Theory and practice of XML processing programming languages

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MPRI Lectures on Theory of Subtyping

Outline of the lecture

1. **XML Programming in** $\mathbb{C}$Duce

2. **Theoretical Foundations**

3. **Polymorphic Subtyping**

4. **Polymorphic Language**

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    - XML Regular Expression Types and Patterns
    - XML Programming in ℂDuce
    - Tools on top of ℂDuce
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   - Subtyping algorithms
   - ℂDuce functional core
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   - Semantic solution
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   - Current status
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4. **Polymorphic Language**
   - Motivating example
   - Formal setting
   - Explicit substitutions
   - Inference System
   - Efficient implementation

PART 1: XML PROGRAMMING IN ℂDUCE

Programming with XML

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- Level 3: XML types taken seriously (aka: related work)
  - XDuce, Xtatic
  - XQuery
  - C_ω (Microsoft)
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- Public release available (0.5.3) in all major Linux distributions.
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Used both for teaching and in production code.

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  - Dynamic dispatch
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- **Type-driven compilation**
  - Optimizations made possible by static types
  - Avoids unnecessary and redundant tests at runtime
  - Allows a more declarative style

Regular Expression
Types and Patterns for XML

Types & patterns: the functional languages perspective

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"`match`" is more interesting than "`let`", since it can test several "`|`"-separated patterns.

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Define types: patterns come for free.

Which types should we start from?

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- **Boolean type constructors are useful for programming:**

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map catalogue with
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  Select in *catalogue* all cars that if used then are guaranteed.

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1. **Define types for XML documents,**
2. **Add boolean type constructors,**
3. **Define patterns as types with capture variables**

XML syntax

```
<bib>
  <book year="1997">
    <title>  Object-Oriented Programming </title>
    <author>
      <last>  Castagna </last>
      <first>  Giuseppe </first>
    </author>
    <price>  56 </price>
     Bikhäuser
  </book>
  <book year="2000">
    <title>  Regexp Types for XML </title>
    <editor>
      <last>  Hosoya </last>
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XML syntax

```
type Bib = <bib>[
             <book year=String>[
               <title>
               <author>[
                 <last>[PCDATA]
                 <first>[PCDATA]
               ]
               <price>[PCDATA]
               PCDATA
             ]
             <book year=String>[
               <title>[PCDATA]
               <editor>[
                 <last>[PCDATA]
                 <first>[PCDATA]
               ]
               PCDATA
             ]
           ]
```

`String = [PCDATA] = [Char*]`

XML syntax

```
type Bib = <bib>[Book Book]
type Book = <book year=String>[
                        Title
                        (Author  | Editor )
                        Price?
                        PCDATA]
type Author = <author>[Last First]
type Editor = <editor>[Last First]
type Title = <title>[PCDATA]
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type First = <first>[PCDATA]
type Price = <price>[PCDATA]
```

XML syntax

```
type Bib = <bib>[Book*]
type Book = <book year=String>[
                        Title                 nested elements
                        (Author+ | Editor+)
                        Price?
                        PCDATA]
type Author = <author>[Last First]
type Editor = <editor>[Last First]
type Title = <title>[PCDATA]
type Last = <last>[PCDATA]
type First = <first>[PCDATA]
type Price = <price>[PCDATA]
```

XML syntax

```
type Bib = <bib>[Book*]
type Book = <book year=String>[
                       Title
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                       Price?
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type Author = <author>[Last First]
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type Title = <title>[PCDATA]
type Last = <last>[PCDATA]
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XML syntax

```
type Bib = <bib>[Book*]
type Book = <book year=String>[
                       Title
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                       Price?                 optional elems
                       PCDATA]
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XML syntax

```
type Bib = <bib>[Book*]
type Book = <book year=String>[
                       Title
                       (Author+ | Editor+)
                       Price?
                       PCDATA]           mixed content
type Author = <author>[Last First]
type Editor = <editor>[Last First]
type Title = <title>[PCDATA]
type Last = <last>[PCDATA]
type First = <first>[PCDATA]
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XML syntax

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type Bib = <bib>[Book*]
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```

This and: singletons, intersections, differences, `Empty`, and `Any`.

Patterns

Patterns = Types + Capture variables

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type Bib = <bib>[Book*]
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TYPES

```
type Bib = <bib>[Book*]
```

PATTERNS

```
    <bib>[x::Book*]
```

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type Bib = <bib>[Book*]
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PATTERNS

```
    <bib>[x::Book*]
```

The pattern binds `x` to the *sequence* of all books in the bibliography

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
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PATTERNS

```
match bibs with
    <bib>[x::Book*] -> x
```

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Patterns = Types + Capture variables

TYPES

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type Bib = <bib>[Book*]
```

PATTERNS

```
match bibs with
    <bib>[x::Book*] -> x
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Returns the content of `bibs`.

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
```

PATTERNS

```
<bib>[( x::<book year="2005">_ | y::_ )*]
```

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
```

PATTERNS

```
<bib>[( x::<book year="2005">_ | y::_ )*]
```

Binds `x` to the sequence of all this year's books, and `y` to all the other books.

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
```

PATTERNS

```
match bibs with
    <bib>[( x::<book year="2005">_ | y::_ )*] -> x@y
```

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
```

PATTERNS

```
match bibs with
    <bib>[( x::<book year="2005">_ | y::_ )*] -> x@y
```

Returns the concatenation (i.e., "`@`") of the two captured sequences

Patterns

Patterns = Types + Capture variables

TYPES

```
type Bib = <bib>[Book*]
type Book = <book year=String>[Title Author+ Publisher]
type Publisher = String
```

PATTERNS

```
<bib>[(x::<book year="1990">[ _* Publisher\"ACM"] | _)*]
```

Patterns

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```

PATTERNS

```
<bib>[(x::<book year="1990">[ _* Publisher\"ACM"] | _)*]
```

Binds `x` to the *sequence* of books published in 1990 from publishers others than “ACM” and discards all the others.

Patterns

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Exact type inference:

E.g.: if we match the pattern `[(x::Int|_)*]` against an expression of type `[Int* String Int]` the type deduced for x is `[Int+]`

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XML-programming in CDuce

Functions: basic usage

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type Program = <program>[ Day* ]
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type Invited = <invited>[ Title Author+ ]
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Extract subsequences (union polymorphism)

```
fun (Invited|Talk -> [Author+])
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fun (Invited|Talk -> [Author+])
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Extract subsequences of non-consecutive elements:

```
fun ([(Invited|Talk|Event)*] -> ([Invited*], [Talk*]))
    [ (i::Invited | t::Talk | _)* ] -> (i,t)
```

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type Program = <program>[ Day* ]
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```

Perl-like string processing (`String` = `[Char*]`)

```
fun parse_email (String -> (String,String))
    | [ local::_* '@' domain::_* ] -> (local,domain)
    | _ -> raise "Invalid email address"
```

Functions: advanced usage

```
type Program = <program>[ Day* ]
type Day = <day date=String>[ Invited? Talk+ ]
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Functions can be **higher-order** and **overloaded**

```
let patch_program
(p :[Program], f :(Invited -> Invited) & (Talk -> Talk)):[Program]
   = xtransform p with (Invited | Talk) & x -> [ (f x) ]
```

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Higher-order, overloading, subtyping provide name/code sharing...

```
let first_author ([Program] -> [Program];
                  Invited -> Invited;
                  Talk -> Talk)
| [ Program ] & p -> patch_program (p,first_author)
| <invited>[ t a _* ] -> <invited>[ t a ]
| <talk>[ t a _* ] -> <talk>[ t a ]
```

Functions: advanced usage

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type Program = <program>[ Day* ]
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Even more compact: replace the last two branches with:

```
<(k)>[ t a _* ] -> <(k)>[ t a ]
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Functions: advanced usage

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Even more compact: replace the last two branches with:

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```

. . . it is all syntactic sugar!

Types

$$t \ ::= \ \texttt{Int} \ \mid \ v \ \mid \ (t, t) \ \mid \ t \rightarrow t \ \mid \ t \vee t \ \mid \ t \wedge t \ \mid \ \neg t \ \mid \ \texttt{Any}$$

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$$p \ ::= \ t \ | \ x \ | \ (p,p) \ | \ p \vee p \ | \ p \wedge p$$

Example:

```
type Book = <book>[Title (Author+|Editor+) Price?]
```

…it is all syntactic sugar!

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Patterns

$$p \ ::= \ t \mid x \mid (p,p) \mid p \vee p \mid p \wedge p$$

Example:

```
type Book = <book>[Title (Author+|Editor+) Price?]
```

encoded as

$$\begin{aligned} Book &= (\,\text{`}book, (\,Title, X \vee Y\,)) \\ X &= (Author, X \vee (Price, \text{`}nil) \vee \text{`}nil) \\ Y &= (Editor, Y \vee (Price, \text{`}nil) \vee \text{`}nil) \end{aligned}$$

Some reasons to consider regular expression types and patterns

Some good reasons to consider regexp patterns/types

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  2. Informative error messages
  3. Error mining
  4. Efficient execution
  5. Compact programs
  6. Logical optimisation of pattern-based queries
  7. Pattern matches as building blocks for iterators
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  9. Type-based query optimisation

Some good reasons to consider regexp patterns/types

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In case of error return a sample value in the difference of the inferred type and the expected one

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List of books of a given year, stripped of the Editors and Price

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```
type Book = <book year=String>[Title (Author+|Editor+) Price?]
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List of books of a given year, stripped of the Editors and Price

```
  fun onlyAuthors (year:Int,books:[Book*]):[Book*] =
```

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fun onlyAuthors (year:Int,books:[Book*]):[Book*] =
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Returns the following error message:

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Use static type information to perform an optimal set of tests

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Computing the optimal solution requires to fully exploit intersections and differences of types

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Specific kind of push-down tree automata

On top of ℂDuce

- Full integration with OCaml
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- Graphical queries
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Need to seamlessly call OCaml code in ℂDuce and viceversa

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What we need:

A mapping between OCaml and ℂDuce types and values

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`ocaml2cduce`: $t \rightarrow \mathbb{T}(t)$

`cduce2ocaml`: $\mathbb{T}(t) \rightarrow t$

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Use `M.f` to call the function `f` exported by the OCaml module `M`

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Example: use ocaml-mysql library in ℂDuce

```
let db = Mysql.connect Mysql.defaults;;

match Mysql.list_dbs db ‘None [] with
| (‘Some,l) -> print [ 'Databases: ' !(string_of l) '\ n' ]
| ‘None -> [];;
```

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Example: use ℂDuce to compute a factorial:

```
(* File cdnum.mli: *)
val fact: Big_int.big_int -> Big_int.big_int

(* File cdnum.cd: *)
let aux ((Int,Int) -> Int)
| (x, 0 | 1) -> x
| (x, n) -> aux (x * n, n - 1)

