MPRI 2–2 TD 3 du 22/01/2019 — With solutions

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1) The goal of this exercise is to understand the structure of the category $\mathbf{PoLR}^!$ (the category of coalgebras of the comonad ! on the category **PoLR**). We refer to the lecture notes for all basic definitions and notations.

1.1) Given a preorder S, we set $h_S = \{(a, u^0) \in |S| \times |!S| \mid \forall a' \in u^0\ a' \leq_S a\}$. Prove that $h_S \in \mathbf{PoLR}(S, !S)$.

Solution ▷ Let $(a, u^0) \in h_S$ and $(b, v^0) \in |S \multimap !S|$ be such that $(b, v^0) \leq_{S \multimap !S} (a, u^0)$, that is, $a \leq_S b$ and $v^0 \leq_{!S} u^0$, we must prove that $(b, v^0) \in h_S$. So let $b' \in v^0$, let $a' \in u^0$ be such that $b' \leq_S a'$ (using $v^0 \leq_{!S} u^0$), we know that $a' \leq_S a$ since $(a, u^0) \in h_S$ and hence $b' \leq_S a' \leq_S a \leq_S b$. So $\forall b' \in v^0\ b' \leq_S b$ as required. ◁

1.2) Is the family of morphisms $(h_S)_S$ natural in S? That is, is it true that $h_T\, t = !t\, h_S$ for all $t \in \mathbf{PoLR}(S, T)$?

Solution ▷ We postpone the answer to Question 1.9. ◁

1.3) Prove that $\mathsf{der}_S\ h_S = \mathsf{Id}_S$.

Solution ▷ We have to prove an equality of sets, so we prove both inclusions. Let $(a, a') \in |S \multimap S|$. Assume first that $(a, a') \in \mathsf{der}_S\ h_S$. So there exists $u^0 \in |!S|$ such that $(a, u^0) \in h_S$ and $(u^0, a') \in \mathsf{der}_S$. By the second condition there is $a'' \in u^0$ such that $a' \leq_S a''$. By the first condition, $a'' \leq_S a$ so $a' \leq_S a$ that is $(a, a') \in \mathsf{Id}_S$.

Conversely assume that $(a, a') \in \mathsf{Id}_S$, that is $a' \leq_S a$. Then $(\{a'\}, a') \in \mathsf{der}_S$ and $(a, \{a'\}) \in h_S$ so $(a, a') \in \mathsf{der}_S\ h_S$. ◁

1.4) Prove that $\mathsf{dig}_S\ h_S = !h_S\, h_S$. So we have shown that (S, h_S) is an object of $\mathbf{PoLR}^!$: any preorder has a canonical structure of coalgebra. We prove now that this structure is unique.

Solution ▷ Assume first that $(a, U^0) \in !h_S\, h_S$. So let $u^0 \in |!S|$ be such that $(a, u^0) \in h_S$ and $(u^0, U^0) \in !h_S$. The second condition means that $\forall v^0 \in U^0\ \exists a' \in u^0\ (a', v^0) \in h_S$. Let $b \in \cup U^0$. Let $v^0 \in U^0$ be such that $b \in v^0$. By the first condition there is $a' \in u^0$ such that $b \leq_S a'$ (a' satisfies this for all the elements of v^0 actually). Then $a' \leq_S a$ by the first condition. Therefore $(\{a\}, U^0) \in \mathsf{dig}_S$ and since $(a, \{a\}) \in h_S$ we have $(a, U^0) \in \mathsf{dig}_S\ h_S$.

Conversely assume that $(a, U^0) \in \mathsf{dig}_S\ h_S$, so let $u^0 \in |!S|$ be such that $(a, u^0) \in h_S$ and $(u^0, U^0) \in \mathsf{dig}_S$, that is $\cup U^0 \leq_{!S} u^0$ (and hence $\forall v^0 \in U^0\ v^0 \leq_{!S} u^0 \leq_{!S} \{a\}$). Hence $(\{a\}, U^0) \in !h_S$ and since $(a, \{a\}) \in h_S$ we have $(a, U^0) \in !h_S\, h_S$ as required. ◁

1.5) Let $h \in \mathbf{PoLR}(S, !S)$ be a coalgebra structure. Using the fact that $\mathsf{der}_S\ h \subseteq \mathsf{Id}_S$ prove that $h \subseteq h_S$ (take $(a, u^0) \in h$ and then for any $a' \in u^0$ observe that $(u^0, a') \in \mathsf{der}_S$).

Solution ▷ Let $(a, u^0) \in h$. Let $a' \in u^0$, we have $(u^0, a') \in \mathsf{der}_S$ and hence $(a, a') \in \mathsf{Id}_S$ since $\mathsf{der}_S\ h \subseteq \mathsf{Id}_S$. Therefore $a' \leq_S a$ which shows that $h \subseteq h_S$. ◁

1.6) Using the fact that $\mathsf{Id}_S \subseteq \mathsf{der}_S\ h$, prove that $(a, \{a\}) \in h$ for all $a \in |S|$ (do not forget that $h \in \mathbf{PoLR}(S, !S)$!).

