Web based models of Linear Logic

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Linear Logic : a refinement of intuitionistic/classical logic where contraction and weakening have a *logical* status.

- Contraction and disjunction exist in two different flavors : *multiplicative* ($\otimes$ and 1, $\parr$ and $\perp$) and *additive* (& and $\top$, $\oplus$ and 0).
- Exponential modalities give a logical status to structural rules (! and ?).
- Negation is *involutive* ($(\cdot)^\perp$).

Linear logic has various *denotational interpretations* where

- formulae are interpreted as objects in some category $\mathcal{C}$
- and proofs as morphisms in $\mathcal{C}$.

The simplest one is the *pure relational interpretation* in the category $\mathbf{Rel}$ of sets and relations:

- objects of $\mathbf{Rel}$ are sets and
- $\mathbf{Rel}(S,T) = \mathcal{P}(S \times T)$ (relations).
- Usual composition or relations.

Interpretation of formulae in Rel.

- $S^{\perp} = S$
- $S \otimes T = S \times T$ and $1 = \{*\}$
- and so $S \parr T = S \times T$ and $\perp = \{*\}$ since $\parr$ is the De Morgan dual of $\otimes$ ($(E \otimes F)^{\perp} = E^{\perp} \parr F^{\perp}$) and similarly for 1 and $\perp$.
- $S \oplus T = S + T$ and $0 = \emptyset$
- and so $S \mathbin{\&} T = S + T$ and $\top = \emptyset$ (duals of $\oplus$ and 0).
- $!S = ?S = \mathcal{M}_{\mathsf{fin}}(S)$ (the finite multisets of elements of S).

Two kinds of non trivial structures which can be superimposed on this simple semantics and which are respected by the interpretations of proofs:

- phase valued coherence structures (with A. Bucciarelli) and
- finiteness structure.

I
Phase valued coherence structures

Idea: generalize Girard's idea of *coherence space* (sets equipped with a graph structure, then proofs are interpreted as cliques), but now coherence takes values in a phase space.

Similar ideas developped in

- Girard : *On denotational completeness*.
- Lamarche : *Generalizing coherence spaces and hypercoherences*.
- De Paiva and Schalk : *Poset-valued sets or How to build models for linear logics*.

The connection with the present work is not very clear (at least to me). . .

A **phase space** is a pair $(M, \perp\!\!\!\perp)$ where M is commutative monoid (multiplicative notations, unit e) and $\perp\!\!\!\perp \subseteq M$ (no other constraint). If $U \subseteq M$, define

$$U^{\perp} = \{m \in M \mid mU \subseteq \perp\!\!\!\perp\}.$$

Then the following facts are straightforward:

- $U \subseteq U^{\perp\perp}$
- $U \subseteq V \Rightarrow V^{\perp} \subseteq U^{\perp}$
- and $U^{\perp\perp\perp} = U^{\perp}$.

U is a **fact** if $U^{\perp\perp} = U$. So $U^{\perp}$ is always a fact.

Phase spaces provide a natural *truth values* semantics of (exponential free) linear logic:

Principle : formulae are interpreted as facts and if a formula is provable, its interpretation contains e.

How it works :

- interpret 1 as $\perp\!\!\!\perp^{\perp}$ and $\perp$ as $\perp\!\!\!\perp$ (which is always a fact);
- if U, V are facts, $U \otimes V = (UV)^{\perp\perp}$ and of course $U \parr V = (U^{\perp}V^{\perp})^{\perp}$.
- Interpret 0 as $M^{\perp}$ and $\top$ as M (which is always a fact);
- $U \oplus V = (U \cup V)^{\perp\perp}$ and $U \mathbin{\&} V = U \cap V$ (which is always a fact).

Exponentials require more structure on M...

Phase valued coherence : given

- a formula of linear logic F whose denotational interpretation as a set (in $\mathbf{Rel}$) is $[F]$
- and a family α of elements of $[F]$ (indexed by some set $J \subseteq I$ where I is a fixed infinite countable set)

we want a fact in some phase space, the "coherence value" of α in F.

Idea : use a phase space which is itself "indexed" by I.

An **indexed phase space** is a pair $(M_0, \perp\!\!\!\perp)$ where

- M_0 is a commutative monoid (multiplicative notations, neutral element e) possessing an absorbing element 0
- and $\perp\!\!\!\perp \subseteq M_0^I$.

This time, $\perp\!\!\!\perp$ has to satisfy some conditions for making the interpretation possible.

Conditions on $\perp\!\!\!\perp \subseteq M_0^I$:

- **Closure under restriction :**
  if $J \subseteq I$, then $e_J \perp\!\!\!\perp \subseteq \perp\!\!\!\perp$
  where e_J is the family which takes value e on J and 0 outside J.
  And also $e_\emptyset \in \perp\!\!\!\perp$.

- **Homogeneity :**
  if $f, g : J \to I$ are *injections*, then $f^*(\perp\!\!\!\perp) = g^*(\perp\!\!\!\perp)$
  where f^* is the reindexing function, $f^*(m)_j = m_{f(j)}$.

The second condition means that $\perp\!\!\!\perp$ “looks the same everywhere”. This condition is necessary only for interpreting the exponentials.

Example.

$M_0 = \{0, e, \tau\}$ with $\tau\tau = \tau$.

For $m \in M_0^I$, decide that $m \in \perp\!\!\!\perp$ if there is at most one index i such that $m_i = \tau$.

A fact F can be described as a *graph* on I, more precisely as a collection of unlabelled symmetric edges between distinct elements of I : $i \neq j$ are related in $\mathcal{G}$ iff there is $m \in F$ with $m_i = m_j = \tau$.

One retreives a notion of *coherence spaces* (very different from standard ones on exponentials!).