let fact (x : Int) : Int = aux(1,x)
```

PART 2: THEORETICAL FOUNDATIONS

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WHY?

Short answer: YOU JUST SAW IT!

Recap:

- to encode XML types
- to define XML patterns
- to precisely type pattern matching

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- **Starting point of what follows: the approach of Hosoya&Pierce.**

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Hosoya and Pierce use the model of values:

$$[\![t]\!]_{\mathcal{V}} = \{v \mid \; \vdash v : t\}$$

Ok because the only values of XDuce are XML documents (no first-class functions)

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Think semantically!

Intuition

$$[\![t{\to}s]\!] = ???$$

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$$[\![t \to s]\!] = \{\text{functions from } [\![t]\!] \text{ to } [\![s]\!]\}$$

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$$[\![t \to s]\!] = \{ f \subseteq \mathcal{D}^2 \mid \forall (d_1, d_2) \in f .\ d_1 \in [\![t]\!] \Rightarrow d_2 \in [\![s]\!] \}$$

Intuition

$$[\![t{\rightarrow}s]\!] = \mathcal{P}(\overline{[\![t]\!] \times \overline{[\![s]\!]}}) \qquad (\ \overline{X} \overset{\text{def}}{=} \text{complement of } X \ )$$

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$$[\![t{\rightarrow}s]\!] = \mathcal{P}(\overline{[\![t]\!] \times \overline{[\![s]\!]}}) \qquad (*)$$

Impossible since it requires $\mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D}$

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KEY OBSERVATION 2:

We need the model to state **how types are related** rather than **what the types are**

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Intuition

$$[\![t{\to}s]\!] = \mathcal{P}(\overline{[\![t]\!] \times \overline{[\![s]\!]}}) \qquad (*)$$

KEY OBSERVATION 2:

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and similarly for any boolean combination of arrow types.

Technically . . .

1. **Take** $[\![\_]\!] : \textbf{Types} \to \mathcal{P}(\mathcal{D})$ such that

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DOES A MODEL EXIST?

Is it possible to define $[\![\_]\!] : \textbf{Types} \to \mathcal{P}(\mathcal{D})$ that satisfies the model conditions, in particular a $[\![]\!]$ such that $[\![t]\!] = \emptyset \Leftrightarrow \mathbb{E}(t) = \emptyset$?

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YES: an example within two slides

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$$[\![s \times t]\!]_{\mathcal{U}} = [\![s]\!]_{\mathcal{U}} \times [\![t]\!]_{\mathcal{U}} \qquad [\![t \rightarrow s]\!]_{\mathcal{U}} = \mathcal{P}_f(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}})$$

It is a model: $\mathcal{P}_f(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}}) = \varnothing \iff \mathcal{P}(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}}) = \varnothing$

Exhibit a model

Does a model exists? (i.e. a $[\![\ ]\!]$ such that $[\![t]\!] = \emptyset \iff \mathbb{E}(t) = \emptyset$)
YES: take $(\mathcal{U}, [\![\ ]\!]_{\mathcal{U}})$ where

1. $\mathcal{U}$ least solution of $X = X^2 + \mathcal{P}_f(X^2)$
2. $[\![\ ]\!]_{\mathcal{U}}$ is defined as:

$$\begin{array}{lll}
[\![\mathbb{0}]\!]_{\mathcal{U}} = \emptyset & [\![\mathbb{1}]\!]_{\mathcal{U}} = \mathcal{U} & [\![\neg t]\!]_{\mathcal{U}} = \mathcal{U} \backslash [\![t]\!]_{\mathcal{U}} \\
[\![s {\vee} t]\!]_{\mathcal{U}} = [\![s]\!]_{\mathcal{U}} \cup [\![t]\!]_{\mathcal{U}} & & [\![s {\wedge} t]\!]_{\mathcal{U}} = [\![s]\!]_{\mathcal{U}} \cap [\![t]\!]_{\mathcal{U}} \\
[\![s {\times} t]\!]_{\mathcal{U}} = [\![s]\!]_{\mathcal{U}} \times [\![t]\!]_{\mathcal{U}} & & [\![t {\to} s]\!]_{\mathcal{U}} = \mathcal{P}_f(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}})
\end{array}$$

It is a model: $\mathcal{P}_f(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}}) = \varnothing \iff \mathcal{P}(\overline{[\![t]\!]_{\mathcal{U}} \times \overline{[\![s]\!]_{\mathcal{U}}}}) = \varnothing$

It is the **best** model: for any other model $[\![\ ]\!]_{\mathcal{D}}$

$$t_1 \leq_{\mathcal{D}} t_2 \quad \Rightarrow \quad t_1 \leq_{\mathcal{U}} t_2$$

Subtyping Algorithms.

Canonical forms

Every (recursive) type

$$t ::= B \mid t{\times}t \mid t{\to}t \mid t{\vee}t \mid t{\wedge}t \mid \neg t \mid \mathbb{0} \mid \mathbb{1}$$

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is equivalent (semantically, w.r.t. $\leq$) to a type of the form (I omitted base types):

$$\bigvee_{(P,N)\in\Pi}((\bigwedge_{s\times t\in P} s{\times}t){\wedge}(\bigwedge_{s\times t\in N}\neg(s{\times}t))) \ \bigvee_{(P,N)\in\Sigma}((\bigwedge_{s\to t\in P} s{\to}t){\wedge}(\bigwedge_{s\to t\in N}\neg(s{\to}t)))$$

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1. Put it in disjunctive normal form, e.g.

$$(a_1\wedge a_2\wedge\neg a_3)\vee(a_4\wedge\neg a_5)\vee(\neg a_6\wedge\neg a_7)\vee(a_8\wedge a_9)$$

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3. Group negative and positive atoms in the intersections:

$$\bigvee_{(P,N)\in S}((\bigwedge_{a\in P} a)\wedge(\bigwedge_{a\in N}\neg a))$$

Decision procedure

$$s \leq t?$$

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$$s \leq t \iff [\![s]\!] \cap \overline{[\![t]\!]} = \varnothing \iff [\![s \wedge \neg t]\!] = \varnothing \iff s \wedge \neg t = \mathbb{0}$$

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1. Consider $s \wedge \neg t$
2. Put it in canonical form

$$\bigvee_{(P,N)\in\Pi} ((\bigwedge_{s\times t\in P} s\times t) \wedge (\bigwedge_{s\times t\in N} \neg(s\times t))) \bigvee_{(P,N)\in\Sigma} ((\bigwedge_{s\to t\in P} s\to t) \wedge (\bigwedge_{s\to t\in N} \neg(s\to t)))$$

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3. Decide (coinductively) whether all the intersections occuring above are empty by applying the set theoretic properties stated in the next slide.

Subtyping decomposition

Decomposition law for products:

$$\bigwedge_{i\in I} t_i{\times}s_i \leq \bigvee_{j\in J} t_j{\times}s_j$$
$$\iff \forall J' \subseteq J.\ \left(\bigwedge_{i\in I} t_i \leq \bigvee_{j\in J'} t_j\right) \text{ or } \left(\bigwedge_{i\in I} s_i \leq \bigvee_{j\in J\setminus J'} s_j\right)$$

Decomposition law for arrows:

$$\bigwedge_{i\in I} t_i{\to}s_i \leq \bigvee_{j\in J} t_j{\to}s_j$$
$$\iff \exists j \in J.\forall I' \subseteq I.\ \left(t_j \leq \bigvee_{i\in I'} t_i\right) \text{ or } \left(I' \neq I \text{ et } \bigwedge_{i\in I\setminus I'} s_i \leq s_j\right)$$

Exercise

Using the laws of the previous slide prove the following equivalences:

$$t_1{\times}s_1 \leq t_2{\times}s_2 \quad\Longleftrightarrow\quad t_1 \leq \emptyset \text{ or } s_1 \leq \emptyset \text{ or } (t_1 \leq t_2 \text{ and } s_1 \leq s_2)$$

$$t_1{\rightarrow}s_1 \leq t_2{\rightarrow}s_2 \quad\Longleftrightarrow\quad t_2 \leq \emptyset \text{ or } \quad \text{or } (t_2 \leq t_1 \text{ and } s_1 \leq s_2)$$

Application to a language.

Language

$$
\begin{array}{rll}
e ::= & x & \text{variable} \\
\mid & \mu f^{(s_1 \to t_1; \dots ; s_n \to t_n)}(x).e & \text{abstraction, } n \geq 1 \\
\mid & e_1 e_2 & \text{application} \\
\mid & (e_1, e_2) & \text{pair} \\
\mid & \pi_i(e) & \text{projection, } i = 1, 2 \\
\mid & (x = e \in t)\,?\,e_1 : e_2 & \text{binding type case}
\end{array}
$$

Typing

$$\frac{\Gamma \vdash e : s \leq_{\mathcal{B}} t}{\Gamma \vdash e : t}\ (\textit{subsumption})$$

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$$\frac{(\forall i)\ \Gamma, (f : s_1 \to t_1 \wedge \ldots \wedge s_n \to t_n), (x : s_i) \vdash e : t_i}{\Gamma \vdash \mu f^{(s_1 \to t_1; \ldots; s_n \to t_n)}(x).e : s_1 \to t_1 \wedge \ldots \wedge s_n \to t_n}\ (abstr)$$

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(for $s_1 \equiv s{\wedge}t$, $s_2 \equiv s{\wedge}\neg t$)

$$\frac{\Gamma \vdash e : s \qquad \Gamma, (x : s_1) \vdash e_1 : t_1 \quad \Gamma, (x : s_2) \vdash e_2 : t_2}{\Gamma \vdash (x = e \in t)?e_1 : e_2 : \bigvee_{\{i \mid s_i \not\simeq \mathbb{0}\}} t_i}\ (\mathit{typecase})$$

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Consider:

$$\mu \mathrm{f}^{(\mathrm{Int}\to\mathrm{Int};\mathrm{Bool}\to\mathrm{Bool})}(x).(y = x \in \mathrm{Int})\,?\,(y+1) : \mathrm{not}(y)$$

Reduction

$$\begin{array}{rcll}
(\boldsymbol{\mu} f^{(\ldots)}\mathbf{(}x\mathbf{).}e)v & \rightarrow & e[x/v, (\boldsymbol{\mu} f^{(\ldots)}\mathbf{(}x\mathbf{).}e)/f] & \\
(x = v \in t)\mathbf{?}e_1 \mathbf{:} e_2 & \rightarrow & e_1[x/v] & \text{if } v \in [\![t]\!] \\
(x = v \in t)\mathbf{?}e_1 \mathbf{:} e_2 & \rightarrow & e_2[x/v] & \text{if } v \notin [\![t]\!]
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where

$$v \ ::= \ \mu f^{(\dots)}\mathbf{(}x\mathbf{)}\mathbf{.}e \ \mid \ (v,v)$$

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The circle is closed

Why does it work?

$$s \leq_{\mathcal{B}} t \qquad \Longleftrightarrow \qquad s \leq_{\mathcal{V}} t \qquad (1)$$

Equation (1) (actually, $\Rightarrow$) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with "enough points"

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For any model $\mathcal{B}$,

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For any model $\mathcal{B}$,

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In particular, thanks to multiple arrows in λ-abstractions:

$$\bigwedge_{i=1..k} s_i \rightarrow t_i \not\leq t$$

then the two types are distinguished by $\mu f^{(s_1\rightarrow t_1;\ldots;s_k\rightarrow t_k)}(x).e$

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is fundamental for meaningful error messages:

Exibit the *v* at issue rather than pointing to the failure of some deduction rule.

Summary of the theory

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5. **Enjoy.**

PART 3: POLYMORPHIC SUBTYPING

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$$(X\times Y\to X)\wedge((X\to Y)\to X\to Y)$$

and define for them an intuitive semantics

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WHY?

Short answers:

- Parametric polymorphism is very useful in practice.
- It covers new needs peculiar to XML processing (*eg*, SOAP envelopes).
- It would make the interface with OCaml complete
- The extension shoud shed new light on the notion of **parametricity**

Concrete answer: an example in web development

We need parametric polymorphism to statically type service registration in the Ocsigen web server:

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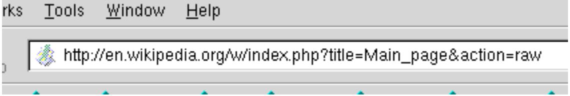
We need parametric polymorphism to statically type service registration in the Ocsigen web server:

- To every page possibly with parameters



http://en.wikipedia.org/w/index.php?title=Main_page&action=raw

corresponds a function that takes the parameters (the query string) and dynamically generates the appropriate Xhtml page:

```
let wikipage (p : WikiParams) : Xhtml = ...

type WikiParams = <params>
                    <title> String </title>
                    <action> "raw"|"edit" <action>
                  </params>
```

- The binding between the URL `$WEBROOT/w/index` and the function `wikipage` is done by the Ocsigen function `register_new_service`:

```
register_new_service(wikipage,"w.index")
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- We would like to give `register_new_service` the type

  $$\forall(\mathtt{X} \leq \mathtt{QueryString}).(\mathtt{X} \to \mathtt{Xhtml}) \times \mathtt{Path} \to \mathtt{unit}$$

  where `QueryString` is the XML type that includes all *query strings* and `Path` specifies the paths of the server.

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  $$\forall(\texttt{X} \leq \texttt{QueryString}).(\texttt{X} \to \texttt{Xhtml}) \times \texttt{Path} \to \texttt{unit}$$

  where `QueryString` is the XML type that includes all *query strings* and `Path` specifies the paths of the server.

Notice

We need both higher-order polymorphic functions

- The binding between the URL `$WEBROOT/w/index` and the function `wikipage` is done by the Ocsigen function `register_new_service`:

  ```
  register_new_service(wikipage,"w.index")
  ```

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Notice

We need both higher-order polymorphic functions and bounded quantification

A very hard problem

Naive solution

$$t \;::=\; B \;\mid\; t\times t \;\mid\; t\to t$$

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$$t ::= B \mid t{\times}t \mid t{\rightarrow}t \mid t{\vee}t \mid t{\wedge}t \mid \neg t \mid \mathbb{0} \mid \mathbb{1}$$

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Now use the previous relation. This is defined for “ground types”

Let $\sigma : \textbf{Vars} \to \textbf{Types}_{\text{ground}}$ denote ground substitutions then define:

$$s \leq t \quad \overset{def}{\Longleftrightarrow} \quad \forall\sigma\,.\, s\sigma \leq t\sigma$$

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- If $t \leq X$, then $X = (X\backslash t)\vee t$ and, therefore, $(t{\times}X) = (t{\times}(X\backslash t))\vee(t{\times}t)$. This union is contained component-wise in the one above.