Solution ▷ Let $a \in |S|$, we have $(a, a) \in \mathsf{Id}_S$ and hence $(a, a) \in \mathsf{der}_S\ h$. So let $u^0 \in |!S|$ be such that $(a, u^0) \in h$ and $(u^0, a) \in \mathsf{der}_S$. By the second property there is $a' \in u^0$ such that $a \leq_S a'$ and hence $\{a\} \leq_{!S} u^0$. Since $(a, u^0) \in h$, $\{a\} \leq_{!S} u^0$ and $h \in \mathbf{PoLR}(S, !S)$, we have $(a, \{a\}) \in h$. ◁

1.7) Prove that $h = h_S$.

Solution ▷ It suffices to prove that $h_S \subseteq h$, so let $(a, u^0) \in h_S$. We have seen that $(a, \{a\}) \in h$ and by our assumption we have $u^0 \leq_S \{a\}$. Since $h \in \mathbf{PoLR}(S, !S)$ we conclude that $(a, u^0) \in h$ as required. ◁

Strangely enough we have not used the equation $\mathsf{dig}_S\ h = !h\,h$. We have shown that any object S of **PoLR** has exactly one structure of !-coalgebra. Observe that one has accordingly $\mathsf{dig}_S = h_{!S}$, for instance, since $(!S, \mathsf{dig}_S)$ is a typical !-coalgebra, the free one generated by S.

A natural question is whether such a phenomenon occurs in all models of LL.

1.8) (Open question) Look for a counter-example in the model of coherence spaces, that is: a coherence space which has no coalgebra structures or which has several coalgebra structures), for the usual "!" comonad on coherence spaces.

1.9) Let S and T be preorders and let $s \in \mathbf{PoLR}(S,T)$, remember that $s \in \mathbf{PoLR}^!((S,h_S),(T,h_T)))$ iff $h_T\, s = !s\, h_S$. Prove that this condition is equivalent to: for all $a \in |S|$ and $b_1,\dots,b_n \in |T|$ (with $n \in \mathbb{N}$), there is $b \in |T|$ such that $(a,b) \in s$ and $b_i \leq_T b$ for all i iff there are $a_1,\dots,a_n \in |S|$ such that $a_i \leq_S a$ and $(a_i,b_i) \in s$ for all i. What does this condition mean when $n = 0$?

Solution $\triangleright$ Let $s \in \mathbf{PoLR}(S,T)$. Let $a \in |S|$ and $v^0 = \{b_1,\dots,b_n\} \in |!T|$. Then $(a, v^0) \in h_T\, s$ means that there is $b \in |T|$ such that $(a,b) \in s$ and $(b, v^0) \in h_T$, that is $b_i \leq_T b$ for all i. And $(a, v^0) \in !s\, h_S$ means that there is $u^0 \in |!S|$ such that $(a, u^0) \in h_S$ (that is $u_0 \leq_{!S} \{a\}$) and $(u^0, v^0) \in !s$, which is equivalent to the existence of $a_1,\dots,a_n \in |S|$ such that $a_i \leq a$ and $(a_i,b_i) \in s$ for each i. This proves the equivalence.

Notice that, since we assume $s \in \mathbf{PoLR}(S,T)$, the $\Rightarrow$ direction of the equivalence is always true: if $(a,b) \in s$ and $b_1,\dots,b_n \leq b$ then it suffices to take $a_i = a$ for $i = 1,\dots,n$. So the criterion for $s \in \mathbf{PoLR}^!(S,T)$ boils down to: if $(a_i,b_i) \in s$ and $a_i \leq_S a$ for $i = 1,\dots,n$, then there exists $b \in |T|$ such that $(a,b) \in s$ and $b_i \leq_T b$ for $i = 1,\dots,n$.

When $n = 0$ this criterion means that, for all $a \in |S|$, there is a $b \in |T|$ such that $(a,b) \in s$. So if $|S| \neq \emptyset$ and $s \in \mathbf{PoLR}^!(S,T)$ then $s \neq \emptyset$. Notice that this provides a negative answer to Question 1.2.

1.10) An *ideal* of S is a downwards-closed directed subset of $|S|$, that is, a subset u of $|S|$ such that

- $u \neq \emptyset$
- $\forall a_1, a_2 \in u\, \exists a \in u\ a_1, a_2 \leq_S a$
- $\forall a \in u\, \forall a' \in |S|\ a' \leq_S a \Rightarrow a' \in u$.

We use $\widehat{\mathcal{I}}(S)$ for the set of all ideals of $|S|$ (sometimes called the *ideal completion* of S), ordered under inclusion. Prove that $\widehat{\mathcal{I}}(S)$ is a cpo (which has not necessarily a least element however). Prove that, for any $a \in |S|$, one has $\downarrow a \in \widehat{\mathcal{I}}(S)$ and that $\downarrow a$ is isolated in $\widehat{\mathcal{I}}(S)$ (see Chapter 5 in the lecture notes). Last prove that $\widehat{\mathcal{I}}(S)$ is algebraic (actually any algebraic cpo D is of shape $\widehat{\mathcal{I}}(S)$ for S the set of isolated elements of D equiped with the induced order relation).