Indexed linear logic.

Phase semantics interprets normally formulae of LL.

Given F a formula of LL and $\alpha \in [F]^J$ (with $J \subseteq I$), can we see α as a formula of some version of LL? **Yes**.

A formula A of *indexed linear logic* has

- an underlying formula $\underline{A}$ of standard LL,
- a domain $\mathsf{d}(A) \subseteq I$
- and a family $\langle A \rangle \in [\underline{A}]^{\mathsf{d}(A)}$.

Basic property : for all formula F of LL and all $\alpha \in [F]^J$, there exists A such that $\underline{A} = F$, $\mathsf{d}(A) = J$ and $\langle A \rangle = \alpha$. But in general, there will be an infinity of such formulae A.

Indexed formulae are defined as follows :

- if $J \subseteq I$ then 1_J and $\perp_J$ are formulae with
  - $\mathsf{d}(1_J) = \mathsf{d}(\perp_J) = J$,
  - $\langle 1_J \rangle_j = \langle \perp_J \rangle_j = * \in [1] = [\perp]$ and
  - $\underline{1_J} = 1$ and $\underline{\perp_J} = \perp$.
- If $\mathsf{d}(A) = \mathsf{d}(B) = J$ then $A \otimes B$ and $A \parr B$ are well defined, with
  - $\mathsf{d}(A \otimes B) = \mathsf{d}(A \parr B) = J$,
  - $\langle A \otimes B \rangle_j = \langle A \parr B \rangle_j = (\langle A \rangle_j, \langle B \rangle_j)$
  - and of course $\underline{A \otimes B} = \underline{A} \otimes \underline{B}$ and $\underline{A \parr B} = \underline{A} \parr \underline{B}$.

- 0 and $\top$ are formulae, with $\mathsf{d}(0) = \mathsf{d}(\top) = \emptyset$, $\langle 0 \rangle = \langle \top \rangle =$ the empty family, and of course $\underline{0} = 0$ and $\underline{\top} = \top$.
- If $\mathsf{d}(A) \cap \mathsf{d}(B) = \emptyset$ then $A \oplus B$ and $A \mathbin{\&} B$ are well defined, with
  - $\mathsf{d}(A \oplus B) = \mathsf{d}(A \mathbin{\&} B) = \mathsf{d}(A) \cup \mathsf{d}(B)$,
  - $\langle A \oplus B \rangle_j = \langle A \mathbin{\&} B \rangle_j = \langle A \rangle_j$ if $j \in \mathsf{d}(A)$ and
    $\langle A \oplus B \rangle_j = \langle A \mathbin{\&} B \rangle_j = \langle B \rangle_j$ if $j \in \mathsf{d}(B)$
  - and of course $\underline{A \oplus B} = \underline{A} \oplus \underline{B}$ and $\underline{A \mathbin{\&} B} = \underline{A} \mathbin{\&} \underline{B}$.

Let us say that a function is *locally finite* if $f^{-1}(i)$ is finite for all i.

- If f : d$(A) \to J$ is locally finite, then $!_f A$ and $?_f A$ are well defined with
  - d$(!_f A) =$ d$(?_f A) = J$,
  - $\langle !_f A \rangle_j = \langle ?_f A \rangle_j =$ the multiset of all $\langle A \rangle_k$ with $f(k) = j$
  - and of course $\underline{!_f A} = !\underline{A}$ and $\underline{?_f A} = ?\underline{A}$.

Local phase space. Given $J \subseteq I$, consider the phase space $(M_0^J, \perp\!\!\!\perp_J)$ where $\perp\!\!\!\perp_J$ is the projection of $\perp\!\!\!\perp \subset M_0^I$ onto M_0^J.

To A we associate $A^\bullet$, a fact of $(M_0^J, \perp\!\!\!\perp_J)$ where $J = \mathsf{d}(A)$.

Main property (thanks to our homogeneity hypothesis) :

this fact depends only on $\underline{A}$ and $\langle A \rangle$.

Multiplicatives : like in ordinary phase semantics, for instance

$$(A \otimes B)^\bullet = (A^\bullet B^\bullet)^{\perp\perp}$$

in the space $(M_0^J, \perp\!\!\!\perp_J)$ where $J = \mathsf{d}(A) = \mathsf{d}(B)$.

Additives : much more interesting.
$L = \mathsf{d}(A)$ and $R = \mathsf{d}(B)$ with $L \cap R = \emptyset$.

$$\begin{aligned}(A \,\&\, B)^\bullet &= \{(p,q) \in M_0^{L\cup R} \simeq M_0^L \times M_0^R \mid \\ &\qquad p \in A^\bullet,\ q \in B^\bullet\} \\ &\simeq A^\bullet \times B^\bullet\end{aligned}$$

This set is always a fact.
$(A \oplus B)^\bullet$ is defined by De Morgan duality.

Exponentials : given $f : K \to J$ locally finite, we can define a monoid morphism $f_* : M_0^K \to M_0^J$ by

$$f_*(m)_j = \prod_{f(k)=j} m_k \,.$$

If $\mathrm{d}(A) = K$, we set

$$(!_f A)^\bullet = (f_*(A^\bullet))^{\perp\perp}$$

and of course $(?_f A)^\bullet$ is defined by De Morgan duality.

Natural question : is there a deduction system for indexed linear logic corresponding to this phase semantics ? **Yes.**

A sequent is $\vdash_J A_1, \ldots, A_n$ with $\mathsf{d}(A_i) = J$ for each i.

