Zero entropy systems

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Introduction

Subject: symbolic systems of zero entropy, focusing on systems of linear complexity. How can we describe them?
The iteration of a (primitive)morphism is well-known way to generate a system of linear complexity. We shall discuss a generalization called S-adic representation.
We will study in more detail a class of systems of linear complexity the so-called tree sets and prove a property of their S-adic representation.
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Outline

- Symbolic systems
- Factor complexity
- S-adic representations
- Tree sets
- S-adic representation of tree sets

Symbolic systems

Consider the set $A^{\mathbb{Z}}$ of biinfinite sequences $x = (x_n)_{n \in \mathbb{Z}}$ with the shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $y = \sigma(x)$ if $y_n = x_{n+1}$.
A symbolic system (or two-sided subshift) is a set $X \subset A^{\mathbb{Z}}$ of biinfinite sequences which is

1. closed for the product topology,
2. invariant by the shift, that is $\sigma(X) \subset X$.

A set of words on the alphabet A is factorial if it contains A and the factors (or substrings) of its elements. A factorial set F is biextendable if for any $w \in F$ there are letters $a, b \in A$ such that $awb \in F$.
The set of words appearing in the sequences of a symbolic system X is a biextendable set and any biextendable set is obtained in this way.
Variant: one sided subshift $X \subset A^{\mathbb{N}}$.

Minimal systems

The symbolic system X is minimal if it does not contain properly another nonempty one.

An infinite factorial set F is said to be uniformly recurrent if for any word $w \in F$ there is an integer $n \geq 1$ such that w is a factor of any word of F of length n.

Remark that a uniformly recurrent set F is recurrent: for every $u, v \in F$, there is some x such that $uxv \in F$.

A system is minimal if and only if the set of its factors is uniformly recurrent.

Factor complexity

The factor complexity of a factorial set F on the alphabet A is the sequence $p_n(F) = \mathrm{Card}(F \cap A^n)$. We have $p_0(F) = 1$ and we assume $p_1(F) = \mathrm{Card}(A)$ for any factorial set.
The sets of bounded complexity are the factors of eventually periodic sequences. The binary Sturmian sets are, by definition, those of complexity $n + 1$ (like the Fibonacci set).

Computing the complexity

Let F be a factorial set on the alphabet A. The multiplicity of $w \in F$ with respect to F is

$$m_F(w) = e_F(w) - \ell_F(w) - r_F(w) + 1$$

where $e_F(w)$ (resp. $\ell_F(w)$, resp. $r_F(w)$) is the number of pairs $a, b \in A$ (resp. the number of $a \in A$) such that $awb \in F$ (resp. $aw \in F$, resp. $wa \in F$).

Example

For $F = A^*$, one has $m_F(w) = (\mathrm{Card}(A) - 1)^2$ for any $w \in F$.

A word w is right-special if $r_F(w) > 1$, left-special if $\ell_F(w) > 1$ and bispecial if it is both right and left special.

Let $s_n = p_{n+1} - p_n$ and $b_n = s_{n+1} - s_n$ be the first and second differences of the sequence $p_n(F)$. The following result shows that the knowledge of special words is the key for computing the complexity.

Theorem (Cassaigne, 1997)

Let F be a factorial set on the alphabet A. One has

$$\begin{aligned} s_n &= \sum_{w \in F \cap A^n} (r(w) - 1), \\ b_n &= \sum_{w \in F \cap A^n} m(w). \end{aligned}$$

Topological entropy

The entropy of a factorial set F is

$$h(F) = \lim \frac{1}{n} \log p_n(F)$$

The limit exists because $\log(p_n(F))$ is subadditive. For example, the entropy of the full shift $A^{\mathbb{Z}}$ on k letters is $\log(k)$.
The following result shows that the entropy of a minimal system can be almost arbitrary.

Theorem (Grillenberger,1972)

Let A be an alphabet with $k \geq 2$ letters. For any $h \in [0, \log k[$ there is a minimal one sided subshift with entropy h.

S-adic representations

Let S be a set of morphisms and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in S with $\sigma_n : A_{n+1}^* \to A_n^*$ and (a_n) be a sequence of letters with $a_n \in A_n$ such that

$$x = \lim \sigma_0 \cdots \sigma_{n-1}(a_n)$$

exists and is an infinite word. The sequence is an S-adic representation of the set of factors of x.
The sequence $\sigma_0\sigma_1 \cdots \in S^\omega$ is the directive sequence of the representation.

Morphic words

A word $x \in A^{\mathbb{N}}$ is morphic if there exist morphisms $\tau : B^* \to B^*$ and $\sigma : B^* \to A^*$ and a letter $b \in B$ such that $x = \sigma\tau^\omega(b)$. It is purely morphic if σ is the identity.

The set of factors of x has an S-adic representation with $S = \{\sigma, \tau\}$ and directive word $\sigma\tau^\omega$.

A morphism $\varphi : A^* \to A^*$ is primitive if there is an integer $n \geq 1$ such that for every pair $a, b \in A$, the letter a appears in $\varphi^n(b)$.

Proposition

The set of factors of a fixed point of a primitive morphism is minimal with at most linear complexity.

Sturmian sets

A set F is Sturmian if it is recurrent, closed under reversal and for every $n \geq 1$ there is exactly one right-special word w of length n, which is such that $r_F(w) = \mathrm{Card}(A)$.
A word x is Sturmian if its set of factors is Sturmian. It is standard if all its left-special factors are prefixes of x.
Any Sturmian set is S-adic with a finite set S. This results from the fact that any standard Sturmian word is obtained by iterating a sequence of morphisms of the form ψ_a for $a \in A$ defined by $\psi_a(a) = a$ and $\psi_a(b) = ab$ for $b \neq a$ (Arnoux,Rauzy, 1991).