Solution $\triangleright$ $\widehat{\mathcal{I}}(S)$ is a cpo: it suffices to prove that, if $\mathcal{D} \subseteq \widehat{\mathcal{I}}(S)$ is directed then $\cup\mathcal{D} \in \widehat{\mathcal{I}}(S)$ since then $\cup\mathcal{D}$ is necessarily the least upper bound of $\mathcal{D}$ in $\widehat{\mathcal{I}}(S)$ (which is ordered under inclusion). Since $\mathcal{D} \neq \emptyset$ and $u \neq \emptyset$ for all $u \in \mathcal{D}$, we have $\cup\mathcal{D} \neq \emptyset$. Next let $a \in \cup\mathcal{D}$ and let $a' \in |S|$ be such that $a' \leq_S a$. Let $u \in \mathcal{D}$ be such that $a \in u$. Since $u \in \widehat{\mathcal{I}}(S)$ we have $a' \in u$ and hence $a' \in \cup\mathcal{D}$. Last let $a_1, a_2 \in \cup\mathcal{D}$. Let $u^1, u^2 \in \mathcal{D}$ be such that $a_i \in u^i$ for $i = 1, 2$. Since $\mathcal{D}$ is directed there is $u \in \mathcal{D}$ such that $u^i \subseteq u$ for $i = 1, 2$ and hence $a_1, a_2 \in u$. But $u \in \widehat{\mathcal{I}}(S)$ hence u is directed, so there is $a \in u$ such that $a_i \leq_S a$ for $i = 1, 2$. We have $a \in \cup\mathcal{D}$ since $u \in \mathcal{D}$ and this ends the proof that $\cup\mathcal{D} \in \widehat{\mathcal{I}}(S)$. $\triangleleft$

Isolated elements: first, if $a \in |S|$ it is clear that $\downarrow a \in \widehat{\mathcal{I}}(S)$, let us prove that it is isolated. So let $\mathcal{D} \subseteq \widehat{\mathcal{I}}(S)$ be directed and such that $\downarrow a \subseteq \cup\mathcal{D}$. Then $a \in \cup\mathcal{D}$ so there is $u \in \mathcal{D}$ such that $a \in u$, but then $\downarrow a \subseteq u$ since $u \in \widehat{\mathcal{I}}(S)$, which ends the proof that $\downarrow a$ is isolated. Conversely let $u_0 \in \downarrow S$ be isolated. Let $\mathcal{D} = \{\downarrow a \mid a \in u_0\}$. Then $\mathcal{D}$ is a directed subset of $\widehat{\mathcal{I}}(S)$ (because u_0 is directed) and clearly $\cup\mathcal{D} = u_0$. So since u_0 is isolated there must be $a \in u_0$ such that $u_0 \subseteq \downarrow a$. Since the converse inclusion holds because $u_0 \in \widehat{\mathcal{I}}(S)$, we must have $u_0 = \downarrow a$.

$\widehat{\mathcal{I}}(S)$ is algebraic: this is obvious since for all $u \in \widehat{\mathcal{I}}(S)$, one has $u = \cup\{\downarrow a \mid a \in u\}$ (the set $\{\downarrow a \mid a \in u\}$ is directed in $\widehat{\mathcal{I}}(S)$, it is the set of all isolated lower bounds of u).

1.11) Exhibit a canonical bijection between $\widehat{\mathcal{I}}(S)$ and $\mathbf{PoLR}^!((1,h_1),(S,h_S))$ (remember that $1 = (\{*\}, =)$ so that simply $h_1 = \{(*,*)\}$). Using it prove that, if $s \in \mathbf{PoLR}^!((S,h_S),(T,h_T))$ and $u \in \widehat{\mathcal{I}}(S)$ one has $s\,u \in \widehat{\mathcal{I}}(T)$ (you can also prove this directly). We use $\mathsf{fun}^!(s)$ for this function $\widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$.

Solution ▷ Canonical bijection: let $u \in \widehat{\mathcal{I}}(S)$, then we claim that $\{*\} \times u \in \mathbf{PoLR}^!((1,h_1),(S,h_S))$. First we have $\{*\} \times u \in \mathbf{PoLR}(1,S)$ because u is downwards closed. By Question 1.9, it suffices to prove if $b_1,\dots,b_n \in u$ then there is $b \in u$ such that $b_i \leq_S b$ for all i which results immediately from the fact that $u \in \widehat{\mathcal{I}}(S)$. Conversely if $s \in \mathbf{PoLR}^!((1,h_1),(S,h_S))$ then $u = \{a \mid (*,a) \in s\} \in \widehat{\mathcal{I}}(S)$ again by 1.9 and by the fact that $s \in \mathbf{PoLR}(1,S)$. We set $\theta(u) = \{*\} \times u$.