A few examples of deduction rules (not all of them) : **multiplicative** rules are without surprises, for instance

$$\frac{\vdash_J \Gamma, A \qquad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

Additives are more interesting

$$\frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, A \oplus B} \qquad \frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, B \oplus A}$$

which implies that $\mathsf{d}(B) = \emptyset$ since we must have $\mathsf{d}(A) = J$ and $\mathsf{d}(A) \cap \mathsf{d}(B) = \emptyset$.

$$\frac{\vdash_L \Gamma_L, A \qquad \vdash_R \Gamma_R, B}{\vdash_{L \cup R} \Gamma, A \,\&\, B}$$

with $L \cap R = \emptyset$, all the formulae of Γ have domain $L \cup R$, Γ_L is obtained by *restricting* its formulae to L and similarly for Γ_R. Restriction is easy to define by induction on formulae.

Restriction : if $\mathsf{d}(C) = J$ and $L \subseteq J$ then C_L satisfies

- $\underline{C_L} = \underline{C}$
- $\mathsf{d}(C_L) = L$
- $\langle C_L \rangle \in [\underline{C}]^L$ is the restriction of $\langle C \rangle \in [\underline{C}]^J$ to L.

Thanks to the closure under restriction of $\perp\!\!\!\perp$, ${C_L}^\bullet$ is the projection of $C^\bullet$ onto M_0^L.

Exponentials : there are two groups of rules, weakening/contraction and dereliction/promotion.

Contraction : take A with domain $L \cup R$ ($L \cap R = \emptyset$) and let $l : L \to J$ and $r : R \to J$ be locally finite functions, and $f : L \cup R \to J$ defined by "cases":

$$f(i) = \begin{cases} l(i) & \text{if } i \in L \\ r(i) & \text{if } i \in R \end{cases}$$

which is also locally finite, then

$$\frac{\vdash_J \Gamma, ?_l A_L, ?_r A_R}{\vdash_J \Gamma, ?_f A}$$

Dereliction uses the idea of changing the localization of a formula. If A has domain K and $f : K \to J$ is a *bijection*, we can define (by induction on A) $f_*(A)$ with

- $\underline{f_*(A)} = \underline{A}$
- $\mathsf{d}(f_*(A)) = J$
- $\langle f_*(A) \rangle_j = \langle A \rangle_{f^{-1}(j)}$

Thanks to the homogeneity of $\perp\!\!\!\perp$, $(f_*(A))^\bullet = f_*(A^\bullet)$.

In these circumstances, dereliction is

$$\frac{\vdash_J \Gamma, f_*(A)}{\vdash_J \Gamma, ?_f A}$$

Promotion involves composing locally finite functions: if $g : K \to J$ is locally finite,

$$\frac{\vdash_K ?_{f_1} A_1, \ldots, ?_{f_n} A_n, B}{\vdash_J ?_{g \circ f_1} A_1, \ldots, ?_{g \circ f_n} A_n, !_g B}$$

Remark : from a proof of $\vdash_J A$, we obtain, by forgetting all indexing sets and functions, a proof π of $\vdash \underline{A}$ in standard linear logic.

For each $j \in \mathsf{d}(A)$, remember that $\langle A \rangle_j \in [\underline{A}]$.

Basic property : $\forall j \in \mathsf{d}(A) \quad \langle A \rangle_j \in [\pi]$

$[\pi]$ is the interpretation of π in the category of sets and relations; it is a subset of $[\underline{A}]$.

And of course the indexed phase semantics of indexed linear logic enjoys **soundness**: if $\vdash_J A$ then $e_J \in A^\bullet$ (which is a fact in the local phase space $(M_0^J, \perp\!\!\!\perp_J)$).

So, given F a formula of LL, it is a good idea to say that a family $\alpha \in [F]^J$ is coherent (w.r.t. $(M_0, \perp\!\!\!\perp)$) if $e_J \in A^\bullet$ for *some* (equivalently, *any*) A such that $\underline{A} = F$ and $\langle A \rangle = \alpha$.

[We have seen that there is always at least one such A.]

Indeed, for any proof π of $\vdash F$ and any family $\alpha \in [F]^J$ such that

$$\forall j \in J \quad \alpha_j \in [\pi]$$

this family α will be coherent with respect to any indexed phase space.

This is *denotational soundness*.

What about (denotational) completeness?

Amounts to truth-value completeness of indexed phase semantics w.r.t. indexed linear logic.

But... this completeness does not hold for two (interesting) reasons:

- the local phase space at $\emptyset$ is trivial, so that $e_\emptyset \in A^\bullet$ whenever $\mathrm{d}(A) = \emptyset$: this is a kind of *partiality* principle;
- $(!_{\mathrm{Id}}A)^\bullet = (\mathrm{Id}_* A^\bullet)^{\perp\perp}$ so that $e_J \in (A \multimap !_{\mathrm{Id}}A)^\bullet$ whereas $A \multimap !_{\mathrm{Id}}A = A^\perp \parr !_{\mathrm{Id}}A$ is not provable in indexed linear logic.

This latter fact has consequences also on exponential-free formulae of indexed linear logic.

For instance, if $\mathsf{d}(A) = \mathsf{d}(A') = L$ and $\mathsf{d}(B) = \mathsf{d}(B') = R$ with $L \cap R = \emptyset$, then the formula

$$(A \mathbin{\&} B) \otimes (A' \mathbin{\&} B') \multimap (A' \mathbin{\&} B) \otimes (A \mathbin{\&} B')$$

is valid in any indexed phase space, but not provable in indexed linear logic. The reason is that

$$!_l A \otimes !_r B =!_{\mathrm{Id}}(A \mathbin{\&} B) = A \mathbin{\&} B$$

in these spaces (l, r the injections of L, R into $L \cup R$).

Nevertheless :

- there is a nice, more liberal, system of indexed linear logic for which completeness holds,
- this system is equivalent to standard indexed linear logic + these 2 additional principles
- and these two additional principles are incompatible with *totality*.

Conjecture. . .

