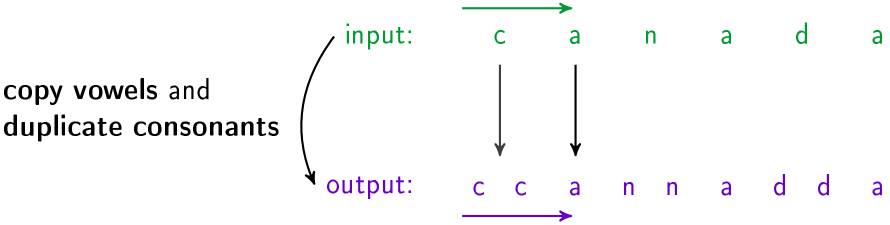

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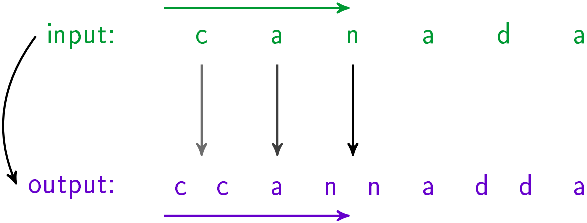

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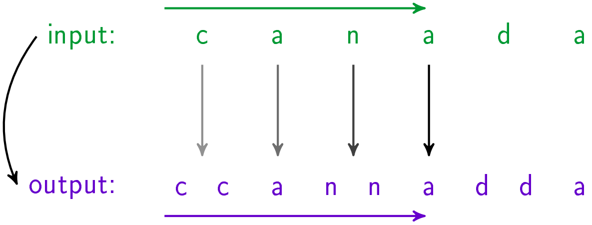

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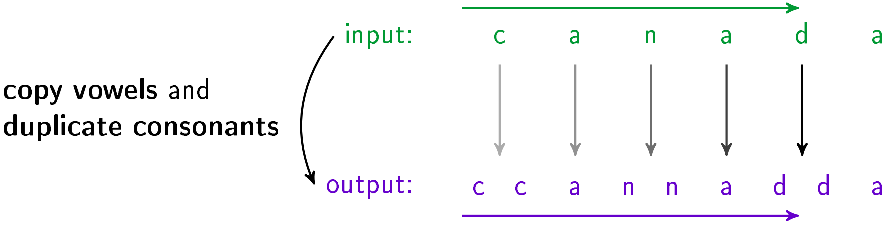

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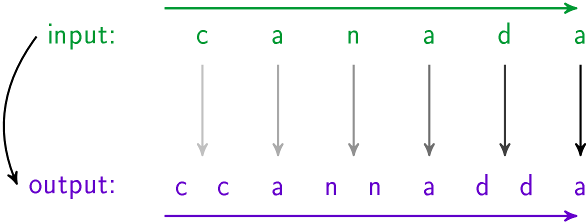

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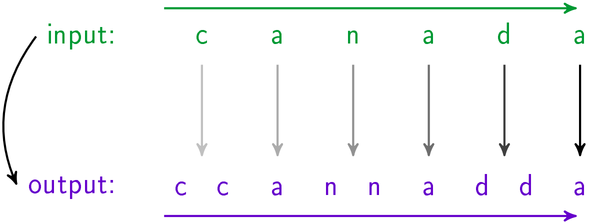

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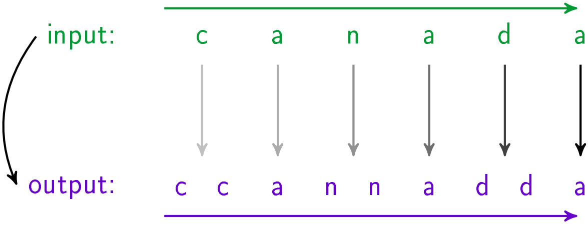

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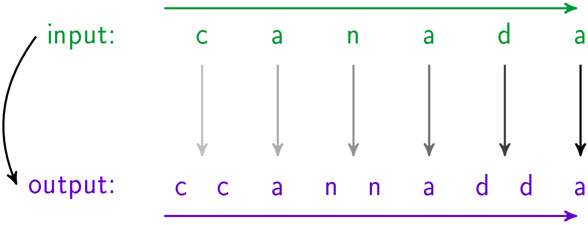

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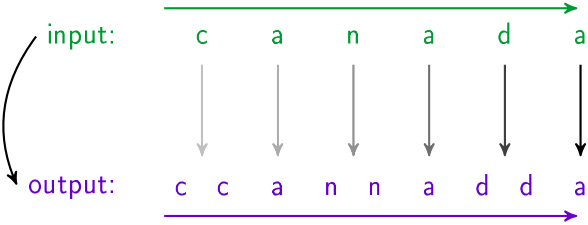

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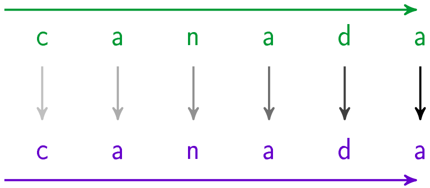

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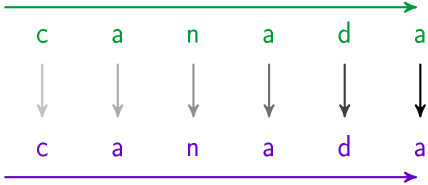

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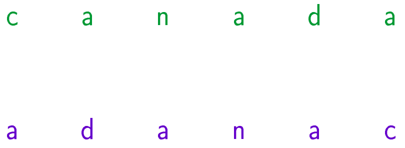

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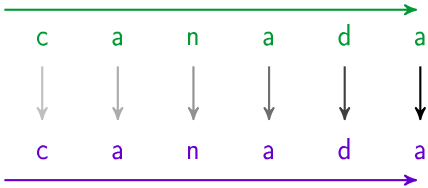

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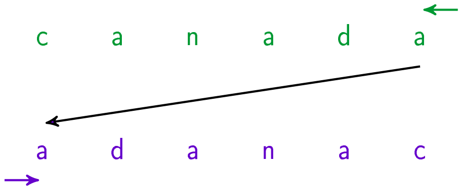

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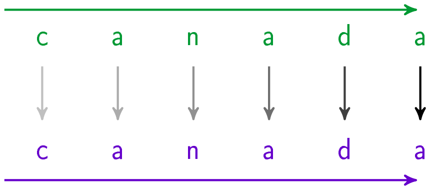

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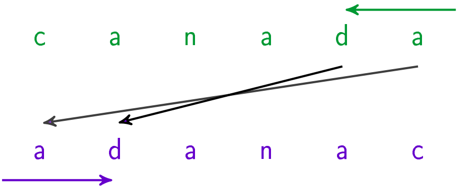

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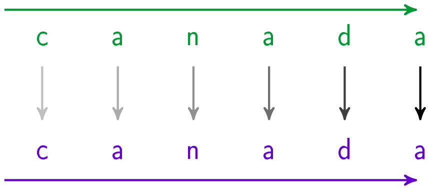

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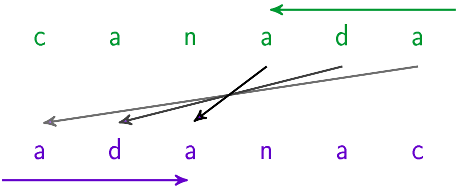

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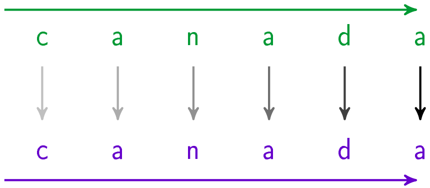

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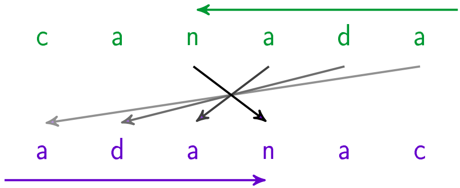

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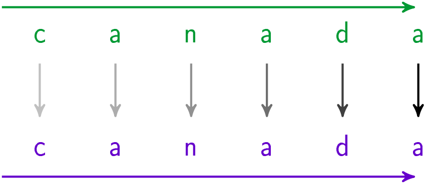

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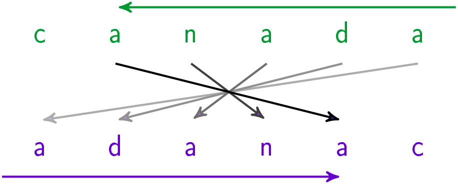

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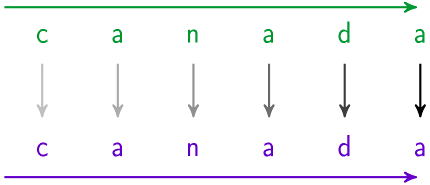

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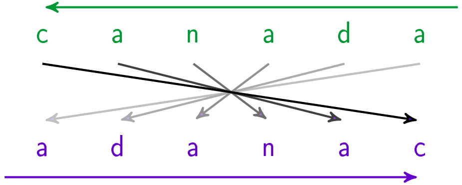

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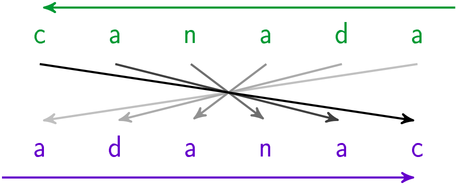

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The mirror of a relation $R \subseteq \Sigma^* \times \Delta^*$ is the relation:

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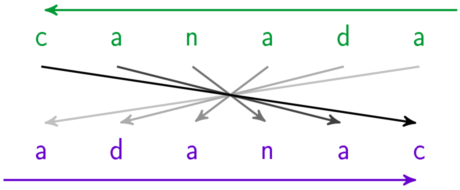

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Definition ($\overline{\mathrm{Id}}_\Sigma \circ \mathrm{Rat}$)

R is mirror rational if $\overline{\mathrm{Id}}_\Sigma \circ R \in \mathrm{Rat}$.