S-adic representations and linear complexity

An S-adic representation (σ_n) is everywhere growing if $\lim |\sigma_0 \cdots \sigma_n(a)| = \infty$ for every $a \in A_{n+1}$.

Theorem (Ferenczi, 1996)

Any minimal symbolic system on a finite alphabet A with at most linear factor complexity has an everywhere growing S-adic representation with S finite.

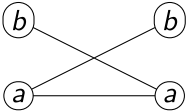
The S-adic conjecture: under which additional condition does a set with a finite S-adic representation have linear complexity?

Extension graphs

Let F be a factorial set. For a given word $w \in F$, set

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in F\}, \\ E(w) &= \{(a,b) \in A \times A \mid awb \in F\}, \\ R(w) &= \{b \in A \mid wb \in F\}. \end{aligned}$$

The extension graph of w in F is the graph on the set vertices which is the disjoint union of $L(w)$ and $R(w)$ and with edges the set $E(w)$.
For example, if $A = \{a, b\}$ and $F \cap A^2 = \{aa, ab, ba\}$, the extension graph of ε is



Tree sets

A factorial set F is a tree set if for any $w \in F$, the extension graph of w is a tree.

Any Sturmian sets is a tree set.

Proposition

The complexity of a tree set F on k letters is $p_n(F) = (k-1)n + 1$.

This results from the fact that $m_F(w) = 0$ for all $w \in F$ since $G(w)$ is a tree.

Elementary automorphisms

The set S_e of elementary positive automorphisms on A is formed by the permutations on A and for every $a, b \in A$ with $a \neq b$ by the morphisms

$$\alpha_{a,b}(c) = \begin{cases} ab & \text{if } c = a, \\ c & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\alpha}_{a,b}(c) = \begin{cases} ba & \text{if } c = a, \\ c & \text{otherwise} \end{cases}$$

Note that $\alpha_{a,b}$ (resp. $\tilde{\alpha}_{a,b}$) places a b after (resp. before) each a.
The monoid generated by elementary positive automorphisms is the monoid of tame positive automorphisms. It is stricly included in the monoid of positive autmorphisms.
The morphisms ψ_a giving the S-adic representation of Sturmian sets are tame.

S-adic representation of tree sets

An S-adic representation (σ_n) is primitive if for all $r \geq 0$ there is an $s > r$ such that every letter of A_r occurs in every $\sigma_r \cdots \sigma_{s-1}(a)$ for $a \in A_s$.

Theorem (BDDLPRR, Discrete Math., 2014)

Any uniformly recurrent tree set has a primitve S_e-adic representation.

The converse is false. For example, let $\varphi : a \mapsto ac, b \mapsto bac, c \mapsto cb$. Then $\varphi = \alpha_{a,c}\alpha_{c,b}\alpha_{b,a}$ although the set F of factors of its fixed point $\varphi^\omega(a)$ is not a tree set since $bb, bc, cb, cc \in F$.
A characterization of tree sets by their S_e-adic representation is known for 3 letters (Leroy, 2014).

Outline of the proof, step 1

A return word to u in a factorial set F is a word v such that $uv \in F$ ends with u and has no proper prefix with the same property (i.e. the first time we see u again).

Theorem (BDDLPRR, Monatsh. Math., 2014)

If F is a uniformly recurrent tree set, the set of return words to any $u \in F$ is a basis of the free group on A.

Outline of the proof, step 2

Let F be a uniformly recurrent tree set and let φ map bijectively B onto the set $\mathcal{R}_F(u)$ of return words to u. The derived set of F is $\varphi^{-1}(F)$. The following generalizes the well-known fact that the derived set of a Sturmian set is Sturmian.

Theorem (BDDLPRR, Discrete Math., 2014)

The derived set of a uniformly recurrent tree set is a uniformly recurrent tree set.

Outline of the proof, step 3

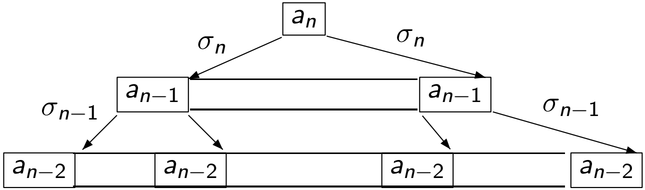
An automorphism of the free group is tame if if it belongs to the submonoid generated by the elementary positive automorphisms (in particular it is positive). A basis X of the free group is tame if there is a tame automorphism α such that $X = \alpha(A)$.

Theorem (BDDLPRR, Discrete Mat., 2014)

Any basis of the free group contained in a uniformly recurrent tree set is tame.

Outline of the proof

Let $a_0 \in A_0 = A$. Let σ_0 map bijectively A_1 onto $\mathcal{R}_F(a_0)$. Then σ_0 is a positive automorphism (by step 1) and tame (by step 3). Then the derived set $T_1 = \sigma_0^{-1}(T)$ is a uniformly recurrent tree set (by step 2) and we can iterate infinitely, choosing $a_1 \in A_1$ and σ_1 mapping bijectively A_2 onto $R_{T_1}(a_1)$, and so on.



The landscape

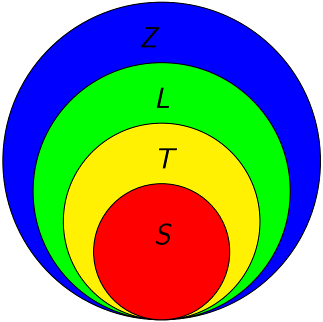


Figure : The classes of uniformly recurrent sets: Sturmian (S), Tree (T), of linear complexity (L) , of zero entropy (Z).