II
Finiteness spaces: a finitary version of Girard's quantitative semantics

Basic observation: the operations which interpret the linear connectives in $\mathbf{Rel}$ are completely standard if we consider the objects of $\mathbf{Rel}$ as *bases* of vector spaces.

E.g. *in finite dimension*, if $\mathcal{B}$ is a basis of E and $\mathcal{C}$ a basis of F, then $\mathcal{B} \times \mathcal{C}$ is a basis of $E \otimes F$, and in $\mathbf{Rel}$, $X \otimes Y = X \times Y$.

$\mathcal{B}+\mathcal{C}$ (disjoint union) is a basis of $E \oplus F$ and in $\mathbf{Rel}$, $X \oplus Y = X+Y$.

$\mathcal{B}^* \simeq \mathcal{B}$ is a basis of E^*, the dual of E, and in $\mathbf{Rel}$, $X^\perp = X$.

Problem: due to the exponentials, we are obliged to consider infinite sets in $\mathbf{Rel}$, that is, infinite-dimensional vector spaces. Some *topology* must be used. Which one?

Finiteness spaces are a simple concrete solution to this problem.

Let $\mathbf{K}$ be a field and X be a finite set, to be considered as a basis of the vector space $\mathbf{K}^X$. If we identify its dual with $\mathbf{K}^X$, then evaluation for $x \in \mathbf{K}^X$ and $x' \in (\mathbf{K}^X)^* = \mathbf{K}^X$ is given by

$$\langle x, x' \rangle = \sum_{a \in X} x_a x'_a$$

Idea : keep this formula valid when X is infinite.

This effect can be obtained quite simply.

Given $U \subseteq I$, define

$$U^{\perp} = \{u' \subseteq I \mid u \cap u' \text{ finite}\}$$

Just as in phase semantics:

- $U \subseteq U^{\perp\perp}$
- $U \subseteq V \Rightarrow V^{\perp} \subseteq U^{\perp}$
- and $U^{\perp\perp\perp} = U^{\perp}$.

A finiteness space is a pair $X = (|X|, \mathsf{F}(X))$ with

$$\mathsf{F}(X) \subseteq \mathcal{P}(|X|) \quad \text{and} \quad \mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$$

this is a typical "linear logical relation" definition, cf.

- the proof of strong normalization in Girard's original TCS paper on LL, linear "candidats de réductibilité", and a lot of other examples in his work (types in the GoI, more recently, ludics)...
- Loader-Hyland-Tan's double gluing.

The nice feature of this kind of definition is that it gives for free a lot of closure properties.

For instance:

- if $u \subseteq |X|$ is finite then $u \in \mathsf{F}(X)$,
- $u \subseteq v \in \mathsf{F}(X) \Rightarrow u \in \mathsf{F}(X)$,
- $u, v \in \mathsf{F}(X) \Rightarrow u \cup v \in \mathsf{F}(X)$,

indeed, one has to check that $u \cup v \in \mathsf{F}(X)^{\perp\perp}$, that is $(u \cup v) \cap u'$ finite for all $u' \in \mathsf{F}(X)^{\perp}$.

Associated vector space:

$$\mathbf{K}\langle X\rangle = \{x \in \mathbf{K}^{|X|} \mid \mathsf{supp}(x) \in \mathsf{F}(X)\}$$

where $\mathsf{supp}(x) = \{a \in |X| \mid x_a \neq 0\}$.

Vector space because $\mathsf{F}(X)$ containes $\emptyset$ and is closed under finite unions.

Topology: define the neighborhoods of 0 and the translate everywhere.

Each $u' \in \mathsf{F}(X)^{\perp}$ determines a basic neighborhood of 0, namely

$$\{x \in \mathbf{K}\langle X\rangle \mid \mathsf{supp}(x) \cap u' = \emptyset\}$$

Limit cases:

- if $\mathsf{F}(X) = \mathcal{P}(|X|)$ then $\mathbf{K}\langle X\rangle = \mathbf{K}^{|X|}$, $\mathsf{F}(X)^{\perp}$ = finite subsets of $|X|$, product topology (like the Cantor or Baire space)
- if $\mathsf{F}(X)$ = finite subsets of $|X|$ then $\mathsf{F}(X)^{\perp} = \mathcal{P}(|X|)$, discrete topology.

and there is a wide spectrum of possibilities between these two limit cases.

Due to the $\mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$ definition, we get for free that this topology is

- Hausdorff
- and complete (each Cauchy sequence converges).

But in general it is not metrizable (and one gets non metrizable spaces when interpreting LL).

Remarks. These topological vector spaces are particular Lefschetz "linear topological vector spaces" (1942). See also the work of Barr, Blute, Scott.

They are very different from the usual topological vector spaces (Banach, Hilbert. . .): the field is taken with the *discrete* topology and the spaces are *totally disconnected*.

Examples of constructions:

- $|X^\perp| = |X|$ and $\mathsf{F}(X^\perp) = \mathsf{F}(X)^\perp$,
- $|X \otimes Y| = |X| \times |Y|$ and
  $\mathsf{F}(X \otimes Y) = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^{\perp\perp}$

and one shows that in fact

$$w \in \mathsf{F}(X \otimes Y) \text{ iff } w \subseteq u \times v \text{ for some } u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y).$$

Then it turns out that

- $\mathbf{K}\langle X^{\perp}\rangle$ is canonically isomorphic to the topological dual of $\mathbf{K}\langle X\rangle$. A vector $x' \in \mathbf{K}\langle X^{\perp}\rangle$ defines a continuous linear form on $\mathbf{K}\langle X\rangle$ by

$$\langle x, x'\rangle = \sum_{a\in|X|} x_a x'_a \quad \text{finite sum!}$$

- More generally $\mathbf{K}\langle X \multimap Y\rangle$ is canonically isomorphic to the space $\mathcal{L}(X,Y)$ of all continuous linear maps $\mathbf{K}\langle X\rangle \to \mathbf{K}\langle Y\rangle$. (Maps faithfully represented by infinite matrices, exactly as in finite dimension.)