Action of morphisms: let $s \in \mathbf{PoLR}^!((S,h_S),(T,h_T))$. Then $\theta(\mathsf{fun}^!(s)(u)) = s\,\theta(u) \in \mathbf{PoLR}^!(1,T)$ and hence $\mathsf{fun}^!(s)(u) \in \widehat{\mathcal{I}}(T)$. ◁

1.12) Prove that $\mathsf{fun}^!(s)$ is Scott-continuous. Conversely, given a Scott-continuous function $f : \widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$, define $\mathsf{tr}^!(f) = \{(a,b) \in |S| \times |T| \mid b \in f(\downarrow a)\}$. Prove that $\mathsf{tr}^!(f) \in \mathbf{PoLR}^!((S,h_S),(T,h_T))$.

Solution ▷ Scott continuity is obvious: by its definition $\mathsf{fun}^!(s)$ is monotonic and commutes with all existing unions, so in particular with directed ones (the only ones which certainly exist in $\widehat{\mathcal{I}}(S)$). Let $f : \widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$ be Scott continuous. Observe first that $\mathsf{tr}^!(f) \in \mathbf{PoLR}(S,T)$ holds by monotonicity of f.

We prove that the criterion of Question 1.9 holds for $\mathsf{tr}^!(f)$, so let $a \in |S|$ and $b_1,\dots,b_n \in |T|$. Assume that we have $a_1,\dots,a_n$ such that $a_i \leq_S a \in |S|$ and $(a_i,b_i) \in \mathsf{tr}^!(f)$ for each i. Then $b_i \in f(\downarrow a_i) \subseteq f(\downarrow a)$ and since $f(\downarrow a) \in \widehat{\mathcal{I}}(T)$ there exists $b \in f(\downarrow a)$ such that $b_i \leq_T b$ for each i. Since $(a,b) \in \mathsf{tr}^!(f)$ we have proven our contention. ◁

1.13) Prove that the operations $\mathsf{fun}^!(\_)$ and $\mathsf{tr}^!(\_)$ are inverse of each other.

Solution ▷ First let $s \in \mathbf{PoLR}^!(S,T)$ and let us prove that $\mathsf{tr}^!(\mathsf{fun}^!(s)) = s$. Let $(a,b) \in s$, then $b \in \mathsf{fun}^!(s)(\downarrow a)$ and hence $(a,b) \in \mathsf{tr}^!(\mathsf{fun}^!(s))$. Conversely assume that $(a,b) \in \mathsf{tr}^!(\mathsf{fun}^!(s))$, which means that $b \in \mathsf{fun}^!(s)(\downarrow a)$, that is $(a',b) \in s$ for some a' such that $a' \leq_S a$. Since $s \in \mathbf{PoLR}(S,T)$ we have $(a,b) \in s$.

Now let $f : \widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$ be Scott continuous and let us prove first that $f = \mathsf{fun}^!(\mathsf{tr}^!(f))$. Let $u \in \widehat{\mathcal{I}}(S)$ and let us prove that $f(u) \subseteq \mathsf{fun}^!(\mathsf{tr}^!(f))(u)$. Let $b \in f(u) = f(\bigcup_{a\in u} \downarrow a) = \bigcup_{a \in u} f(\downarrow a)$ by Scott continuity (remember that, since $u \in \widehat{\mathcal{I}}(S)$, the set $\{\downarrow a \mid a \in u\}$ is directed in $\widehat{\mathcal{I}}(S)$). So there exists $a \in u$ such that $(a,b) \in \mathsf{tr}^!(f)$. Hence $b \in \mathsf{fun}^!(\mathsf{tr}^!(f))(u)$ and we are done. Conversely we prove that $\mathsf{fun}^!(\mathsf{tr}^!(f))(u) \subseteq f(u)$. Let $b \in \mathsf{fun}^!(\mathsf{tr}^!(f))(u)$. Let $a \in u$ be such that $(a,b) \in \mathsf{tr}^!(f)$, that is $b \in f(\downarrow a)$. Since $a \in u$ we have $\downarrow a \subseteq u$ and hence $b \in f(u)$ by monotonicity of f. ◁

1.14) Prove that $\mathbf{PoLR}^!$ is cartesian (with cartesian product defined using $\otimes$ and not $\&$) and also co-cartesian (with co-product defined using $\oplus$). Describe the corresponding operations on cpos. Compare with what happens in $\mathbf{PoLR}$ for $\&$ and $\oplus$.

Solution ▷ If S is a preorder, we use simply the notation S for the unique associated object (S,h_S) of $\mathbf{PoLR}^!$.

Cartesian product: Observe first that 1 is the terminal object of $\mathbf{PoLR}^!$. Indeed $\mathbf{PoLR}^!(S,1)$ has exactly one element, namely $\{(a,*) \mid a \in |S|\}$. More generally, let $(S_i)_{i\in I}$ be a family of preorders (for I finite). Let T be the preorder defined by $|T| = \prod_{i\in I} |S_i|$ and $(a_i)_{i\in I} \leq_T (a'_i)_{i\in I}$ if $a_i \leq_{S_i} a'_i$ for all $i \in I$ (in other words $T = \bigotimes_{i\in I} S_i$). We define projections $\mathsf{pr}^!_j \in \mathbf{PoLR}^!(T,S_j)$ as $\mathsf{pr}^!_j = \{((a_i)_{i\in I}, a') \mid a' \leq_{S_j} a_j\}$. Clearly $\mathsf{pr}^!_j \in \mathbf{PoLR}(T,S_j)$, let us check that $\mathsf{pr}^!_j \in \mathbf{PoLR}^!(T,S_j)$. So let $\vec{a} = (a_i)_{i\in I} \in |T|$ and $(\vec{a}(l), a'_l) \in \mathsf{pr}^!_j$ with $\vec{a}(l) \leq_T \vec{a}$ for $l = 1,\dots,k$. Then we have $a'_l \leq a(l)_j \leq_S a_j$ for all l, and $(\vec{a}, a_j) \in \mathsf{pr}^!_j$, showing that the criterion of 1.9 holds for $\mathsf{pr}^!_j$.