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If we can give a semantic characterization of models in which this stuttering is absent, then this should yield a subtyping relation that is:

- Semantic
- Intuitive for the programmer
- Decidable

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with

$$\begin{array}{lcl@{\qquad\qquad}lcl}
[\![X]\!]\eta & = & \eta(X) & [\![\neg t]\!]\eta & = & \mathcal{D}\setminus[\![t]\!]\eta \\
[\![t_1 \vee t_2]\!]\eta & = & [\![t_1]\!]\eta \cup [\![t_2]\!]\eta & [\![t_1 \wedge t_2]\!]\eta & = & [\![t_1]\!]\eta \cap [\![t_2]\!]\eta \\
[\![\mathbb{0}]\!]\eta & = & \emptyset & [\![\mathbb{1}]\!]\eta & = & \mathcal{D}
\end{array}$$

Subtyping relation

In this framework the natural definition of subtyping is

$$s \leq t \quad \stackrel{def}{\Longleftrightarrow} \quad \forall\eta\,.\; [\![s]\!]\eta \subseteq [\![t]\!]\eta$$

It just remains to find the uniformity condition to recover parametricity.

The magic property

Consider only models of semantic subtyping in which the following **convexity** property holds

$$\forall\eta.(\llbracket t_1\rrbracket\eta{=}\varnothing \text{ or } \llbracket t_2\rrbracket\eta{=}\varnothing) \iff (\forall\eta.\llbracket t_1\rrbracket\eta{=}\varnothing) \text{ or } (\forall\eta.\llbracket t_2\rrbracket\eta{=}\varnothing)$$

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- An intuitive relation: the algorithm returns intuitive results (actually, it helps to better understand twisted examples)

Examples of subtyping relations

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We can internalize properties such as:

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We can prove relevant relations on infinite types. Consider generic lists:

$$\alpha \ \mathsf{list} = \mu x.(\alpha{\times}x) \vee \mathsf{nil}$$

It contains both the α-lists with an even number of elements

$$\mu x.(\alpha\times(\alpha\times x)) \vee \mathsf{nil} \;\leq\; \mu x.(\alpha\times x) \vee \mathsf{nil}$$

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And we can prove far more complicated relations (see later).

Subtyping algorithm

Subtyping Algorithm

Step 1: **Transform the subtyping problem into an emptiness decision problem:**

$$t_1 \leq t_2 \iff \forall\eta.[\![t_1]\!]\eta \subseteq [\![t_2]\!]\eta \iff \forall\eta.[\![t_1{\wedge}\neg t_2]\!]\eta{=}\varnothing \iff t_1{\wedge}\neg t_2 \leq \mathbb{0}$$

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Step 2: **Put the type whose emptiness is to be decided in disjunctive normal form.**

$$\bigvee_{i\in I}\bigwedge_{j\in J}\ell_{ij}$$

where $a ::= b \mid t\times t \mid t\rightarrow t \mid \mathbb{0} \mid \mathbb{1} \mid \alpha$ and $\ell ::= a \mid \neg a$

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Step 3: **Simplify mixed intersections:**

Consider each summand of the union: cases such as $t_1{\times}t_2 \wedge t_1{\rightarrow}t_2$ or $t_1{\times}t_2 \wedge \neg(t_1{\rightarrow}t_2)$ are straightforward.

Solve: $$\bigwedge_{i\in I} a_i \bigwedge_{j\in J} \neg a'_j \bigwedge_{h\in H} \alpha_h \bigwedge_{k\in K} \neg\beta_k$$

where all a are of the same kind.

Step 4: **Eliminate toplevel negative variables.**,

$$\forall\eta.[\![t]\!]\eta=\varnothing \iff \forall\eta.[\![t\{{}^{\neg\alpha}/_{\alpha}\}]\!]\eta=\varnothing$$

so replace $\neg\beta_k$ for β_k (forall $k\in K$)

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Step 5: Eliminate toplevel variables.

$$\bigwedge_{t_1\times t_2\in P} t_1\times t_2 \bigwedge_{h\in H} \alpha_h \quad\leq\quad \bigvee_{t'_1\times t'_2\in N} t'_1\times t'_2$$

holds if and only if

$$\bigwedge_{t_1\times t_2\in P} t_1\sigma \times t_2\sigma \bigwedge_{h\in H} \gamma_h^1 \times \gamma_h^2 \quad\leq\quad \bigvee_{t'_1\times t'_2\in N} t'_1\sigma \times t'_2\sigma$$

where $\sigma = \{{}^{(\gamma_h^1\times\gamma_h^2)\vee\alpha_h}/_{\alpha_h}\}_{h\in H}$ (similarly for arrows)

Step 6: **Eliminate toplevel constructors, memoize, and recurse**.
Thanks to *convexity* and the product decomposition rules

$$\bigwedge_{t_1\times t_2\in P} t_1\times t_2 \quad\leq\quad \bigvee_{t'_1\times t'_2\in N} t'_1\times t'_2 \tag{3}$$

is equivalent to

$$\forall N'\subseteq N. \left(\bigwedge_{t_1\times t_2\in P} t_1 \leq \bigvee_{t'_1\times t'_2\in N'} t'_1\right) \text{ or } \left(\bigwedge_{t_1\times t_2\in P} t_2 \leq \bigvee_{t'_1\times t'_2\in N\setminus N'} t'_2\right)$$

(similarly for arrows)

PART 4: POLYMORPHIC LANGUAGE

Motivating example

A motivating example in Haskell

[no XML]

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map f l = case l of
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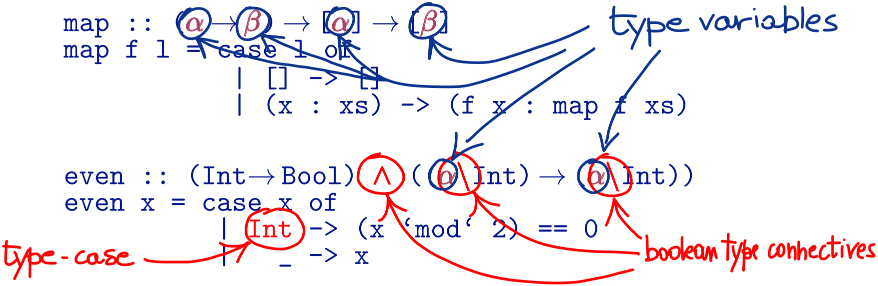

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- Expression: if the argument is an integer then return the Boolean expression otherwise return the argument
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Typical function used to modify some nodes of an XML tree leaving the others unchanged.

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The combination of type-case and intersections yields statically typed dynamic overloading.

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$(\,\mathtt{Int}\ \mathsf{list} \to \mathtt{Bool}\ \mathsf{list}\,) \wedge$
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Difficult because of expansion: needs *a set of type substitutions* — rather than just one— to unify the domain and the argument types.

Formal framework

Formal calculus

Exprs $e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i\to t_i}x.e \mid e{\in}t\,?\,e : e$

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Explicitly-typed functions:

- Needed by the type-case [e.g. $\mu f.\lambda x.f{\in}(\mathbb{1}{\to}\mathtt{Int})\,?\,\mathtt{true} : 42$]
- More expressive with the result type (parameter type not enough)

$\lambda^{\wedge_{i\in I} s_i \to t_i} x.e$: well typed if for all $i{\in}I$ from $x : s_i$ we can deduce $e : t_i$.

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Constructors: $[\![\mathtt{Int}]\!]{=}\{0, 1, -1, ...\}$. $[\![s \to t]\!] = \lambda$-abstractions that when applied to arguments in $[\![s]\!]$ return only results in $[\![t]\!]$.

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- it is *defined* as set-containment: $s \leq t \overset{def}{\iff} [\![s]\!] \subseteq [\![t]\!]$;

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Subtyping with type variables:

- it is *defined* as set-containment: $s \leq t \overset{def}{\iff} [\![s]\!] \subseteq [\![t]\!]$;
- it is such that forall type-substitutions σ: $s \leq t \;\Rightarrow\; s\sigma \leq t\sigma$;
- it is decidable. [ICFP2011].