About mirrors

Proposition

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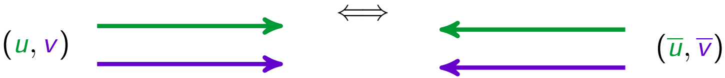

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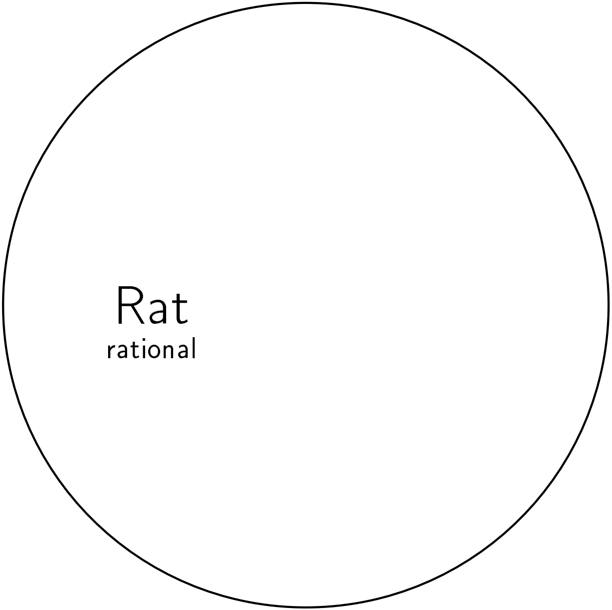

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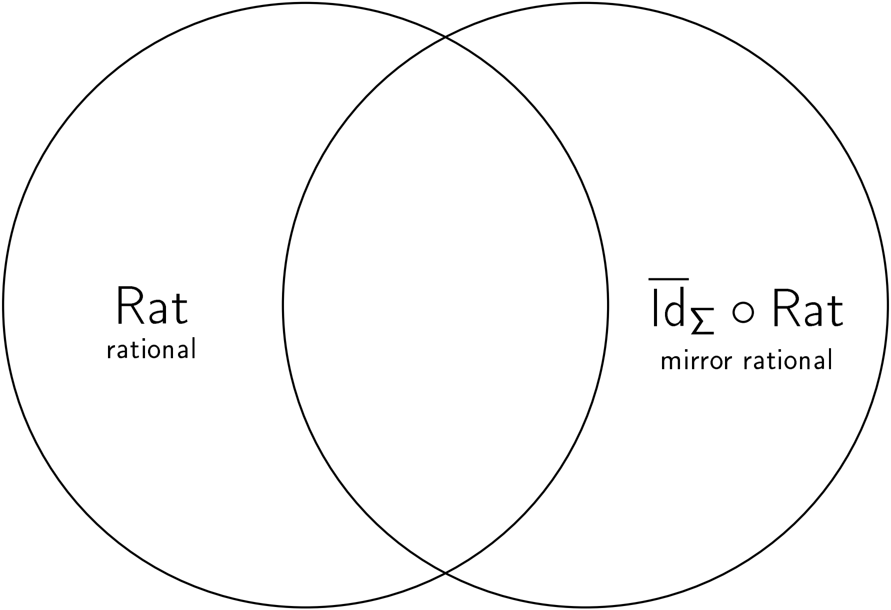

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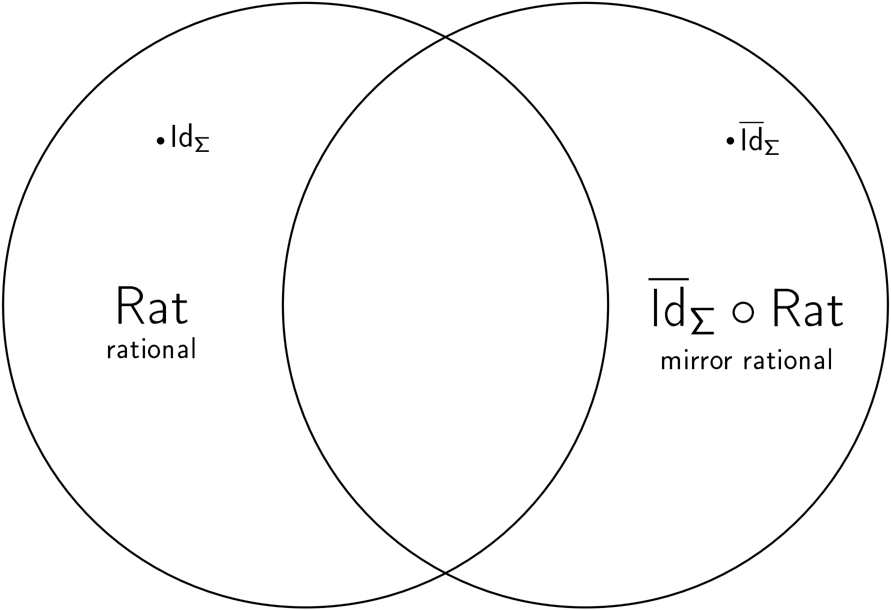

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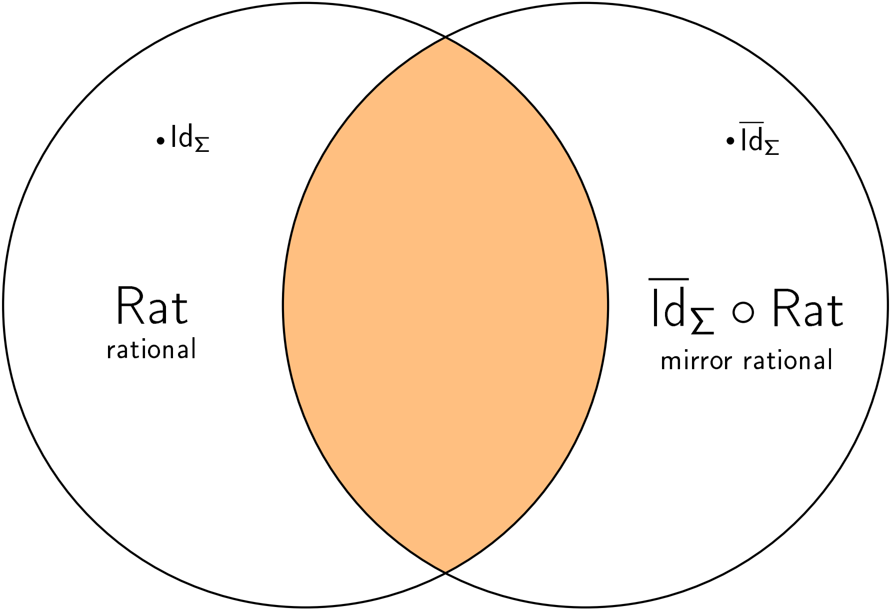