Now let $s_i \in \mathbf{PoLR}^!(U,S_i)$ for $i = 1,\dots,n$. We define $t \subseteq |U| \times |T|$ by $t = \{c,(a_i)_{i\in I} \mid (c,a_i) \in s_i \text{ for } i = 1,\dots,n\}$. The fact that $t \in \mathbf{PoLR}(U,T)$ results easily from the fact that $s_i \in \mathbf{PoLR}(U,S_i)$ for each i. We prove that $t \in \mathbf{PoLR}^!(U,T)$, so let $c \in |U|$ and $(c_l, \vec{a}(l)) \in t$ with $c_l \leq_U c$ for $l = 1,\dots,k$. Let $i \in \{1,\dots,n\}$, we have $(c_l, a(l)_i) \in s_i$ for $l = 1,\dots,k$ so, applying 1.9 to s_i, we can find $a_i \in |S_i|$ such that $(c,a_i) \in s_i$ and $a(l)_i \leq a_i$ for $l = 1,\dots k$. Now $(c, \vec{a} = (a_i)_{i\in I}) \in t$ by definition of t, and $\vec{a}(l) \leq_T \vec{a}$ for $l = 1,\dots,k$. We have proven that t satisfies Criterion 1.9 so $t \in \mathbf{PoLR}^!(U,T)$. The fact that $\mathsf{pr}^!_i \circ t = s_i$ for $i = 1,\dots,n$ immediately results from the definitions. Uniqueness of t also results from the fact that we must have $\mathsf{pr}^!_i \circ t = s_i$ for $i = 1,\dots,n$.

Cpo description of the product: one checks easily (do it!) that $\widehat{\mathcal{I}}(\bigotimes_{i\in I} S_i) = \prod_{i\in I} \widehat{\mathcal{I}}(S_i)$ up to canonical order isomorphism (check the details).

Coproduct: given a (potentially infinite) family $(S_i)_{i\in I}$ of preorders, we show that $T = \bigoplus_{i\in S_i}$ (as defined in the course, that is $|T| = \cup_{i\in I}\{i\} \times |S_i|$) together with the usual injections $\mathsf{in}_i \in \mathbf{PoLR}(S_i, T)$ is the coproduct of the S_i's in $\mathbf{PoLR}^!$. One needs first to prove that $\mathsf{in}_i \in \mathbf{PoLR}^!(S_i, T)$ so let $a \in S_i$ and $(a_1, (j_1, a'_1)), \dots, (a_k, (j_k, a'_k)) \in \mathsf{in}_i$ with $a_1, \dots, a_k \leq_S a$. Then by definition of in_i we know that $j_1 = \dots = j_k = i$ and $a'_1, \dots, a'_k \leq_{S_i} a$, so $(j_1, a'_1), \dots, (j_k, a'_k) \leq_T (i, a)$ and Criterion 1.9 holds since $(a, (i, a)) \in \mathsf{in}_i$.

Then let $s_i \in \mathbf{PoLR}^!(S_i, U)$, we define $t \in \mathbf{PoLR}(T, U)$ as in $\mathbf{PoLR}$, setting $t = \{((i, a), c) \mid (a, c) \in s_i\}$. We have to prove that $t \in \mathbf{PoLR}^!(T, U)$ and for this we use again Criterion 1.9. Let $(i, a) \in |T|$ (so that $a \in |S_i|$) and $((j_1, a_1), c_1), \dots, ((j_k, a_k), c_k) \in t$ with $(j_l, a_l) \leq_T (i, a)$ for $l = 1, \dots k$. This means that $j_l = i$ and $a_l \leq_{S_i} a$ for $l = 1, \dots, k$. So we actually have $(a_l, c_l) \in s_i$ for $l = 1, \dots, k$ and hence, by Criterion 1.9 applied to s_i, there exists $c \in |U|$ such that $(a, c) \in s_i$ and hence $((i, a), c) \in t$, so that Criterion 1.9 holds for t. The fact that $t\,\mathsf{in}_i = s_i$ and that t is unique with these properties are obvious.