Formal calculus: **new stuff**

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Polymorphic functions.

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Polymorphic functions: The novelty of this work is that type variables can occur in the *interfaces*.

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Meaning: types obtained by subsumption *and by instantiation*

- $\lambda^{\alpha\to\alpha}x.x : \mathbb{0} \to \mathbb{1}$ subsumption
- $\lambda^{\alpha\to\alpha}x.x : \neg\mathtt{Int}$ subsumption
- $\lambda^{\alpha\to\alpha}x.x : \mathtt{Int} \to \mathtt{Int}$ instantiation New
- $\lambda^{\alpha\to\alpha}x.x : \mathtt{Bool} \to \mathtt{Bool}$ instantiation New

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Define an explicitly typed, polymorphic calculus
with intersection types and dynamic type-case

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Four simple points to show why dealing with this blend is quite problematic

1. Polymorphism needs instantiation:

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To apply $\lambda^{\alpha\to\alpha}x.x$ to 42 we must use the instance obtained by the type substitution $\{\texttt{Int}/_{\alpha}\}$:

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we *relabel* the function by instantiating its interface.

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2. Type-case needs explicit relabeling:

$(\lambda^{\alpha\to\alpha\to\alpha}x.\lambda^{\alpha\to\alpha}y.x)42 \quad \in \mathtt{Int}\to\mathtt{Int}$
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3. Relabeling must be applied also on function bodies:

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To apply it to 42, relabeling the outer λ by $\{\mathtt{Int}/_{\alpha}\}$ does not suffice:

$$(\lambda^{\alpha\to\alpha}y.42)42$$

is not well typed. The body must be relabeled as well, by applying the $\{\mathtt{Int}/_{\alpha}\}$ yielding: $(\lambda^{\mathtt{Int}\to\mathtt{Int}}y.42)42$

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Let us see why
it is not well typed

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This “dependent relabeling” is the stumbling block for the definition of an explicitly-typed λ-calculus with intersection types.

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Details follow, but remember we want to program in this language

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Before proceeding we can already check our first yardstick:

$$\begin{aligned}
\texttt{even} &= \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})} x \,.\, x{\in}\texttt{Int} \,?\, (x \bmod 2) = 0 : x \\
\texttt{map} &= \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\ f \,. \\
&\qquad\qquad \lambda^{[\alpha]\to[\beta]} \ell \,.\, \ell{\in}\texttt{nil} \,?\, \texttt{nil} : (f(\pi_1 \ell), mf(\pi_2 \ell))
\end{aligned}$$

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Some examples:

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$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i\to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

Relabeling operation $e@[\sigma_j]_{j\in J}$: [Pushes σ's down into λ's]

$$\begin{array}{rcl}
x@[\sigma_j]_{j\in J} & \overset{\text{def}}{=} & x \\
(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_k]_{k\in K}} x.e)@[\sigma_j]_{j\in J} & \overset{\text{def}}{=} & \lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_k]_{k\in K}\circ[\sigma_j]_{j\in J}} x.e \\
(e_1 e_2)@[\sigma_j]_{j\in J} & \overset{\text{def}}{=} & (e_1@[\sigma_j]_{j\in J})(e_2@[\sigma_j]_{j\in J}) \\
(e{\in}t\,?\,e_1 : e_2)@[\sigma_j]_{j\in J} & \overset{\text{def}}{=} & e@[\sigma_j]_{j\in J}{\in}t\,?\,e_1@[\sigma_j]_{j\in J} : e_2@[\sigma_j]_{j\in J} \\
(e[\sigma_k]_{k\in K})@[\sigma_j]_{j\in J} & \overset{\text{def}}{=} & e@([\sigma_k]_{k\in K}\circ[\sigma_j]_{j\in J})
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Reduction semantics

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(e_1 e_2)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & (e_1@[\sigma_j]_{j\in J})(e_2@[\sigma_j]_{j\in J}) \\
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Reduction semantics

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\end{array}$$

Notions of reduction:

$$\begin{array}{rcl}
e[\sigma_j]_{j\in J} & \rightsquigarrow & e@[\sigma_j]_{j\in J} \\
(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_j]_{j\in J}} x.e)v & \rightsquigarrow & (e@[\sigma_j]_{j\in P})\{v/_x\} \qquad P=\{j{\in}J \mid \exists i{\in}I, \vdash v : t_i\sigma_j\} \\
v{\in}t\,?\,e_1 : e_2 & \rightsquigarrow & \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}
\end{array}$$

Reduction semantics

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i\to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

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(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_j]_{j\in J}} x.e)v & \rightsquigarrow & (e@[\sigma_j]_{j\in P})\{{}^{v}\!/\!{}_{x}\} \qquad P = \{j{\in}J \mid \exists i{\in}I, \vdash v : t_i\sigma_j\} \\
v{\in}t\,?\,e_1 : e_2 & \rightsquigarrow & \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}
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Relabeling operation $e@[\sigma_j]_{j\in J}$**:** [Pushes σ's down into λ's]

$$x@[\sigma_j]_{j\in J} \overset{\text{def}}{=} x$$

$(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_k]_{k\in K}} x.e)@[\sigma_j]$ $\overset{\text{def}}{=}$ $\lambda^{\wedge_{i\in I} t_i\to s_i}$ $x.e$

$(e_1 e_2)@[$

$(e{\in}t\,?\,e_1 : e_2)@[$ $: e_2@[\sigma_j]_{j\in J}$

$(e[\sigma_k]_{k\in K})@[$

Only keep the substitutions that make the type of the argument v match at least one input type of the interface

Notions of reduction:

$$e[\sigma_j]_{j\in J} \rightsquigarrow e@[\sigma_j]_{j\in J}$$

$$(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_j]_{j\in J}} x.e)v \rightsquigarrow (e@[\sigma_j]_{j\in P})\{{}^{v}\!/\!_{x}\} \qquad P = \{j{\in}J \mid \exists i{\in}I, \vdash v : t_i\sigma_j\}$$

$$v{\in}t\,?\,e_1 : e_2 \rightsquigarrow \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}$$

Example

$$(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)$$

Example

$$\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)z$$

Example

$$\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{{}^{\texttt{Int}}\!/\!_{\alpha}\}, \{{}^{\texttt{Bool}}\!/\!_{\alpha}\}]z)42$$

Example

$$(\lambda^{(\mathtt{Int}\to\mathtt{Int})\wedge(\mathtt{Bool}\to\mathtt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\mathtt{Int}/\alpha\},\{\mathtt{Bool}/\alpha\}]z)42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\mathtt{Int}/\alpha\},\{\mathtt{Bool}/\alpha\}]42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\texttt{Int}/_{\alpha}\},\{\texttt{Bool}/_{\alpha}\}]z)42$$

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$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}_{[\{\texttt{Int}/_{\alpha}\},\{\texttt{Bool}/_{\alpha}\}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

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$$\rightsquigarrow\quad(\lambda^{\alpha\to\alpha}_{[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

$$\rightsquigarrow\quad(\lambda^{\texttt{Int}\to\texttt{Int}}y.42)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

$$\rightsquigarrow\quad (\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]42$$

$$\rightsquigarrow\quad (\lambda^{\alpha\to\alpha}_{[\{{}^{\texttt{Int}}/_{\alpha}\},\underline{\underline{\{{}^{\texttt{Bool}}/_{\alpha}\}}}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

$$\rightsquigarrow\quad (\lambda^{\boxed{\texttt{Int}\to\texttt{Int}}}y.42)42$$

no **Bool** here

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

$$\leadsto\quad(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]42$$

$$\leadsto\quad(\lambda^{\alpha\to\alpha}_{[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

$$\leadsto\quad(\lambda^{\texttt{Int}\to\texttt{Int}}y.42)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}_{[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

REJECTED

$$\rightsquigarrow \quad (\lambda^{\texttt{Int}\to\texttt{Int}}y.42)42 \qquad\qquad \equiv (((\lambda^{\alpha\to\alpha}y.x)x)@[\{{}^{\texttt{Int}}/_{\alpha}\}])\{{}^{42}/_{x}\}$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

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$$\rightsquigarrow \quad 42$$