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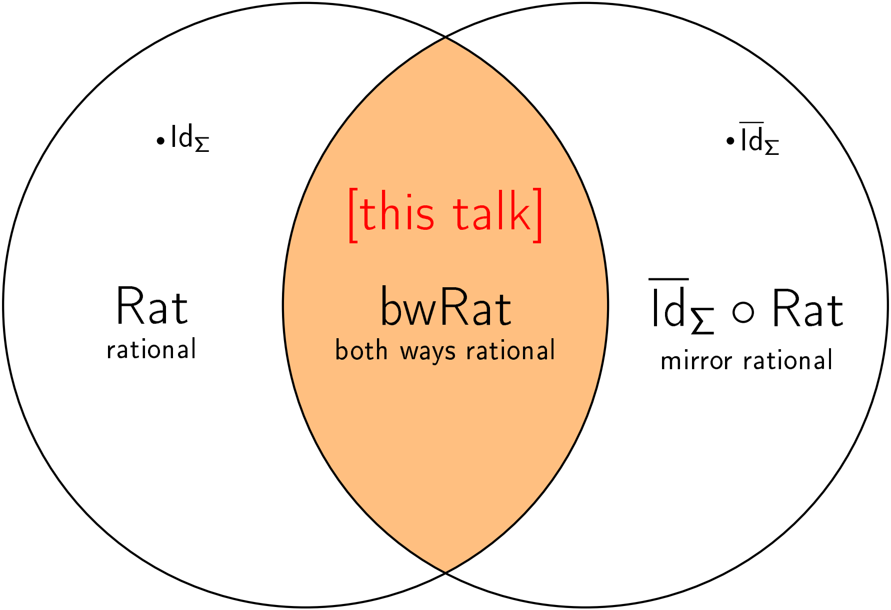

Examples of both ways rational relations

F : **copy only** a'**s**

input: c a n a d a

output: a a a

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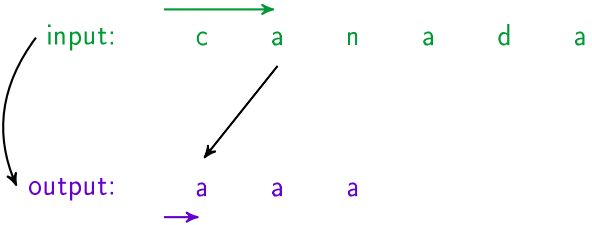

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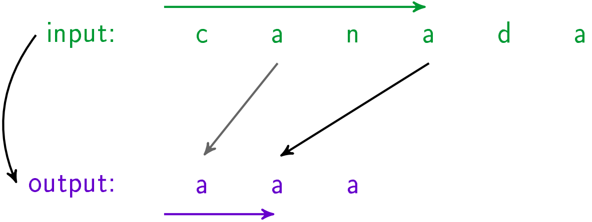

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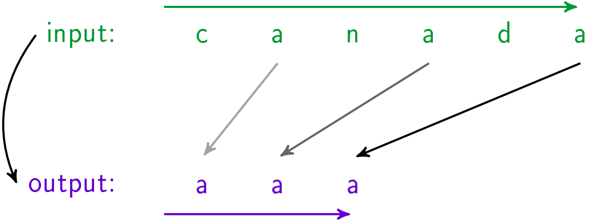

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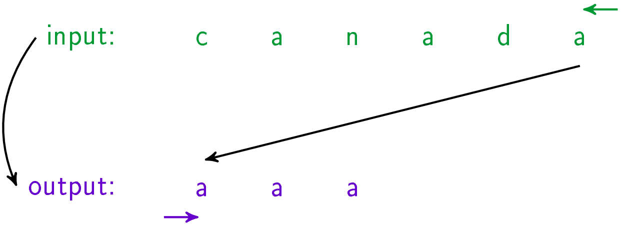

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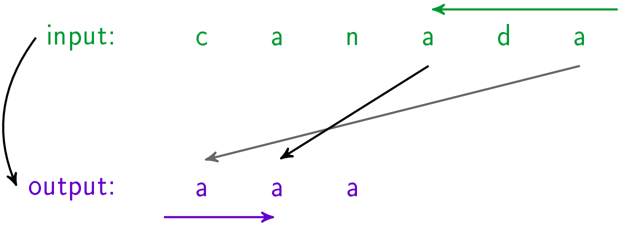

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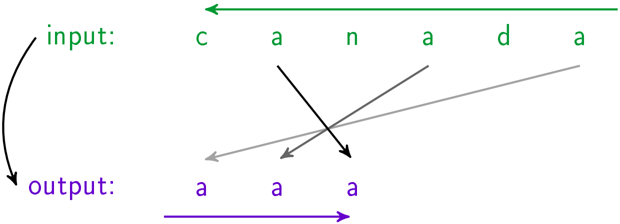

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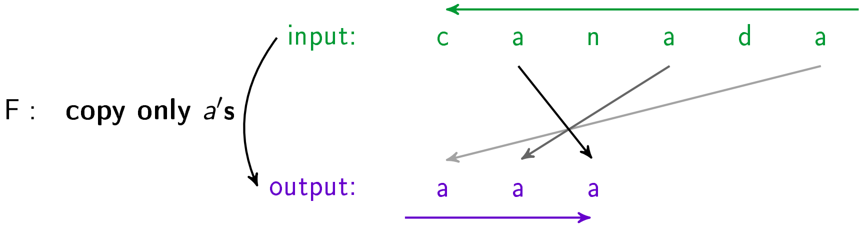

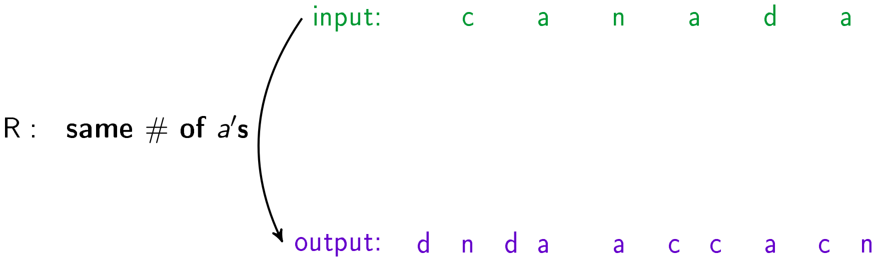

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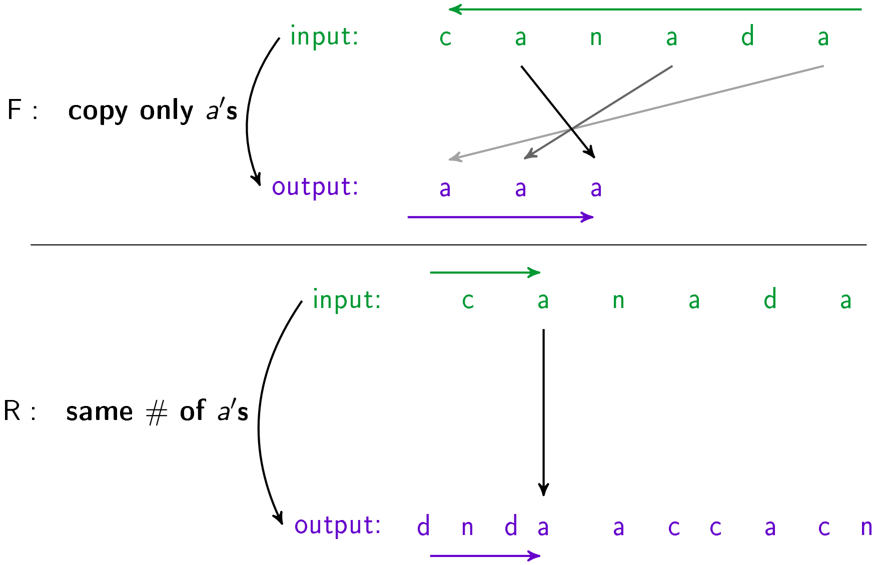

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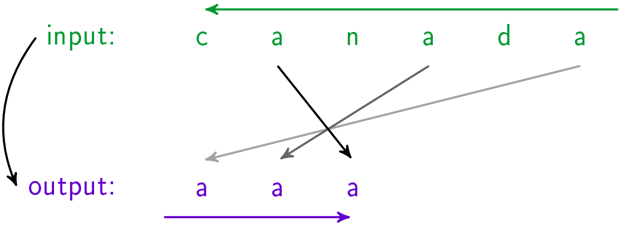

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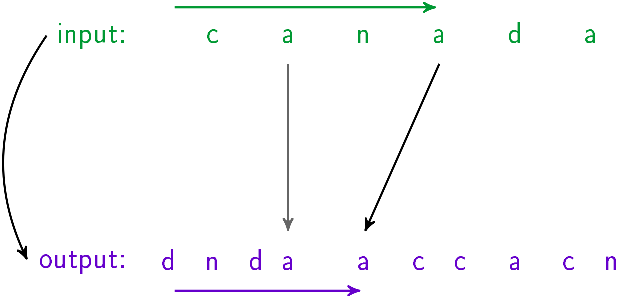

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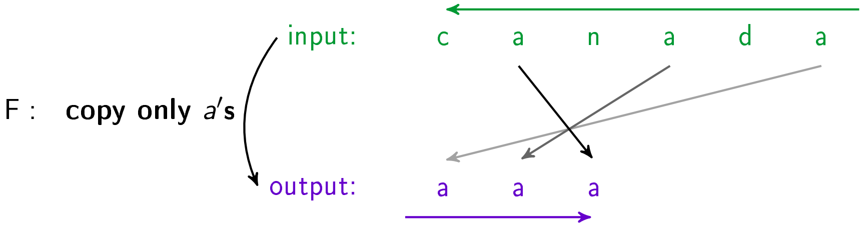