Linear implication: $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$.

Remark: one might expect that the map

$$\begin{array}{rcl} \mathbf{K}\langle X\rangle \times \mathbf{K}\langle X^{\perp}\rangle & \to & \mathbf{K} \\ (x, x') & \mapsto & \langle x, x'\rangle \end{array}$$

is continuous w.r.t. the product topology, but this is not the case (as soon as $|X|$ is infinite).

Multilinear maps satisfy a weaker condition (hypocontinuity).

Exponentials. $|!X|$ is the set of all finite multisets on $|X|$ (the interpretation of "!" in $\mathbf{Rel}$), one has to define $\mathsf{F}(!X)$.

Given $u \subseteq |X|$, define $u^! \subseteq |!X|$ as the set of all finite multisets on u. Then set

$$\mathsf{F}(!X) = \{u^! \mid u \in \mathsf{F}(X)\}^{\perp\perp}$$

and one shows that in fact

$$w \in \mathsf{F}(!X) \text{ iff } w \subseteq u^! \text{ for some } u \in \mathsf{F}(X).$$

Given $x \in \mathbf{K}\langle X\rangle$, define $x^! \in \mathbf{K}^{|!X|}$ by

$$x^!_m = \prod_{a\in|X|} x_a^{m(a)} = x^m$$

where $m \in |!X|$ is seen as a function $m : |X| \to \mathbf{N}$.

Then: supp$(x^!) = ($supp$(x))^!$, hence $x^! \in \mathbf{K}\langle !X\rangle$.

If $X = \{*\}$ then $m \in \mathbf{N}$, $x \in \mathbf{K}$ and x^m is just the standard "x^m".
In general, $m \in |!X|$ is seen as a *multi-exponent*.

So if $A \in \mathbf{K}\langle(!X)^\perp\rangle$ and $x \in \mathbf{K}\langle X\rangle$, the sum

$$\langle x^!, A\rangle = \sum_{m\in|!X|} A_m x^m$$

is finite, and we can see A as a kind of $\mathbf{K}$-valued power series.

Remark: for any given x, the sum is finite, but the number of terms depends on x, so A is not a polynomial in general.

In general, an $f \in \mathcal{L}(!X, Y)$ defines a $\mathbf{K}\langle Y\rangle$-valued power series defined on $\mathbf{K}\langle X\rangle$ by

$$F(x) = f(x^!)\,.$$

Such a power series can be *differentiated*.

This corresponds to new constructions on the exponentials.

Standard structure of the exponentials:

- $!$ is a functor,
- $!$ has a structure of co-monad:
  dereliction $!X \multimap X$ and *digging* $!X \multimap !!X$,
- each $!X$ is a co-monoid:
  weakening $!X \multimap 1$ and *contraction* $!X \multimap !X \otimes !X$.

Differential operations correspond to a new structure on $!X$, in the opposite direction.

Also it strongly uses the possibility of adding morphisms of the same type: remember that in our model, $\mathcal{L}(X, Y)$ is a vector space.

For that reason, in this setting, $\oplus$ and $\&$ are the same operation.

"Differential" structure of the exponential.

We have a *co-dereliction* $\mathsf{d}^0 \in \mathcal{L}(X, !X)$ with matrix

$$\mathsf{d}^0{}_{a,m} = \begin{cases} 1 & \text{if } m = [a] \\ 0 & \text{otherwise} \end{cases}$$

If $f \in \mathcal{L}(!X, Y)$, seen as a power series F from $\mathbf{K}\langle X\rangle$ to $\mathbf{K}\langle Y\rangle$

$$F(x) = f(x^!)\,,$$

then $f \circ \mathsf{d}^0$ (composed as linear continuous maps) is the derivative of F at 0

$$f \circ \mathsf{d}^0 = F'(0) \in \mathcal{L}(X, Y)$$

We have a *co-weakening* $\mathsf{u} \in \mathcal{L}(1, !X)$ with matrix

$$\mathsf{u}_{*,m} = \begin{cases} 1 & \text{if } m = [] \\ 0 & \text{otherwise} \end{cases}$$

If $f \in \mathcal{L}(!X, Y)$, seen as a power series F from $\mathbf{K}\langle X \rangle$ to $\mathbf{K}\langle Y \rangle$, then $f \circ \mathsf{u}$ (composed as linear continuous maps) is the value of F at 0

$$f \circ \mathsf{u} = F(0) \in \mathcal{L}(1, Y) \simeq \mathbf{K}\langle Y \rangle$$

And we have a *co-contraction* $\mathsf{m} \in \mathcal{L}(!X \otimes !X, !X)$ with matrix

$$\mathsf{m}_{(l,r),m} = \begin{cases} \binom{m}{l} & \text{if } m = l + r \quad \text{[A binomial coefficient]} \\ 0 & \text{otherwise} \end{cases}$$

If $f \in \mathcal{L}(!X, Y)$, seen as a power series F from $\mathbf{K}\langle X\rangle$ to $\mathbf{K}\langle Y\rangle$, then $f \circ \mathsf{m} \in \mathcal{L}(!X \otimes !X, Y)$ (composed as linear continuous maps) is the the power series

$$\begin{array}{rcl} G : \mathbf{K}\langle X\rangle \times \mathbf{K}\langle X\rangle & \to & \mathbf{K}\langle Y\rangle \\ (x, y) & \mapsto & F(x + y) \end{array}$$

Combining these morphims, we can for instance define a morphism

$$\mathsf{d} \in \mathcal{L}(!X \otimes X, !X)$$

by composing $!X \otimes \mathsf{d}^0 \in \mathcal{L}(!X \otimes X, !X \otimes !X)$ and $\mathsf{m} \in \mathcal{L}(!X \otimes !X, !X)$.