Type system

$$(subsumption)\ \frac{\Gamma \vdash e : t_1 \quad t_1 \leq t_2}{\Gamma \vdash e : t_2} \qquad (inst)\ \frac{\Gamma \vdash e : t \qquad \sigma_j \sharp \Gamma}{\Gamma \vdash e[\sigma_j]_{j\in J} : \bigwedge_{j\in J} t\sigma_j}$$

$$(appl)\ \frac{\Gamma \vdash e_1 : t_1 \to t_2 \qquad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$$

$$(abstr)\ \frac{\Gamma, x : t_i\sigma_j \vdash e@[\sigma_j] : s_i\sigma_j}{\Gamma \vdash \lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_j]_{j\in J}} x.e : \bigwedge_{i\in I, j\in J} t_i\sigma_j \to s_i\sigma_j}\ \begin{array}{l} i \in I \\ j \in J \end{array}$$

[plus the rules for type-case and variables]

Type system

$$(subsumption)\ \frac{\Gamma \vdash e : t_1 \quad t_1 \leq t_2}{\Gamma \vdash e : t_2} \qquad (inst)\ \frac{\Gamma \vdash e : t \qquad \sigma_j \sharp\ \Gamma}{\Gamma \vdash e[\sigma_j]_{j\in J} : \bigwedge_{j\in J} t\sigma_j}$$

$$(appl)\ \frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \qquad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$$

$$(abstr)\ \frac{\Gamma, x : t_i\sigma_j \vdash e@[\sigma_j] : s_i\sigma_j}{\Gamma \vdash \lambda^{\wedge_{i\in I} t_i \rightarrow s_i}_{[\sigma_j]_{j\in J}} x.e : \bigwedge_{i\in I, j\in J} t_i\sigma_j \rightarrow s_i\sigma_j}\ \begin{array}{l} i \in I \\ j \in J \end{array}$$

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$$(\textit{abstr})\quad \frac{\Gamma, x : t_i \quad \vdash e \qquad : s_i}{\Gamma \vdash \lambda^{\wedge_{i\in I} t_i \to s_i} x.e : \bigwedge_{i\in I} \quad t_i \quad \to s_i}\; i \in I$$

[plus the rules for type-case and variables]

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[plus the rules for type-case and variables]

Properties

Theorem (Subject Reduction)

For every term e *and type* t*, if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$*, then* $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e *be a well-typed closed term. If* e *is not a value, then there exists a term* e' *such that* $e \rightsquigarrow e'$.

Properties

Theorem (Subject Reduction)

For every term e *and type* t, *if* $\Gamma \vdash e : t$ *and* $e \leadsto e'$, *then* $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e *be a well-typed closed term. If* e *is not a value, then there exists a term* e' *such that* $e \leadsto e'$.

Theorem

Let $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $\lceil e \rceil$ *the type erasure of* e. *If* $\vdash_{BCD} a : t$, *then* $\exists e$ *s.t.* $\vdash e : t$ *and* $\lceil e \rceil = a$.

Properties

Theorem (Subject Reduction)

For every term e and type t, if $\Gamma \vdash e : t$ and $e \rightsquigarrow e'$, then $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e be a well-typed closed term. If e is not a value, then there exists a term e' such that $e \rightsquigarrow e'$.

Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\lceil e \rceil$ the type erasure of e. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\lceil e \rceil = a$.

Note that

$$e \quad ::= \quad x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

Properties

Theorem (Subject Reduction)

For every term e *and type* t*, if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$*, then* $\Gamma \vdash e' : t$*.*

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satisfies the above theorem and is closed by reduction.

Properties

Theorem (Subject Reduction)

For every term e *and type* t, *if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$, *then* $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e *be a well-typed closed term. If* e *is not a value, then there exists a term* e' *such that* $e \rightsquigarrow e'$.

Theorem

Let $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $\lceil e \rceil$ *the type erasure of* e. *If* $\vdash_{BCD} a : t$, *then* $\exists e$ *s.t.* $\vdash e : t$ *and* $\lceil e \rceil = a$.

Note that

$$e \quad ::= \quad x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid \cancel{e{\in}t\,?\,e : e} \mid \cancel{e[\sigma_i]_{i\in I}}$$

satisfies the above theorem and is closed by reduction, too.

Properties

Theorem (Subject Reduction)

For every term e *and type* t*, if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$*, then* $\Gamma \vdash e' : t$*.*

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Let e *be a well-typed closed term. If* e *is not a value, then there exists a term* e' *such that* $e \rightsquigarrow e'$*.*

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Let $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $\lceil e \rceil$ *the type erasure of* e*. If* $\vdash_{BCD} a : t$*, then* $\exists e$ *s.t.* $\vdash e : t$ *and* $\lceil e \rceil = a$*.*

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The first n terms ($n = 3, 4, 5$) form an explicitly-typed λ-calculus with intersection types subsuming BCD.

Properties

The definitions we gave:

$$\begin{array}{rcl}
\texttt{even} & = & \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})}x\,.\,x{\in}\texttt{Int}\,?\,(x \bmod 2) = 0 : x \\
\texttt{map} & = & \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\ f\,. \\
 & & \qquad\lambda^{[\alpha]\to[\beta]}\ell\,.\,\ell{\in}\texttt{nil}\,?\,\texttt{nil} : (f(\pi_1\ell), mf(\pi_2\ell))
\end{array}$$

are well typed.

Properties

The definitions we gave:

$$\begin{array}{rcl} \texttt{even} & = & \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})} x\,.\, x{\in}\texttt{Int}\,?\,(x \bmod 2) = 0 : x \\ \texttt{map} & = & \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\, f\,. \\ & & \qquad \lambda^{[\alpha]\to[\beta]}\ell\,.\,\ell{\in}\texttt{nil}\,?\,\texttt{nil} : (f(\pi_1\ell), mf(\pi_2\ell)) \end{array}$$

are well typed.

A yardstick for the language

- ✓ Can define both `map` and `even`
- ✓ Can *check* the types specified in the signature
- ? Can *deduce* the type of the partial application `map even`

Inference of explicit type-substitutions

Two problems:

1. **Local type-substitution inference:** Given a term of

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t\ ?\ e : e$$

a sound & complete algorithm that, whenever possible, inserts sets of type-substitutions that make it a well-typed term of

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2. **Type recostruction:** Given a term

   $$\lambda x.e$$

   find, if possible, a set of type-substitutions $[\sigma_j]_{j\in J}$ such that

   $$\lambda^{\alpha\to\beta}_{[\sigma_j]_{j\in J}} x.e$$

   is well typed

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The reason is purely practical:

- $\lambda^{\alpha\to\alpha}x.3$ must return a static type error
- If we infer decorations, then it can be typed: $\lambda^{\alpha\to\alpha}_{\{\mathtt{Int}/\alpha\}}x.3$

The rule for applications

1. In the type system: [with explicit type-subst.]

$$\textsc{(Appl)}\ \frac{\Gamma \vdash e_1 : s \to u \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1 e_2 : u}$$

[The type of the function is subsumed to an arrow and the type of the argument is subsumed to the domain of this arrow].

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$$\frac{\Gamma \vdash_{\mathcal{A}} e_1 : t \qquad \Gamma \vdash_{\mathcal{A}} e_2 : s}{\Gamma \vdash_{\mathcal{A}} e_1 e_2 : \min\{u \mid t \leq s \to u\}}\ \begin{array}{l} t \leq \mathbb{0} \to \mathbb{1} \\ s \leq \mathrm{dom}(t) \end{array}\ (\textsc{Appl-algorithm})$$

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3. **Inference of type substitutions** [w/o explicit type-subst.]

$$\textsc{(Appl-inference)}\quad \frac{\exists [\sigma_i]_{i\in I}, [\sigma'_j]_{j\in J} \quad \Gamma \vdash_{\mathcal{I}} e_1 : t \qquad \Gamma \vdash_{\mathcal{I}} e_2 : s}{\Gamma \vdash_{\mathcal{I}} e_1 e_2 : \min\{u \mid t[\sigma'_j]_{j\in J} \leq s[\sigma_i]_{i\in I} \to u\}} \quad \begin{array}{l} t[\sigma'_j]_{j\in J} \leq \mathbb{0} \to \mathbb{1} \\ s[\sigma_i]_{i\in I} \leq \mathrm{dom}(t[\sigma'_j]_{j\in J}) \end{array}$$

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Tallying problem

The problem of inferring the type of an application is thus to find for s and t given, $[\sigma_i]_{i\in I}, [\sigma'_j]_{j\in J}$ such that:

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This can be reduced to solving a suite of tallying problems

Definition (Type tallying)

Let $C = \{(s_1, t_1), ..., (s_n, t_n)\}$ a *constraint set*. A type-substitution σ is a solution for the *tallying* of C iff $s\sigma \leq t\sigma$ for all $(s, t) \in C$.