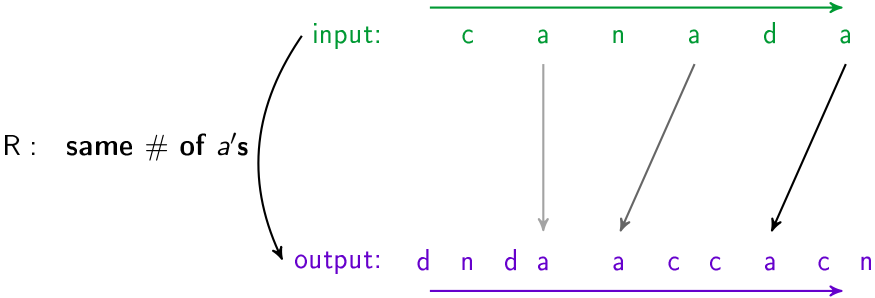

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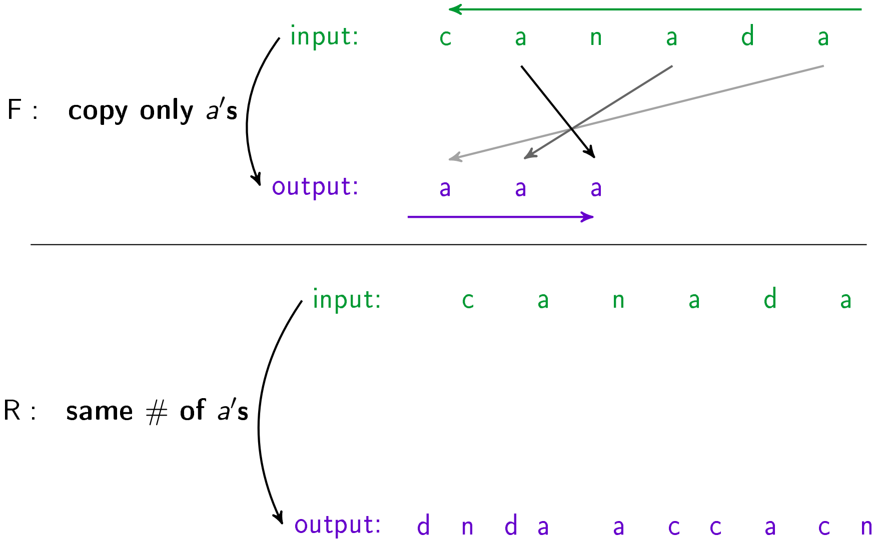

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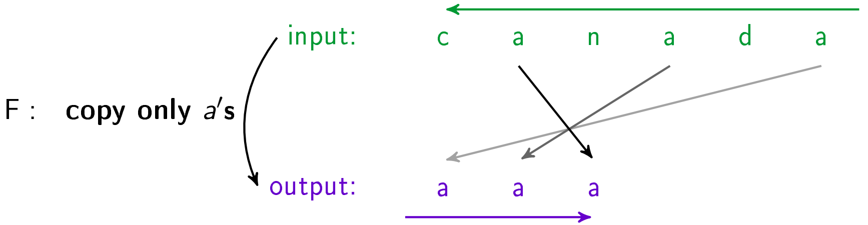

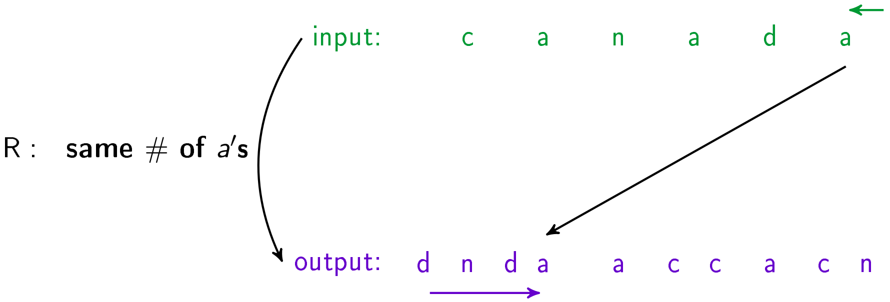

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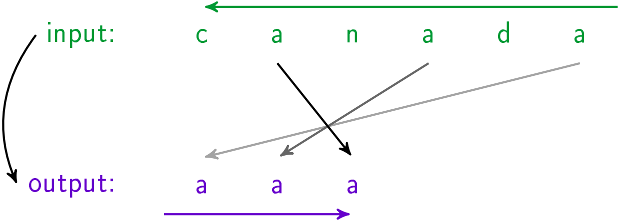

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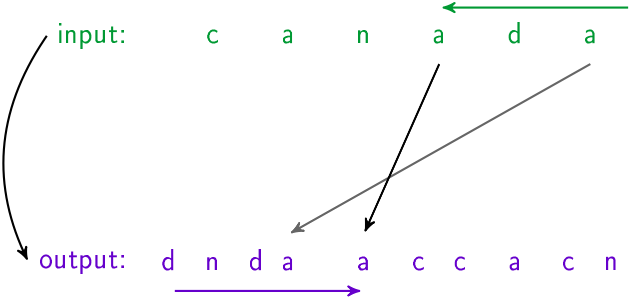

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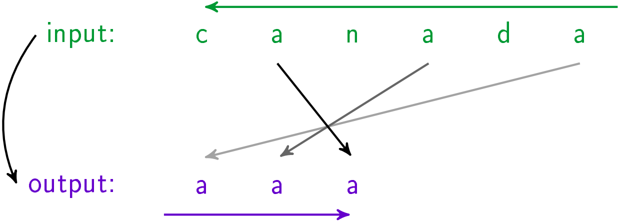

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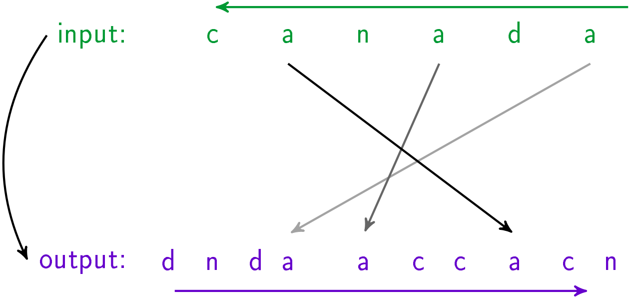

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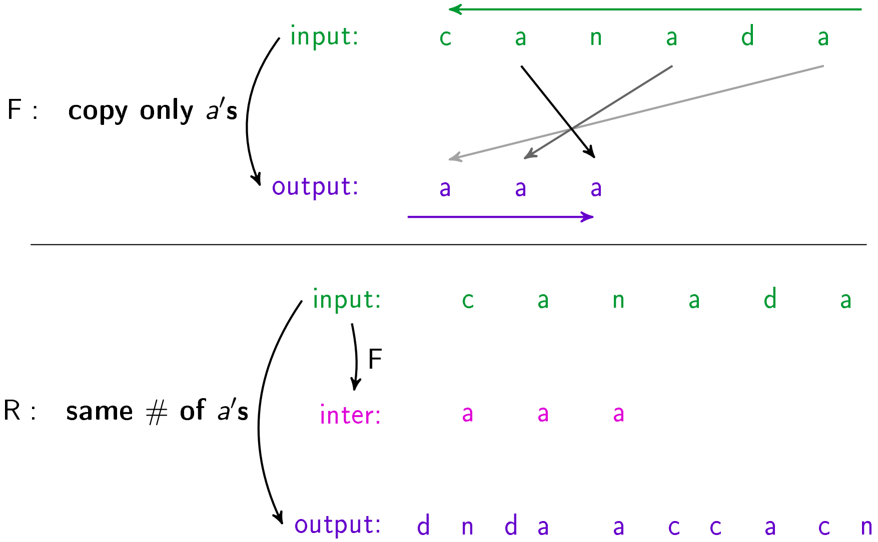

Examples of both ways rational relations

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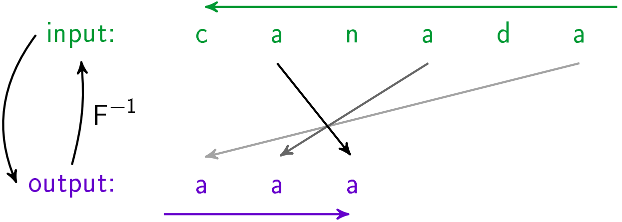
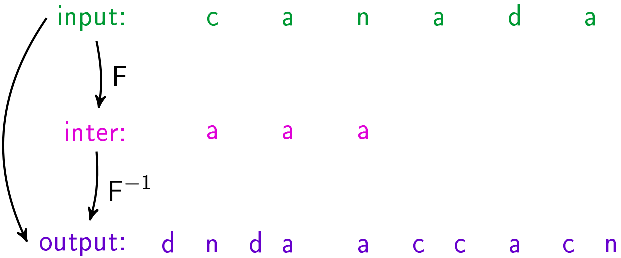