This (and curryfication) allows to transform a power series F from $\mathbf{K}\langle X\rangle$ to $\mathbf{K}\langle Y\rangle$ into a power series F' from $\mathbf{K}\langle X\rangle$ to $\mathbf{K}\langle X \multimap Y\rangle$, the derivative of F:

compose the corresponding linear map $f \in \mathcal{L}(!X, Y)$ with d.

So $!X$ is not only a co-algebra, but also an algebra, and is equipped with a co-dereliction.

Starting from these observations, we introduced the *differential lambda-calculus* and *differential interaction nets*. Work with L. Regnier.

III
Differential interaction nets

This is an extension of MLL proof-nets (or better, Lafont's interaction nets, for convenience)

- with $\parr$ and $\otimes$ (the MLL part),
- with dereliction, weakening and contraction,
- with *co-dereliction*, *co-weakening* and *co-contraction*
- and without promotion (can be added, but the system already exists without it). In linear logic, promotion is the only rule which introduces a "!", whereas here both co-weakening and co-dereliction will introduce a "!".

Reduction rules are imposed by the vector space semantics.

Without promotion, the calculus is fundamentally finitary (strongly normalizes as soon as a DR-correctness criterion is satisfied), even *without types*.

Can be seen as a linear logic version of various *linearizations* of the lambda-calculus (Boudol's resource calculus, Kfoury's linearization).

Striking feature: the $\parr/\otimes$ symmetry of MLL extends to

$$\begin{array}{rcl}
\text{dereliction } (!X \multimap X) & / & \text{co-dereliction}(X \multimap !X) \\
\text{weakening } (!X \multimap 1) & / & \text{co-weakening}(1 \multimap !X) \\
\text{contraction } (!X \multimap !X \otimes !X) & / & \text{co-contraction } (!X \otimes !X \multimap !X)
\end{array}$$

in the sense that the reduction rules preserve the symmetry between these logical rules.

An interaction net consists of

- A collection of *cells*. Each cell has a *type* which determines its *arity* $\in \mathbb{N}$. The *ports* of a cell c are $(c, 0), \ldots, (c, n)$ where n is the arity of (the type of) c.
- $(c, 0)$ is the *principal port* of c, the others are *auxiliary ports*.
- A collection of *free ports*.
- A collection of *wires*; each wire connects 2 ports (or 0 port: loop) and each port is connected to exactly one wire.

Typing

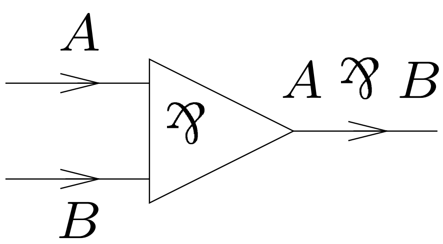
Label oriented wires with LL formulae. For a wire w, the two following typings are identified:

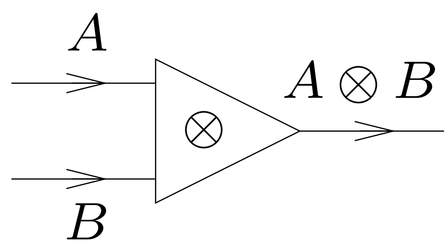
$$\overset{A}{\underset{w}{\longrightarrow}} \qquad\qquad \overset{A^{\perp}}{\underset{w}{\longleftarrow}}$$

The types associated to the cells determine the typing rules.

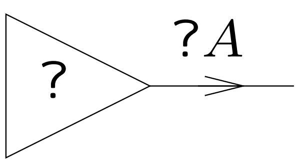
The cells

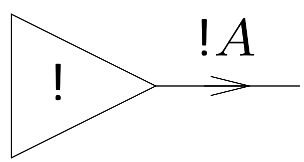
Par/Tensor:





Weakening/Co-weakening:





Dereliction/Co-dereliction:

Contraction/Co-contraction:

These interaction nets will be called **simple nets**.

General nets are formal linear combinations of simple nets *with the same free ports*. In particular, there is a 0 net (for each family of free ports).

Nets can be connected to other nets through free ports: this is "obviously" defined for simple nets and **extended by linearity** to arbitrary nets.

In particular, if we connect a 0 net (with free ports Γ, Λ) to a simple net (with free ports Λ, Δ) through ports Λ, we get a 0 net with free ports Γ, Δ.

Reduction rules

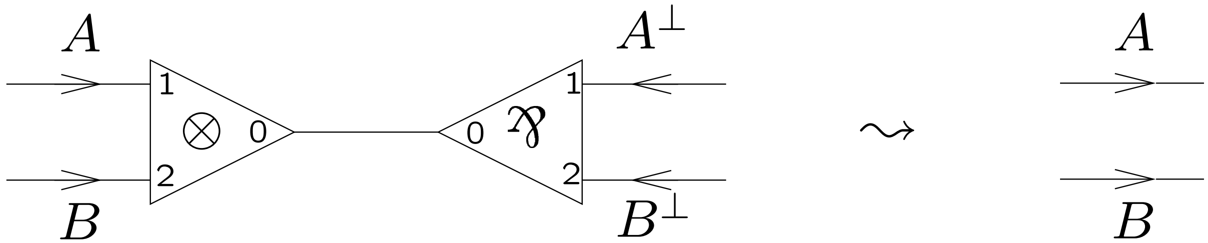
General principle of interaction net: a redex is a simple sub-net consisting of two cells connected through their principal ports.