Cpo description of the coproduct: one checks easily (do it!) that $\widehat{\mathcal{I}}(\bigoplus_{i\in I} S_i)$ is (isomorphic to) the disjoint union of the $\widehat{\mathcal{I}}(S_i)$'s $\bigcup_{i\in I}\{i\} \times \widehat{\mathcal{I}}(S_i)$ with the disjoint union of the order relations. Notice that this means that such a coproduct (if non-trivial) has no least element. Notice also that the unit of this coproduct is the prorder 0 such that $|0| = \emptyset$ and that $\widehat{\mathcal{I}}(0) = \emptyset$ (whereas $\mathcal{I}(0) = \{\emptyset\}$). $\triangleleft$

1.15) Prove that $\widehat{\mathcal{I}}(!S) = \mathcal{I}(S)$. Using this observation explain how the canonical inclusion functor $\mathbf{PoLR}_! \to \mathbf{PoLR}^!$ (from free coalgebras into general ones), which maps S to $!S$ and $s \in \mathbf{PoLR}_!(S, T)$ to $s^! = {!s}\,\mathsf{dig}_S$ can simply be described as an inclusion of categories in that special case (using the characterization of $\mathbf{PoLR}_!(S, T)$ as the set of Scott-contuous functions $\mathcal{I}(S) \to \mathcal{I}(T)$).

Solution $\triangleright$ We define a function $\varphi_S : \widehat{\mathcal{I}}(!S) \to \mathcal{P}(|S|)$ by $\varphi_S(U) = \bigcup U = \{a \in |S| \mid \exists u \in U\ a \in u\}$. We prove that $\varphi_S(U) \in \mathcal{I}(S)$. Let $a \in \varphi_S(U)$ and $a' \in |S|$ such that $a' \leq_S a$. Let $u \in U$ be such that $a \in u$. We have $\{a'\} \leq_{!S} u$ and hence $\{a'\} \in U$ since $U \in \widehat{\mathcal{I}}(!S)$, therefore $a' \in \varphi_S(U)$. So $\varphi_S : \widehat{\mathcal{I}}(!S) \to \mathcal{I}(S)$. It is clear that φ_S is monotonic. Conversely let $\psi_S : \mathcal{I}(S) \to \mathcal{P}(|!S|)$ be defined by $\psi_S(u) = u^! = \mathcal{P}_{\mathrm{fin}}(u)$ (the set of finite subsets of u). We prove that $\psi_S(u) \in \widehat{\mathcal{I}}(!S)$. Let $u^0 \in \mathcal{P}_{\mathrm{fin}}(u)$ and let $v^0 \in |!S|$ such that $v^0 \leq_{!S} u^0$. Let $a \in v^0$. There is $a' \in u^0$ such that $a \leq_S a'$. We have $a' \in u$ and since $u \in \mathcal{I}(S)$ it follows that $a \in u$. Therefore $v^0 \in \psi_S(u)$. So $\psi_S(u) \in \mathcal{I}(!S)$. We have $\emptyset \in \psi_S(u)$ and hence $\psi_S(u) \neq \emptyset$. Last let $u^1, u^2 \in \psi_S(u)$, we have $u^1 \cup u^2 \in \psi_S(u)$ and hence $\psi_S(u)$ is directed, so $\psi_S(u) \in \widehat{\mathcal{I}}(S)$. We have shown that $\psi_S : \mathcal{I}(S) \to \widehat{\mathcal{I}}(!S)$. This map ψ_S is obviously monotonic.

Now we prove that $\psi_S \circ \varphi_S = \mathsf{Id}$ so let $U \in \widehat{\mathcal{I}}(!S)$. Let $u^0 = \{a_1, \dots, a_n\} \in \psi_S(\varphi_S(U))$, that is $a_i \in \psi_S(U)$ for each $i = 1, \dots, n$. So for each i there is $u^i \in U$ such that $a_i \in u^i$. Since U is directed there is $v^0 \in U$ such that $u^i \subseteq v^0$ for $i = 1, \dots, n$. It follows that $u^0 \leq_{!S} v^0$ and hence $u^0 \in U$ since $U \in \widehat{\mathcal{I}}(!S)$. Conversely let $u^0 \in U$, we have $u^0 \subseteq \bigcup U$, that is $u^0 \in \psi_S(\varphi_S(U))$. We have shown that $\psi_S \circ \varphi_S = \mathsf{Id}$. Conversely, let $u \in \mathcal{I}(S)$, we have $\varphi_S(\psi_S(u)) = \bigcup \mathcal{P}_{\mathrm{fin}}(u) = u$. So $\varphi_S \circ \psi_S = \mathsf{Id}$.

Let $s \in \mathbf{PoLR}_!(S, T)$, we have $\mathsf{Fun}(s) : \mathcal{I}(S) \to \mathcal{I}(T)$ defined by $\mathsf{Fun}(s)(u) = s\,u^!$. Let $I : \mathbf{PoLR}_! \to \mathbf{PoLR}^!$ be the mentioned inclusion functor. Then, thanks to the above isomorphism, $\mathsf{fun}^!(I(s)) : \widehat{\mathcal{I}}(!S) \to \widehat{\mathcal{I}}(!T)$ can be considered as a Scott continuous function $\mathcal{I}(S) \to \mathcal{I}(T)$. More precisely, this latter function is $f = \varphi_T \circ \mathsf{fun}^!(I(s)) \circ \psi_S$. We have $f(u) = \psi_T(\mathsf{fun}^!(I(s))(u^!)) = \psi_T(I(s)\,u^!) = \psi_T(s^!\,u^!) = \psi_T((s\,u^!)^!) = \psi_T(\varphi_T(s\,u^!)) = s\,u^! = \mathsf{Fun}(s)(u)$. So, up to these isos, we have $\mathsf{Fun}(s) = \mathsf{fun}^!(I(s))$ so that I is the inclusion functor from the category of Scott continuous functions on prime-algebraic lattices (the lattices of shape $\mathcal{I}(S)$) into the category of Scott continuous functions on algebraic cpos (those of shape $\widehat{\mathcal{I}}(S)$).