R : **same # of a's**

$R = \mathsf{F} \circ \mathsf{F}^{-1}$



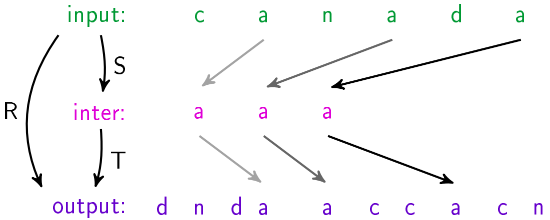
Factorizable relations

Definition (Fact)

$R \subseteq \Sigma^* \times \Delta^*$ is factorizable if there exist

- $S \subseteq \Sigma^* \times a^*$ rational
- $T \subseteq a^* \times \Delta^*$ rational

such that $R = S \circ T$.



bwRat versus Fact

Proposition (closure properties)

If R and S belongs to Fact *(resp.* bwRat*), then so do:*

- $R \cup S$
- $R \circ S$
- R^{-1}
- $\overline{\mathrm{Id}}_{\Sigma} \circ R$
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for $T \in \mathsf{Rec}$

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for $T \in \mathsf{Rec}$ $\left(T = \bigcup_{\textit{finite}} X_i \times Y_i \right)$, $X_i \in \mathsf{Rec}(\Sigma^*)$, $Y_i \in \mathsf{Rec}(\Delta^*)$

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Remark

- $|\Sigma| = 1 \Rightarrow \mathrm{Rat} = \mathrm{Fact} = \mathrm{bwRat}$.
- $|\Delta| = 1 \Rightarrow \mathrm{Rat} = \mathrm{Fact} = \mathrm{bwRat}$.

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Corollary

- $\mathrm{Fact} \subseteq \mathrm{bwRat}$

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Do we have $\mathrm{Fact} = \mathrm{bwRat}$*?*

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Theorem

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$$R \in \mathrm{bwRat} \quad \Leftrightarrow \quad R \in \mathrm{Fact}$$

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Do we have Fact = bwRat*?* ***No.***

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If R is a rational (partial) function then:

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where $x, y, z \in \Delta^*$

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$$\Leftarrow$$

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Proof of $\mathrm{R} \in \mathrm{bwRat} \Rightarrow \mathrm{image}(\mathrm{R}) = \bigcup_{\mathit{finite}} xy^*z$

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Consider the function: $\mathsf{R}^{-1} \circ \mathsf{R} \circ \overline{\mathsf{Id}}_{\Delta}$

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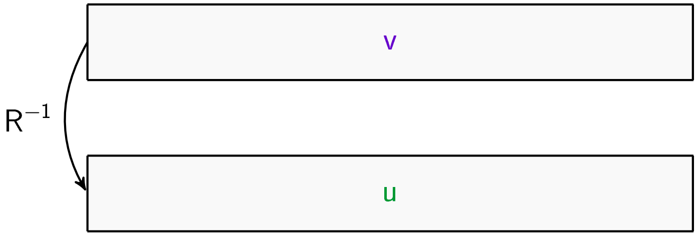

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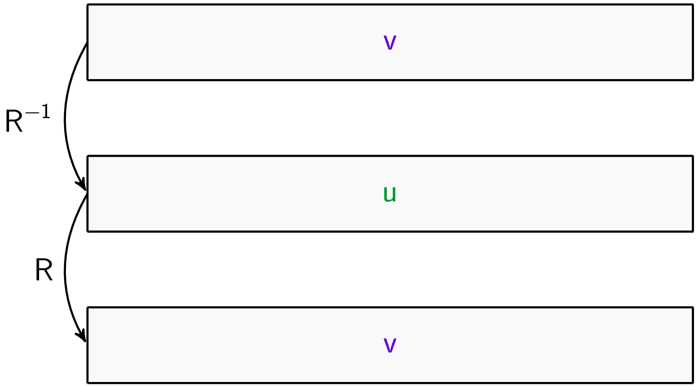

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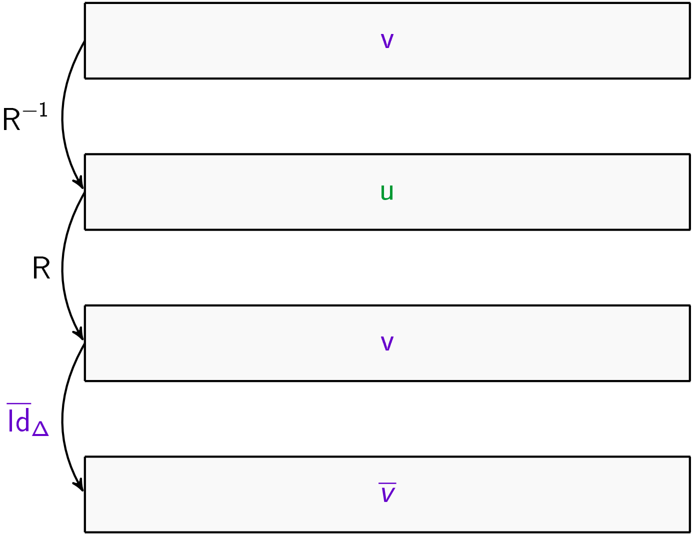

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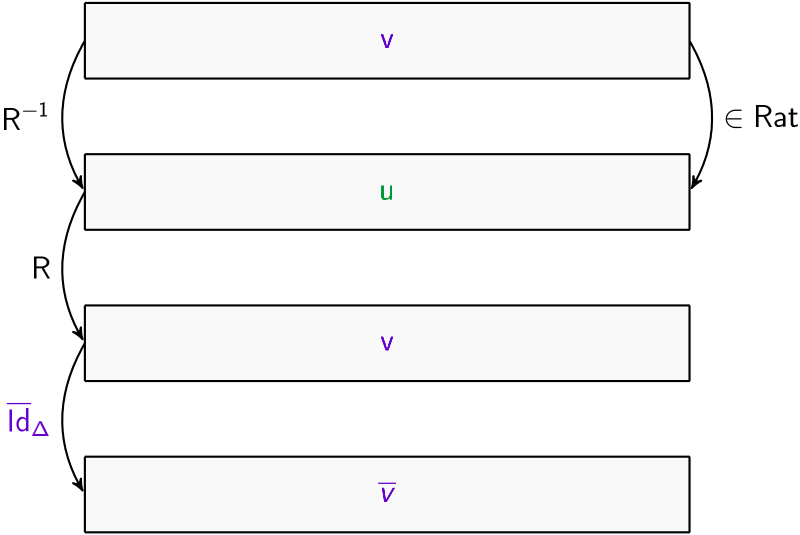

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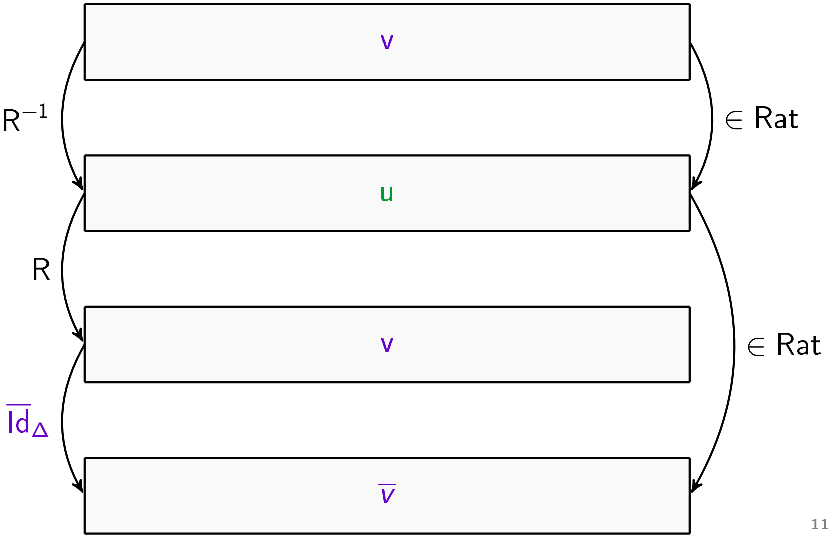