Such a redex reduces to a net (**here**: not necessarily simple) having the same free ports as the redex.

This net replaces the redex in the context where the redex has been singled out.

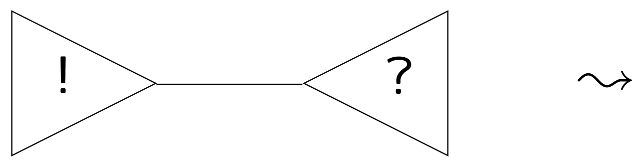
In this kind of setting, confluence is (almost) for free: *no critical pairs*.

The unique MLL reduction:

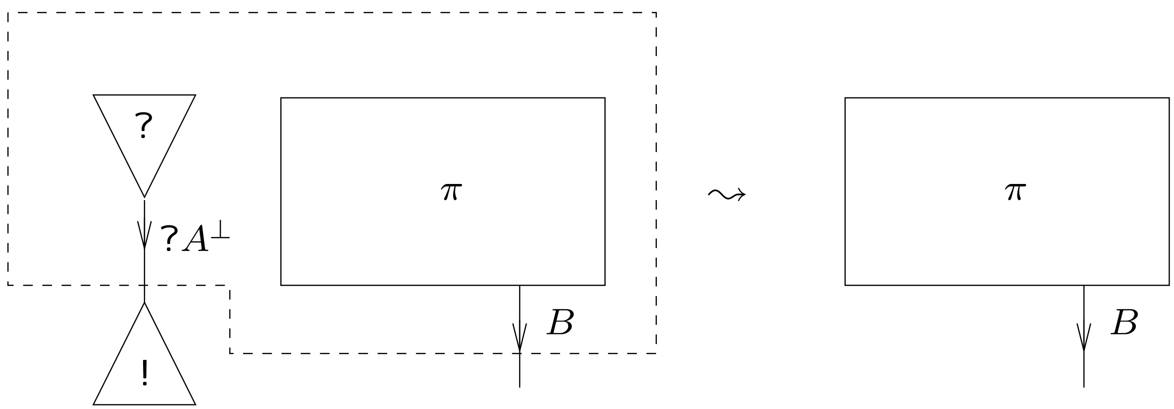