2) Remember that $\mathcal{Z} \in \mathbf{PoLR}_!((S \Rightarrow S) \Rightarrow S, (S \Rightarrow S) \Rightarrow S)$ has been defined during a lesson as a morphism such that, setting $F = \mathsf{Fun}\,\mathcal{Z}$, one has $\mathsf{Fun}\,(F(Y))(s) = \mathsf{Fun}\,s(\mathsf{Fun}\,Y(s))$ for all $s \in \mathbf{PoLR}_!(S, S)$.

2.1) [There was a mistake in this question!] Given $t \in \mathbf{PoLR}_!(T, T)$, we set $\varphi(t) = \bigcup_{n=0}^{\infty}(\mathsf{Fun}\,t)^n(\emptyset) \in \mathcal{I}(T)$, the least fixed point of $\mathsf{Fun}\,t$. Prove that $\varphi(t)$ is the least element of $\mathcal{I}(T)$ such that $(\mathsf{Fun}\,t)(u) \subseteq u$.

Solution $\triangleright$ Let $u \in \mathcal{I}(T)$ be such that $(\mathsf{Fun}\,t)(u) \subseteq u$, one proves by induction on n that $(\mathsf{Fun}\,t)^n(\emptyset) \subseteq u$ for all n, and hence $\varphi(t) \subseteq u$ (write down the details).

Observe that, if $(u^0, b) \in t$ with $u^0 \subseteq \varphi(t)$ then $b \in \varphi(t)$ because $(\mathsf{Fun}\,t)(\varphi(t)) \subseteq \varphi(t)$. Conversely if $b \in \varphi(t)$, there is n such that $b \in (\mathsf{Fun}\,t)^n(\emptyset)$. We cannot have $n = 0$ and hence there is u^0 such that $(u^0, b) \in t$ and $u^0 \subseteq (\mathsf{Fun}\,t)^{n-1}(\emptyset) \subseteq \varphi(t)$.

So $b \in \varphi(t) \Leftrightarrow \exists u^0\ (u^0, b) \in t$ and $u^0 \subseteq \varphi(t)$. ◁

2.2) We set $Y_0 = \varphi(\mathcal{Z}) \in \mathcal{I}((S \Rightarrow S) \Rightarrow S)$. Prove that $\mathsf{Fun}\, Y_0(s) = \varphi(\mathsf{Fun}\, s)$ for all $s \in \mathcal{I}(S \Rightarrow S)$. To this end, prove that $\mathsf{Fun}\,(F^n(\emptyset))(s) = (\mathsf{Fun}\, s)^n(\emptyset)$ by induction on n. Use also the fact that $\mathsf{Fun}\,\_$ is an order isomorphism (between $\mathbf{PoLR}_!(T, U)$ ordered by inclusion and $\mathbf{PoC}(\mathcal{I}(T), \mathcal{I}(U))$ ordered by the pointwise ordering on functions).

Solution ▷ We prove $\mathsf{Fun}\,(F^n(\emptyset))(s) = (\mathsf{Fun}\, s)^n(\emptyset)$ by induction on n. For $n = 0$ both sides are $\emptyset$. Assume that the equations holds for n. We have

$$\begin{aligned}\mathsf{Fun}\,(F^{n+1}(\emptyset))(s) &= \mathsf{Fun}\,(F(Y))(s) \quad \text{where } Y = F^n(\emptyset)\\ &= \mathsf{Fun}\, s(\mathsf{Fun}\, Y(s))\\ &= \mathsf{Fun}\, s((\mathsf{Fun}\, s)^n(\emptyset)) \quad \text{by inductive hypothesis}\\ &= (\mathsf{Fun}\, s)^{n+1}(\emptyset)\end{aligned}$$

as required. Then $\mathsf{Fun}\, Y_0(s) = \mathsf{Fun}\,(\bigcup_{n=0}^{\infty} F^n(\emptyset))(s) = \bigcup_{n=0}^{\infty} \mathsf{Fun}\,(F^n(\emptyset))(s)$ by the mentioned property of $\mathsf{Fun}\,\_$ and this proves our contention. ◁

2.3) Prove that $(V^0, b) \in Y_0$ iff there exists u^0 such that $(u_0, b) \in \downarrow V^0$ and $\forall b' \in u^0\ (V^0, b') \in Y_0$.