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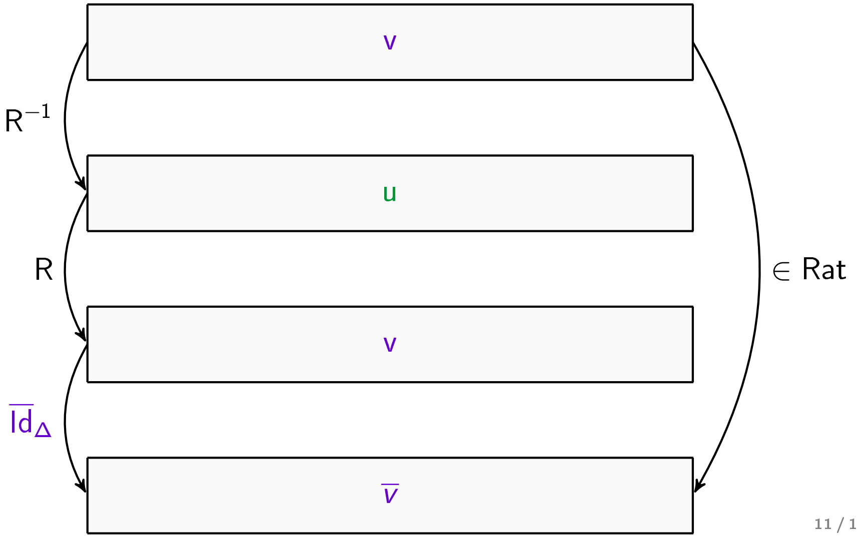

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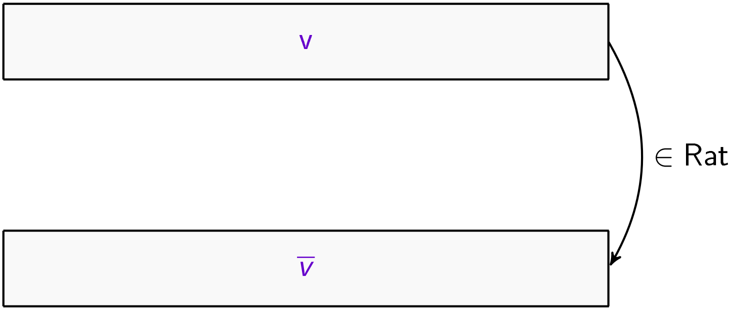

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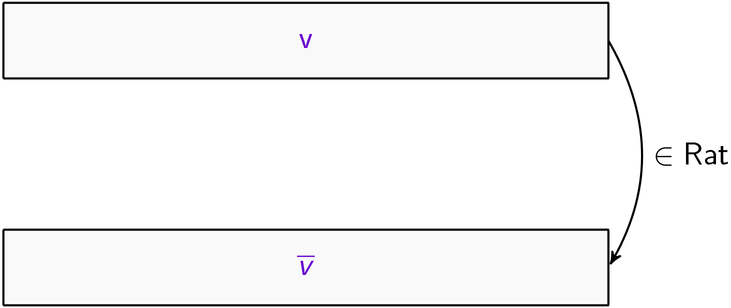
Consider the function: $\mathsf{R}^{-1} \circ \mathsf{R} \circ \overline{\mathsf{Id}}_{\Delta}$

$$= \{(v, \overline{v}) \mid v \in \mathsf{image}(R)\} \in \mathsf{Rat}$$



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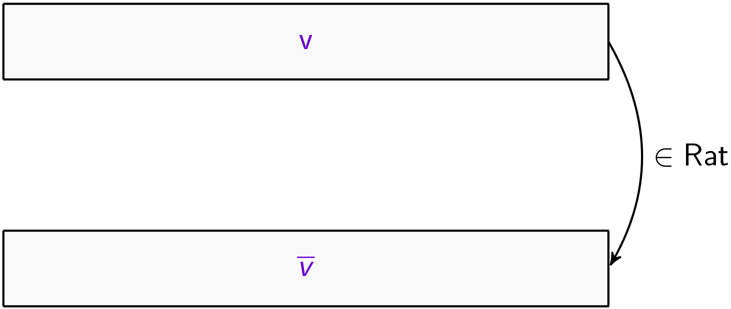

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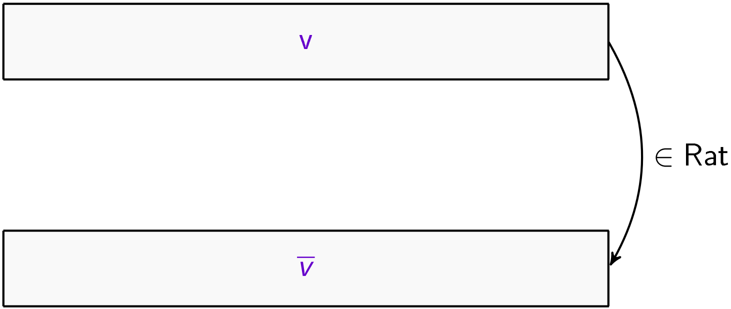
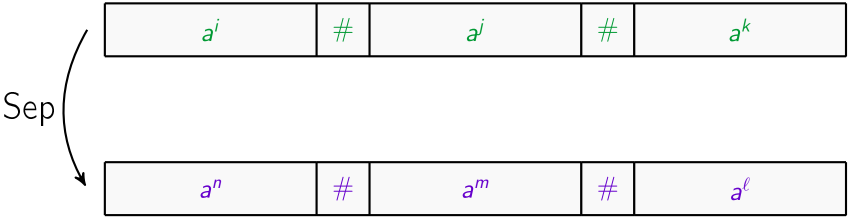
→ decidable



Counter example: bwRat $\not\subseteq$ Fact



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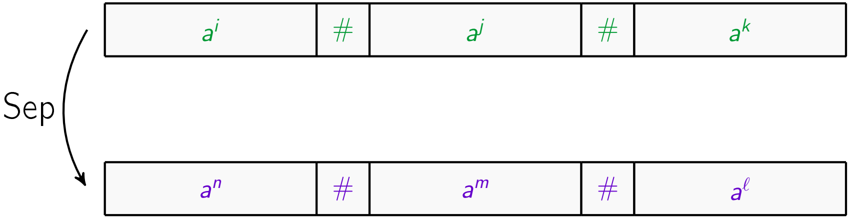
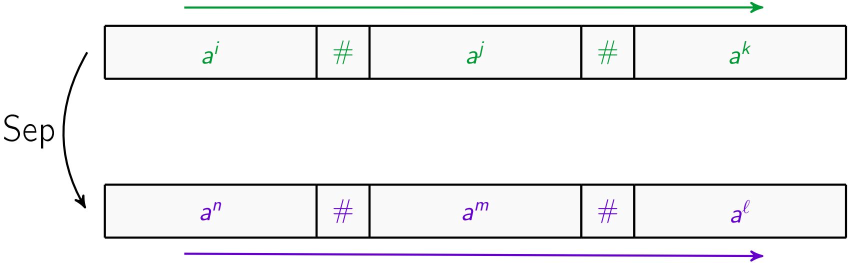


belongs to Sep if

- $i \neq \ell$
- or $k \neq \ell$
- or (i=n and j=m)

Counter example: bwRat $\not\subseteq$ Fact



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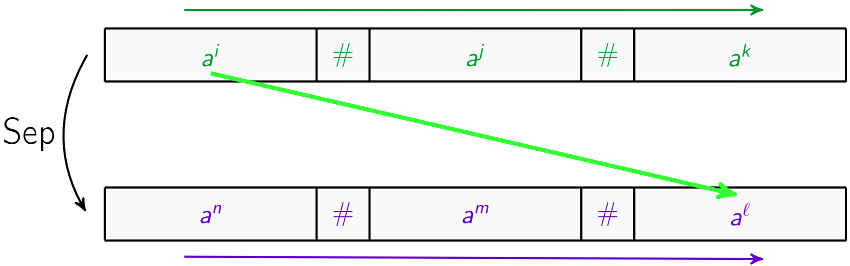

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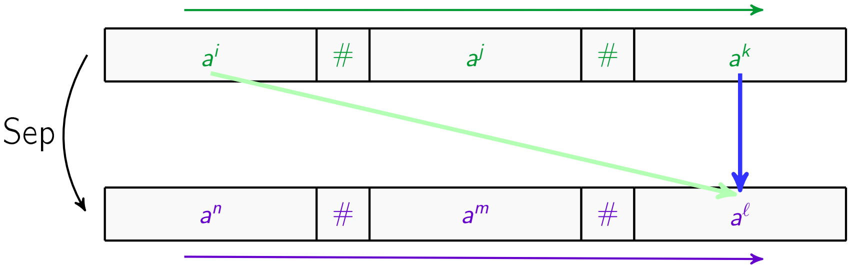