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A sound and complete set of solutions for every tallying problem can be effectively found in three simple steps.

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Use the set-theoretic decomposition rules to transform C into a set of constraint sets whose constraints are of the form (α, t) or (t, α).

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Step 2: Merge constraints on the same variable.

- if (α, t_1) and (α, t_2) are in C, then replace them by $(\alpha, t_1 \wedge t_2)$;
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Step 3: Transform into a set of equations.

After Step 2 we have constraint-sets of the form $\{s_i \leq \alpha_i \leq t_i \mid i \in [1..n]\}$ where α_i are pairwise distinct.

1. select $s \leq \alpha \leq t$ and replace it by $\alpha = (s \vee \beta) \wedge t$ with β fresh.
2. in all other constraints in replace every α by $(s \vee \beta) \wedge t$
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At the end we have a sets of equations $\{\alpha_i = u_i \mid i \in [1..n]\}$ that (with some care) are *contractive*. By Courcelle there exists a solution, *ie*, a substitution for $\alpha_1, \ldots, \alpha_n$ into (possibly recursive regular) types $t_1, \ldots, t_n$ (in which the fresh β's are free variables).

The application problem

Definition (Inference application problem)

Given s and t types, find $[\sigma_i]_{i\in I}$ and $[\sigma'_j]_{j\in J}$ such that:

$$\bigwedge_{i\in I} t\sigma_i \leq \mathbb{0}\to\mathbb{1} \quad \text{and} \quad \bigwedge_{j\in J} s\sigma_j \leq \mathrm{dom}(\bigwedge_{i\in I} t\sigma_i)$$

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   $\bigwedge_{i\in I}(t\rho_i)\sigma'_i \leq \mathbb{0}\to\mathbb{1}$ and $\bigwedge_{j\in J}(s\rho_j)\sigma'_j \leq \mathrm{dom}(\bigwedge_{i\in I}(t\rho_i)\sigma'_i)$;

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➡ **Every solution for γ is a solution for the application.**

Example: map even

Start with the following tallying problem:

$$\{(\alpha_1\to\beta_1)\to[\alpha_1]\to[\beta_1] \leq t\to\gamma\}$$

where $t = (\mathtt{Int}\to\mathtt{Bool})\wedge(\alpha\setminus\mathtt{Int}\to\alpha\setminus\mathtt{Int})$ is the type of `even`

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- At step 2 the algorithm generates 9 constraint-sets: one is unsatisfiable ($t \leq \mathbb{0}$); four are implied by the others; remain
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- Four solutions for γ:
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- The last two are minimal and we take their intersection:
  $\{\gamma = ([\alpha \setminus \mathtt{Int}] \to [\alpha \setminus \mathtt{Int}]) \wedge ([\alpha \vee \mathtt{Int}] \to [(\alpha \setminus \mathtt{Int}) \vee \mathtt{Bool}])\}$

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Decidability: The algorithm is a semi-decision procedure. We conjecture decidability (N.B.: the problem is unrelated to type-reconstruction for intersection types since we have recursive types).

On completeness and decidability

The algorithm produces a set of solutions that is sound (it finds only correct solutions) and complete (any other solution can be derived from them).

Decidability: The algorithm is a semi-decision procedure. We conjecture decidability (N.B.: the problem is unrelated to type-reconstruction for intersection types since we have recursive types).

Completeness: For every solution of the inference problem, our algorithm finds an equivalent or more general solution. However, this solution is not necessary the first solution found.

In a dully execution of the algorithm on `map even` the good solution is the second one.

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Principality: This raises the problem of the existence of principal types: may an infinite sequence of increasingly general solutions exist?

Type reconstruction

- Solve *sets* of contraint-sets by the tallying algorithm:

$$\frac{}{\Gamma \vdash_{\mathcal{R}} x : \Gamma(x) \rightsquigarrow \{\varnothing\}} \qquad \frac{\Gamma, x : \alpha \vdash_{\mathcal{R}} e : t \rightsquigarrow \mathcal{S}}{\Gamma \vdash_{\mathcal{R}} \lambda x.e : \alpha \to \beta \rightsquigarrow \mathcal{S} \sqcap \{\{(t \leq \beta)\}\}}$$

$$\frac{\Gamma \vdash_{\mathcal{R}} e_1 : t_1 \rightsquigarrow \mathcal{S}_1 \qquad \Gamma \vdash_{\mathcal{R}} e_2 : t_2 \rightsquigarrow \mathcal{S}_2}{\Gamma \vdash_{\mathcal{R}} e_1 e_2 : \alpha \rightsquigarrow \mathcal{S}_1 \sqcap \mathcal{S}_2 \sqcap \{\{(t_1 \leq t_2 \to \alpha)\}\}} \quad + \quad \text{rule for typecase}$$

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- Sound. it's a variant: fix interfaces and infer decorations

$$\lambda^{\alpha\to\beta}_{[?]} x.e$$

  Not complete: reconstruction is undecidable

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- Sound. it's a variant: fix interfaces and infer decorations

$$\lambda^{\alpha\to\beta}_{[?]} x.e$$

  Not complete: reconstruction is undecidable

- It types more than ML

$$\lambda x.xx : \mu X.(\alpha \wedge (X \to \beta)) \to \beta \qquad (\leq \alpha \wedge (\alpha \to \beta)) \to \beta)$$

  for functions typable in ML it deduces a type at least as good:

$$map : ((\alpha \to \beta) \to [\alpha] \to [\beta]) \wedge ((\mathbb{0} \to \mathbb{1}) \to [] \to [])$$

Type Reconstruction Algorithm

$$\frac{}{\Gamma \vdash_{\mathcal{R}} c : b_c \leadsto \{\varnothing\}}(\textsc{R-const}) \qquad \frac{}{\Gamma \vdash_{\mathcal{R}} x : \Gamma(x) \leadsto \{\varnothing\}}(\textsc{R-var})$$

$$\frac{\Gamma \vdash_{\mathcal{R}} m_1 : t_1 \leadsto \mathcal{S}_1 \qquad \Gamma \vdash_{\mathcal{R}} m_2 : t_2 \leadsto \mathcal{S}_2}{\Gamma \vdash_{\mathcal{R}} m_1 m_2 : \alpha \leadsto \mathcal{S}_1 \sqcap \mathcal{S}_2 \sqcap \{\{(t_1 \leq t_2 \to \alpha)\}\}}(\textsc{R-appl})$$

$$\frac{\Gamma, x : \alpha \vdash_{\mathcal{R}} m : t \leadsto \mathcal{S}}{\Gamma \vdash_{\mathcal{R}} \lambda x.m : \alpha \to \beta \leadsto \mathcal{S} \sqcap \{\{(t \leq \beta)\}\}}(\textsc{R-abstr})$$

$(\textsc{R-case})$

$$\frac{\begin{array}{rl} \mathcal{S} = & (\mathcal{S}_0 \sqcap \{\{(t_0 \leq \mathbb{0})\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_1 \sqcap \{\{(t_0 \leq t), (t_1 \leq \alpha)\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_2 \sqcap \{\{(t_0 \leq \neg t), (t_2 \leq \alpha)\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_1 \sqcap \mathcal{S}_2 \sqcap \{\{(t_1 \vee t_2 \leq \alpha)\}\}) \end{array} \\ \Gamma \vdash_{\mathcal{R}} m_0 : t_0 \leadsto \mathcal{S}_0 \qquad \Gamma \vdash_{\mathcal{R}} m_1 : t_1 \leadsto \mathcal{S}_1 \qquad \Gamma \vdash_{\mathcal{R}} m_2 : t_2 \leadsto \mathcal{S}_2}{\Gamma \vdash_{\mathcal{R}} (m_0 {\in} t \,?\, m_1 : m_2) : \alpha \leadsto \mathcal{S}}$$

where α, α_i and β in each rule are fresh type variables.