Solution ▷ $(V^0, b) \in Y_0$ iff $b \in \mathsf{Fun}\,(Y_0)(\downarrow V^0) = \varphi(\downarrow V_0)$. The equivalence to be proven is then a special case of the observations of the first question (write down the details). ◁

3) Using the semantic typing system of LPCF, compute the Scott semantics of the following terms (given with their types).

- $\vdash \Omega^\iota : \iota$ where $\Omega^A = \mathsf{fix}\, x^A \cdot x$.
- $\vdash \mathsf{fix}\, x^\iota \cdot \underline{\mathsf{succ}}(x) : \iota$ (give a recursive description of the interpretation of this term).
- $\vdash \lambda x^\iota\, \mathsf{if}(x, \Omega^\iota, z \cdot \underline{0}) : \iota \to \iota$.
- $\vdash \lambda x^\iota\, \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota\, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z)) : \iota \to \iota \to \iota$

Solution ▷ We deal only with the last question the others are simpler.

Let $M = \lambda x^\iota\, \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota\, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z))$ and $N = \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z))$.

Any derivation of $\vdash M : (u^0, (v^0, n)) : \iota \to \iota \to \iota$ is of shape

$$\dfrac{\dfrac{\rho}{x : u^0 : \iota \vdash \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota\, N : (v^0, n)}}{\vdash M : (u^0, (v^0, n)) : \iota \to \iota \to \iota}$$

where ρ is of shape

$$\dfrac{\dfrac{\dfrac{\lambda}{x : u^0 : \iota, a : U^0 : \iota \to \iota, y : v^0 : \iota \vdash N : n : \iota}}{x : u^0 : \iota, a : U^0 : \iota \to \iota \vdash \lambda y^\iota\, N : (v^0, n) : \iota \to \iota} \qquad (\rho(v^1, n'))_{(v^1, n') \in U^0}}{x : u^0 : \iota \vdash \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota\, N : (v^0, n)}$$

with one derivation $\rho(v^0, n')$ for each $(v^1, n') \in U^0$. Coming back to the definition of N, we see that there are two possibilities as to λ. The first one is

$$\dfrac{\dfrac{\zeta \in v^0}{x : u^0 : \iota, a : U^0 : \iota \to \iota, y : v^0 : \iota \vdash y : \zeta : \iota} \qquad \dfrac{\{n\} \leq_{!\mathsf{L}} u^0}{x : u^0 : \iota, a : U^0 : \iota \to \iota, y : v^0 : \iota \vdash x : n : \iota}}{x : u^0 : \iota, a : U^0 : \iota \to \iota, y : v^0 : \iota \vdash N : n : \iota}$$

where we use the observation that if ζ is comparable in L with an element n' of $|\mathsf{L}|$ then $n' = \zeta$.

The second possibility for λ is (with $n = \overline{\mathsf{suc}}\, w^1$)

$$\dfrac{\dfrac{\{\overline{\mathsf{suc}}\,w^0\} \leq_{!\mathsf{L}} v^0}{\Phi \vdash y : \overline{\mathsf{suc}}\,w^0 : \iota} \qquad \dfrac{(\mu(n'))_{n' \in w^1}}{\Phi, z : w^0 : \iota \vdash \underline{\mathsf{succ}}((a)\,z) : \overline{\mathsf{suc}}\,w^1 : \iota}}{\Phi = (x : u^0 : \iota, a : U^0 : \iota \to \iota, y : v^0 : \iota) \vdash N : \overline{\mathsf{suc}}\,w^1 : \iota}$$

where, for each $n' \in w^1$, $\mu(n')$ is the derivation

$$\dfrac{\dfrac{\{(v^1, n')\} \leq_{!\mathsf{L} \multimap \mathsf{L}} U^0}{\Phi, z : w^0 : \iota \vdash a : (v^1, n') : \iota \to \iota} \qquad (\nu(n', m))_{m \in v^1}}{\Phi, z : w^0 : \iota \vdash (a)\,z : n' : \iota}$$

where for each $m \in v^1$ the derivation $\nu(n', m)$ is

$$\dfrac{\{m\} \leq_{!\mathsf{L}} w^0}{\Phi, z : w^0 : \iota \vdash z : m : \iota}$$

Hence on has $\vdash M : (u^0, (v^0, n)) : \iota \to \iota \to \iota$ iff one of the two following conditions hold:

- $\zeta \in v^0$ and $\{n\} \leq_{!\mathsf{L}} u_0$ (indeed in that case we can take $U^0 = \emptyset$).
- $n = \overline{\mathsf{suc}}\,w^1$ and there is $w^0 \in |!\mathsf{L}|$ such that $\{\overline{\mathsf{suc}}\,w^0\} \leq_{!\mathsf{L}} v^0$. Moreover, for each $n' \in w^1$ there is $v^1 \in |!\mathsf{L}|$ such that $\vdash M : (u^0, (v^1, n')) : \iota \to \iota \to \iota$ and $\{m\} \leq_{!\mathsf{L}} w^0$ for each $m \in v^1$.

This equivalence characterizes $[M]$: it is the least set of tuples $(u^0, (v^0, n))$ which satisfies this equivalence.

$\triangleleft$