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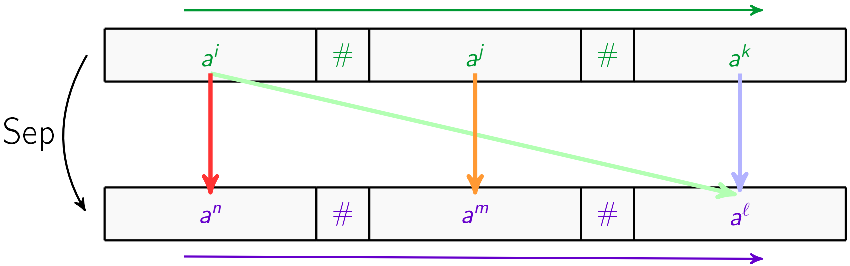

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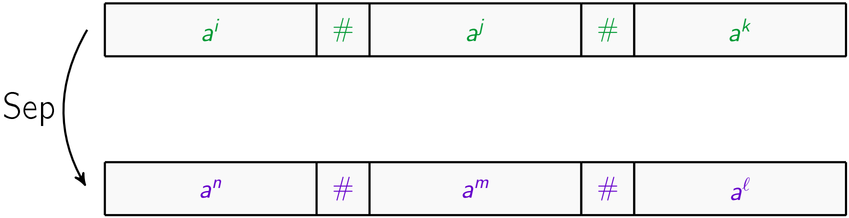

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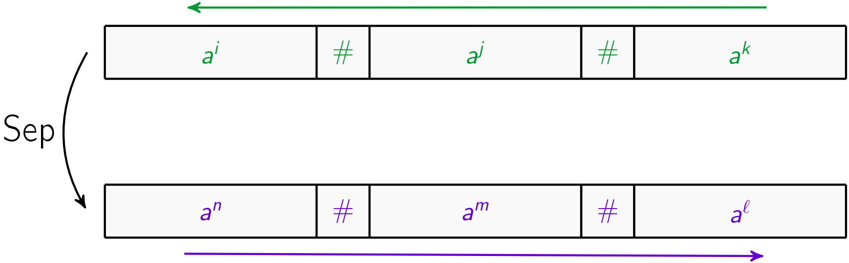

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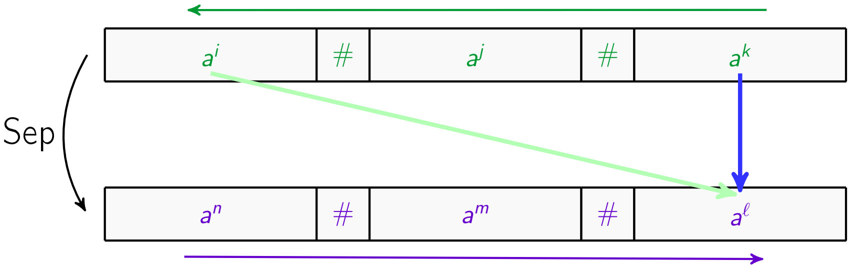

belongs to Sep if

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belongs to Sep if and only if

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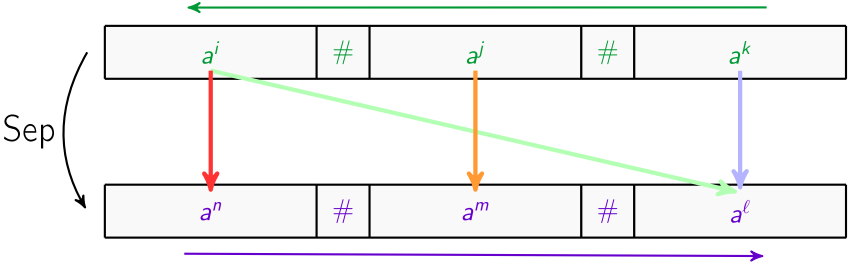
Counter example: bwRat $\not\subseteq$ Fact



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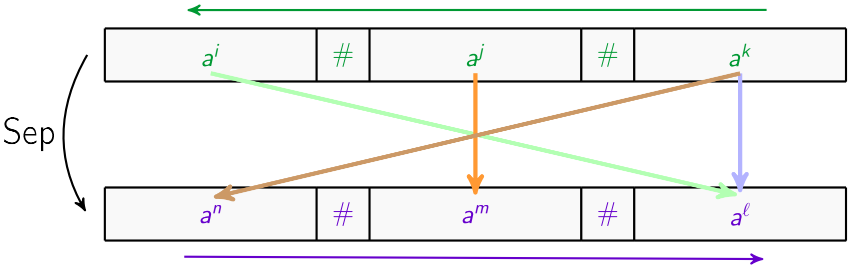

belongs to Sep if

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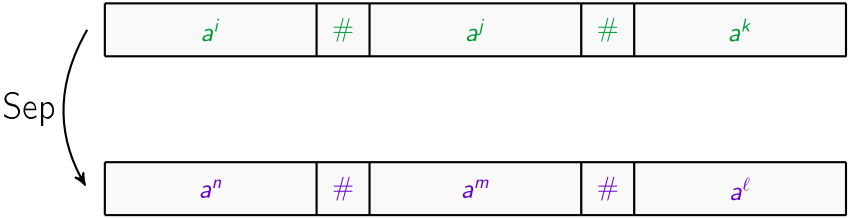

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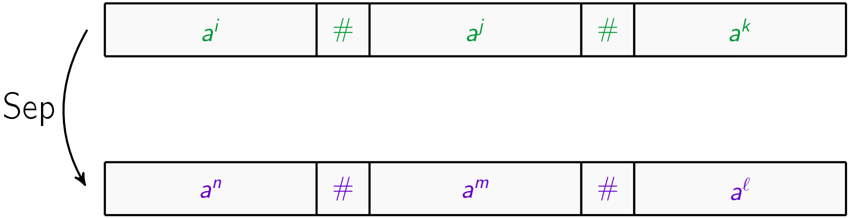
Proposition Sep $\in$ bwRat

Counter example: bwRat $\not\subseteq$ Fact



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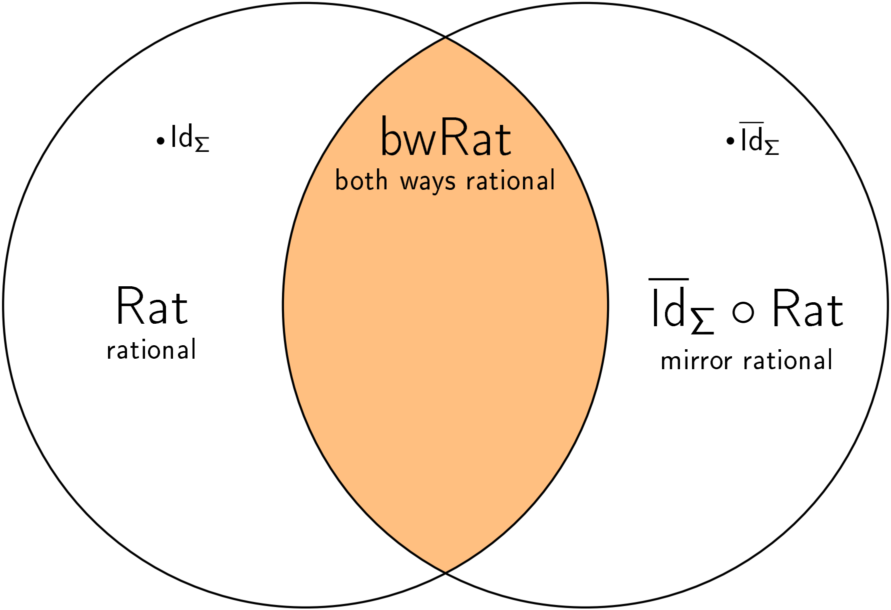
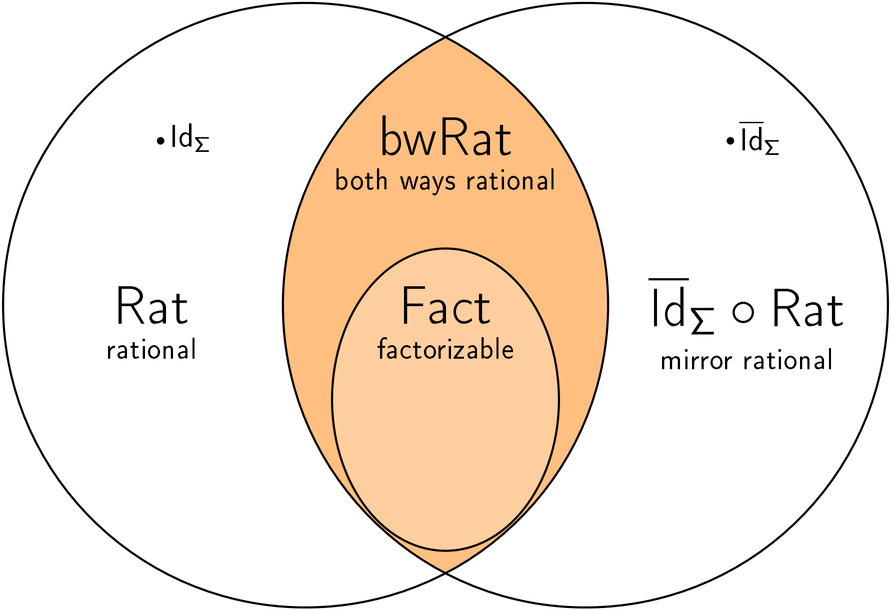
Proposition Sep $\in$ bwRat *but* Sep $\notin$ Fact