Efficient evaluation

Monomorphic language

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e \in t\, ?\, e : e \\
v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\end{array}
$$

Monomorphic language

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}x.e \mid ee \mid e \in t\,?\,e : e \\
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\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathrm{m}} \lambda^{t}x.e \Downarrow \langle \lambda^{t}x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply})\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow \langle \lambda^{t}x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathrm{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathrm{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 e_2 \Downarrow v}
$$

Monomorphic language

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}x.e \mid ee \mid e \in t \mathrel{?} e : e \\
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\end{array}
$$

save the environment

$$
(\text{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathrm{m}} \lambda^{t}x.e \Downarrow \langle \lambda^{t}x.e, \mathcal{E} \rangle}
$$

$$
(\text{Apply})\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow \langle \lambda^{t}x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathrm{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathrm{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 e_2 \Downarrow v}
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save the environment

$$(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathsf{m}} \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}$$

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restore the environment

Monomorphic language

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e \in t\,?\,e : e \\ v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle \end{array}$$

$$\text{(CLOSURE)}\ \frac{}{\mathcal{E} \vdash_{\mathrm{m}} \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}$$

$$\text{(APPLY)}\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathrm{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathrm{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 e_2 \Downarrow v}$$

$$\text{(TYPECASE TRUE)}\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow v_0 \qquad v_0 \in_{\mathrm{m}} t \qquad \mathcal{E} \vdash_{\mathrm{m}} e_2 \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 \in t\,?\,e_2 : e_3 \Downarrow v}$$

$$\text{(TYPECASE FALSE)}\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow v_0 \qquad v_0 \notin_{\mathrm{m}} t \qquad \mathcal{E} \vdash_{\mathrm{m}} e_3 \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 \in t\,?\,e_2 : e_3 \Downarrow v}$$

$$\begin{array}{rcl} c \in_{\mathrm{m}} t & \overset{\mathrm{def}}{=} & \{c\} \leq t \\ \langle \lambda^s x.e, \mathcal{E} \rangle \in_{\mathrm{m}} t & \overset{\mathrm{def}}{=} & s \leq t \end{array}$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e \quad ::= \quad c \mid x \mid \lambda^{t}_{\sigma_I} x.e \mid ee \mid e \in t\, ?\, e : e \mid e\sigma_I$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\ v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle \end{array}$$

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$$(\textsc{Closure})\ \frac{}{\sigma_I ; \mathcal{E} \vdash_{\mathsf{p}} \lambda^{t}_{\sigma_J} x.e \Downarrow \langle \lambda^{t}_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle}$$

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v & ::= & c \mid \langle \lambda^{t}_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle
\end{array}
$$

save the environment

$$
(\textsc{Closure})\ \frac{}{\sigma_I; \boxed{\mathcal{E}} \vdash_{\mathsf{p}} \lambda^{t}_{\sigma_J} x.e \Downarrow \langle \lambda^{t}_{\sigma_J} x.e, \boxed{\mathcal{E}}, \sigma_I \rangle}
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$$

save current type-substitutions

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{rcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I \\ v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^{t}_{\sigma_J}x.e\Downarrow\langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}\qquad(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e\sigma_J\Downarrow v}$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

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$$(\textsc{Apply})\ \frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle\quad\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0\quad\sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L : v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lll} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I \\ v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle \end{array}$$

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restore the environment

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\boxed{\mathcal{E}'},\sigma_H\rangle\qquad\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0\qquad\sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L: v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

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$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^{t}_{\sigma_J}x.e\Downarrow\langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e\sigma_J\Downarrow v}$$

restore the environment

$$(\textsc{Apply})\ \frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L: v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

restore the type substitutions

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\ v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^{t}_{\sigma_J}x.e\Downarrow\langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}\qquad(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e\sigma_J\Downarrow v}$$

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle\quad\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0\quad\sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L: v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I \\ v & ::= & c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} \lambda^{t}_{\sigma_J}x.e \Downarrow \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}$$

$$(\textsc{Apply})$$

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow \langle \lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e, \mathcal{E}', \sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}', x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J \mid \exists \ell\in L : v_0\in_{\mathrm{p}} s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\ v & ::= & c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} \lambda^{t}_{\sigma_J}x.e \Downarrow \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}$$

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid \exists\ell\in L: v_0\in_{\mathrm{p}} s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

1. **compose** of two sets of type-substitution

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$
\begin{array}{rcl}
e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\
v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle
\end{array}
$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^{t}_{\sigma_J}x.e\Downarrow\langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e\sigma_J\Downarrow v}$$

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle\qquad\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0\qquad\sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L: v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

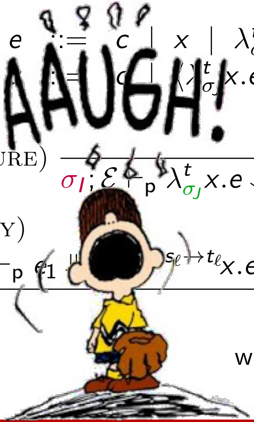
Problem:

At every application compute σ_P:

1. **compose** of two sets of type-substitution
2. **select** the substitutions compatible with the argument v_0

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)



$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$

[illegible] $::= c \mid \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle$

(CLOSURE) $$\frac{}{\sigma_I;\mathcal{E}\vdash_p \lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}$$

(INSTANCE) $$\frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_p e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_p e\sigma_J\Downarrow v}$$

(APPLY)

$\sigma_I;\mathcal{E}\vdash_p e_1$ [illegible] $s_\ell\to t_\ell\, x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_p e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_p e\Downarrow v$

$$\overline{\qquad\sigma_I;\mathcal{E}\vdash_p e_1e_2\Downarrow v\qquad}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J \mid \exists\ell\in L: v_0\in_p s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

1. **compose** of two sets of type-substitution
2. **select** the substitutions compatible with the argument v_0

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\ v & ::= & c \mid \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^{t}_{\sigma_J}x.e\Downarrow\langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}\qquad(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e\sigma_J\Downarrow v}$$

$$(\textsc{Apply})\ \frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle\quad\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_2\Downarrow v_0\quad\boxed{\sigma_P};\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}}e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid\exists\ell\in L: v_0\in_{\mathrm{p}}s_\ell\sigma_j\}$

Solution:

Compute compositions and selections lazily.

Intermediate language as compilation target

$$\begin{array}{lcl} e & ::= & c \mid x \mid \lambda^{t} x.e \mid ee \mid e \in t\,?\,e : e \\ v & ::= & c \mid \langle \lambda^{t} x.e, \mathcal{E} \rangle \end{array}$$

$$(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash \lambda^{t} x.e \Downarrow \langle \lambda^{t} x.e, \mathcal{E} \rangle}$$

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$$(\textsc{Typecase True})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \in t \qquad \mathcal{E} \vdash e_2 \Downarrow v}{\mathcal{E} \vdash e_1 \in t\,?\,e_2 : e_3 \Downarrow v} \qquad (\textsc{Typecase False})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \notin t \qquad \mathcal{E} \vdash e_3 \Downarrow v}{\mathcal{E} \vdash e_1 \in t\,?\,e_2 : e_3 \Downarrow v}$$

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symbolic substitutions

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watch here !

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The only difference!

$$c \in t \overset{\text{def}}{=} \{c\} \leq t$$

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Compilation

1. Compile into the intermediate language

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\begin{array}{rcl}
[\![x]\!]_\Sigma & = & x \\
[\![\lambda^t_{\sigma_I} x.e]\!]_\Sigma & = & \lambda^t_{\mathtt{comp}(\Sigma,\sigma_I)} x.[\![e]\!]_{\mathtt{sel}(x,t,\mathtt{comp}(\Sigma,\sigma_I))} \\
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2. For $\langle \lambda^{s}_{\Sigma} x.e, \mathcal{E} \rangle \in t \stackrel{\text{def}}{=} s(\text{eval}(\mathcal{E}, \Sigma)) \leq t$ we have $s(\text{eval}(\mathcal{E}, \Sigma)) \neq s$ *only if* $\lambda^{s}_{\Sigma} x.e$ results from the partial application of a polymorphic function (*ie*, in s there occur free variables bound in the context).

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   Execution is slowed *only* when testing the type of the result of a partial application of a polymorphic function.

3. This holds also with products (used to encode lists records and XML), whose testing accounts for most of the execution time.

Conclusion

Theory: All the theoretical machinery necessary to design and implement a programming language is there. The practical relevance of the open theoretical issues is negligible.

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Languages: The polymorphic extension of $\mathbb{C}$Duce is being implemented. Future applications: polymorphic extensions of XQuery and embedding some of this type machinery in ML.

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Runtime: Relabeling cannot be avoided but it is materialized only in case of partial polymorphic applications that end up in type-cases, that is, just when it is needed.

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Type reconstruction: Full usage needs more research, expecially about the production of human readable types and helpful error messages, but it is mature enough to use it to type local functions.