Conclusion



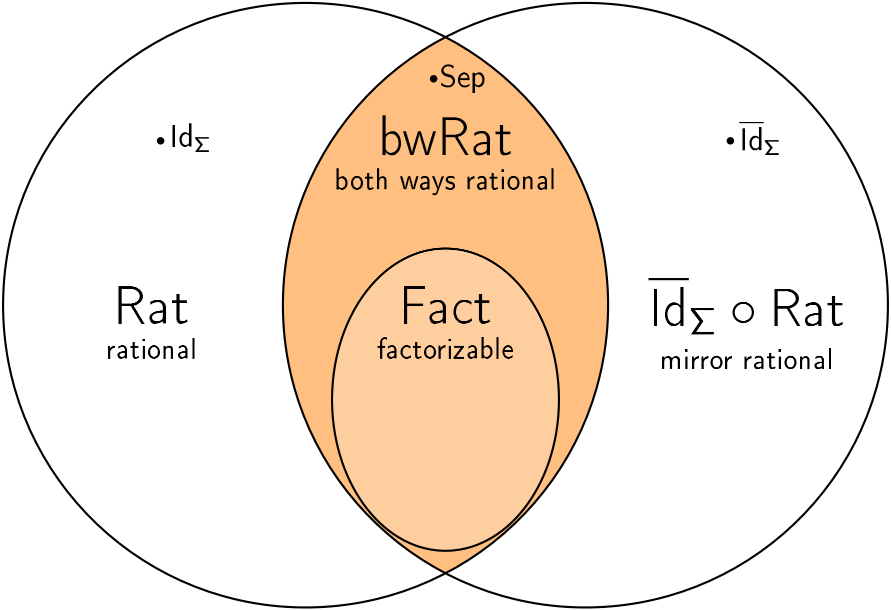
$\bullet \mathrm{Id}_\Sigma$
bwRat
both ways rational
$\bullet \overline{\mathrm{Id}}_\Sigma$
Rat
rational
$\overline{\mathrm{Id}}_\Sigma \circ \mathrm{Rat}$
mirror rational

Conclusion



$\bullet\, \mathsf{Id}_\Sigma$
bwRat
both ways rational
$\bullet\, \overline{\mathsf{Id}}_\Sigma$
Rat
rational
Fact
factorizable
$\overline{\mathsf{Id}}_\Sigma \circ \mathsf{Rat}$
mirror rational

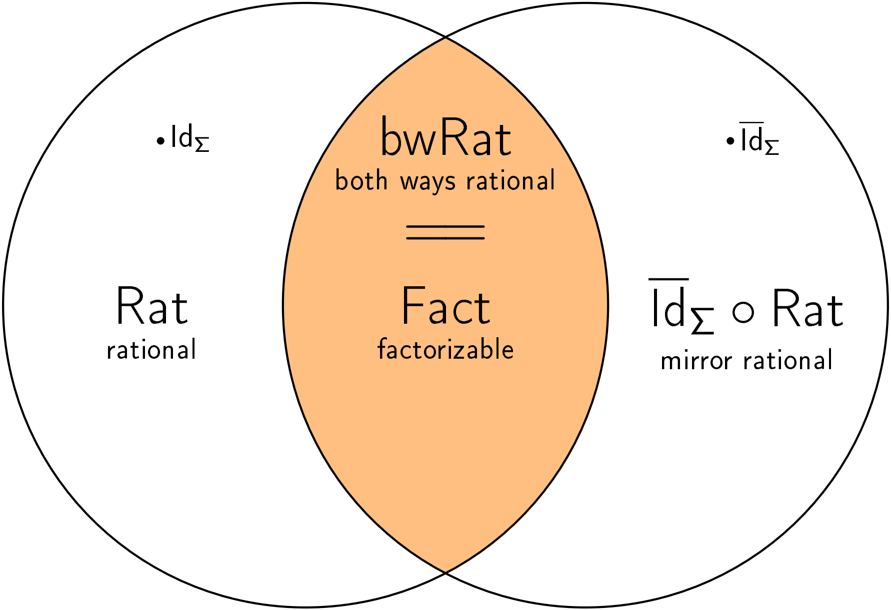
Conclusion



•Sep
bwRat
both ways rational
•Id_Σ
•$\overline{\mathrm{Id}}_\Sigma$
Rat
rational
Fact
factorizable
$\overline{\mathrm{Id}}_\Sigma \circ$ Rat
mirror rational

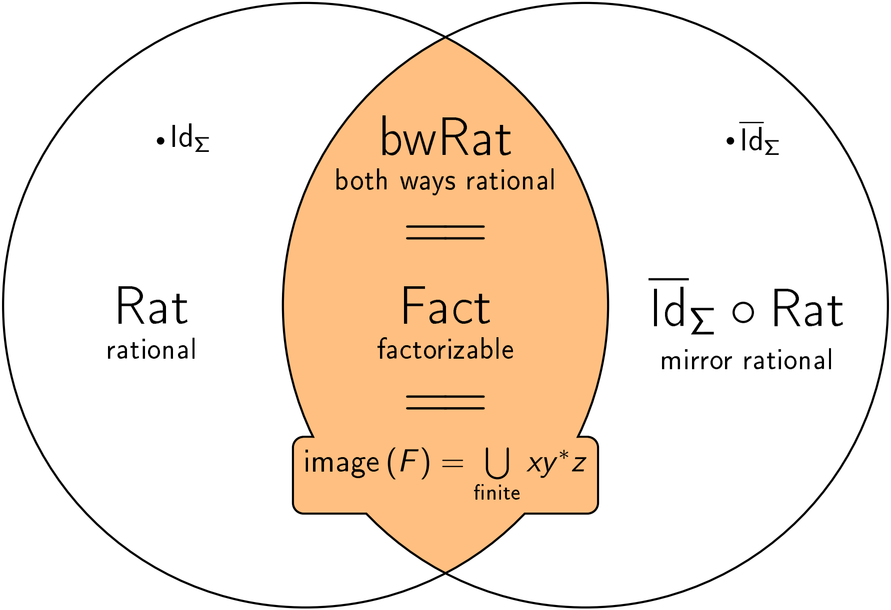
Conclusion

Functional case



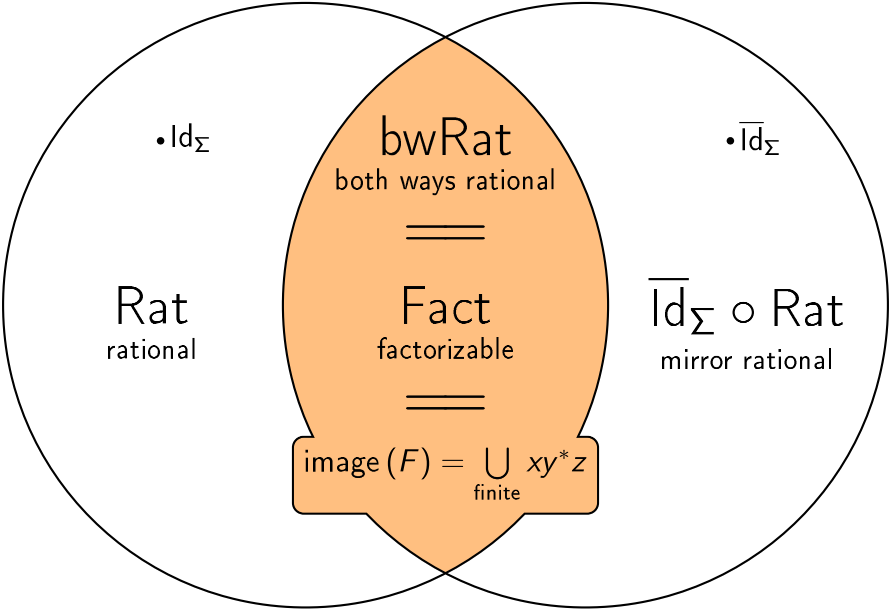
Conclusion

Functional case



Conclusion

Functional case



Thanks for your attention.