Finitary exponential reductions

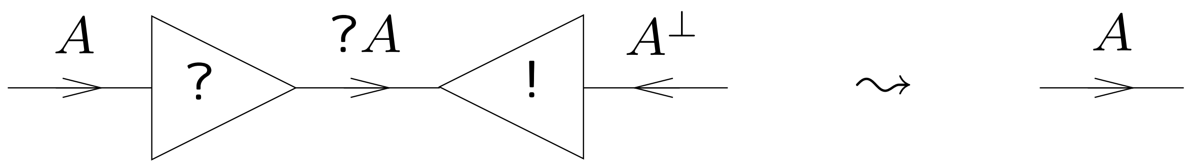
Weakening/Co-weakening

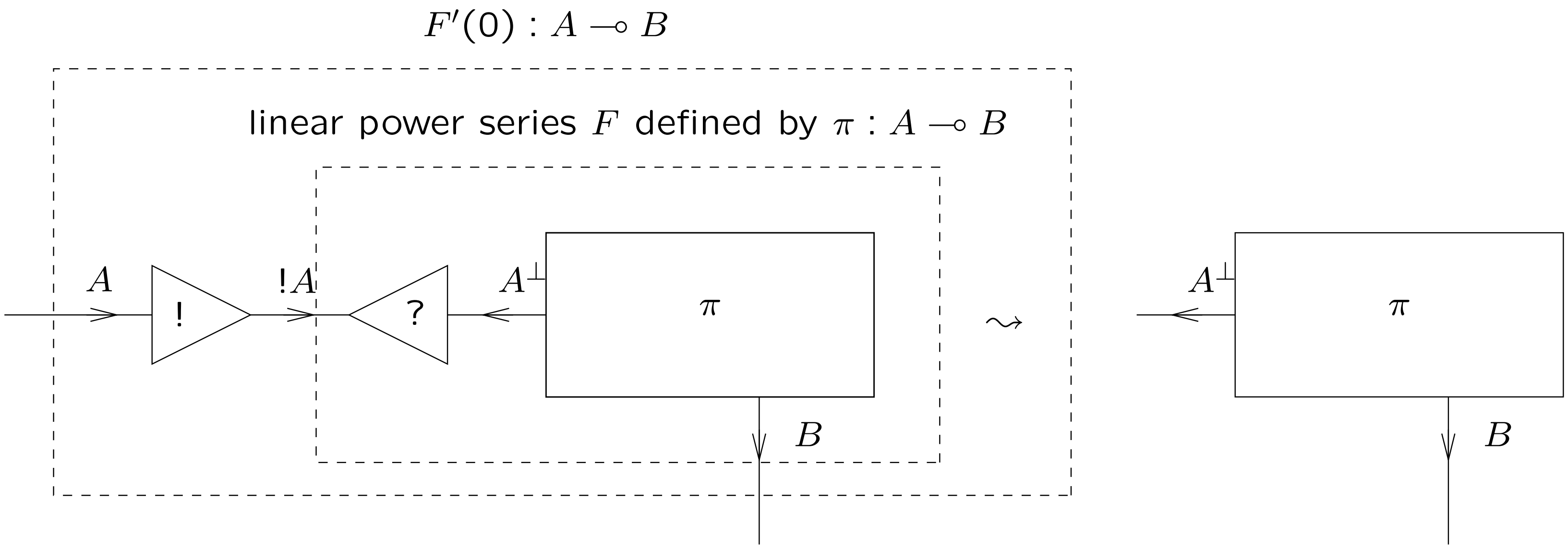


constant power series applied to 0

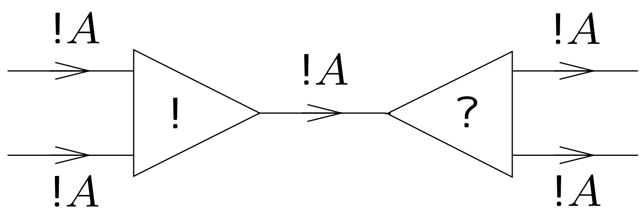


Dereliction/Co-dereliction

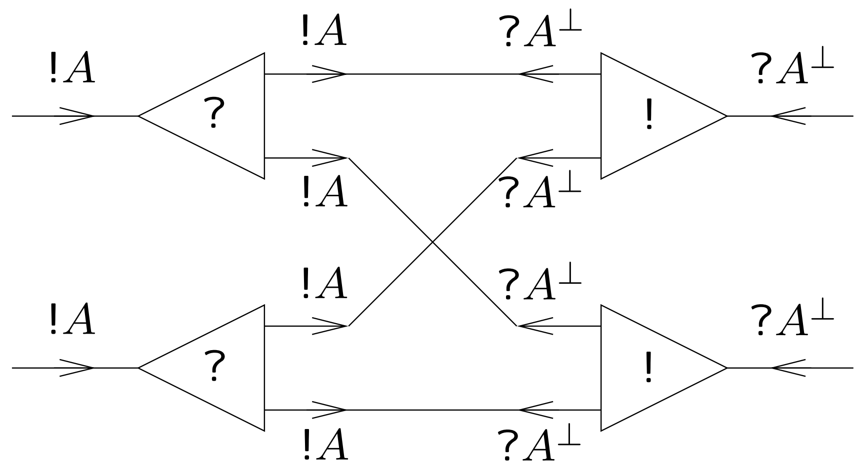


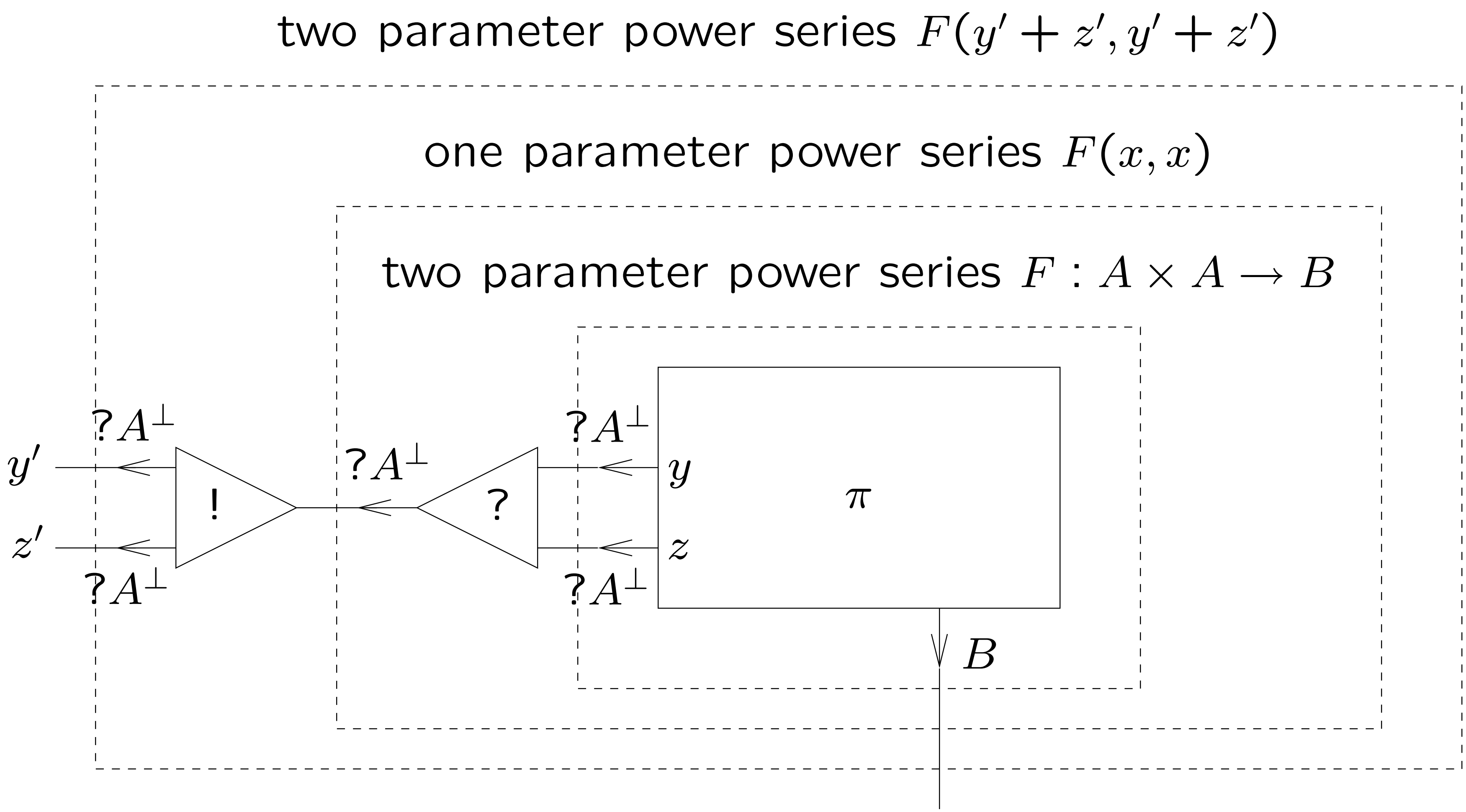


Contraction/Co-contraction



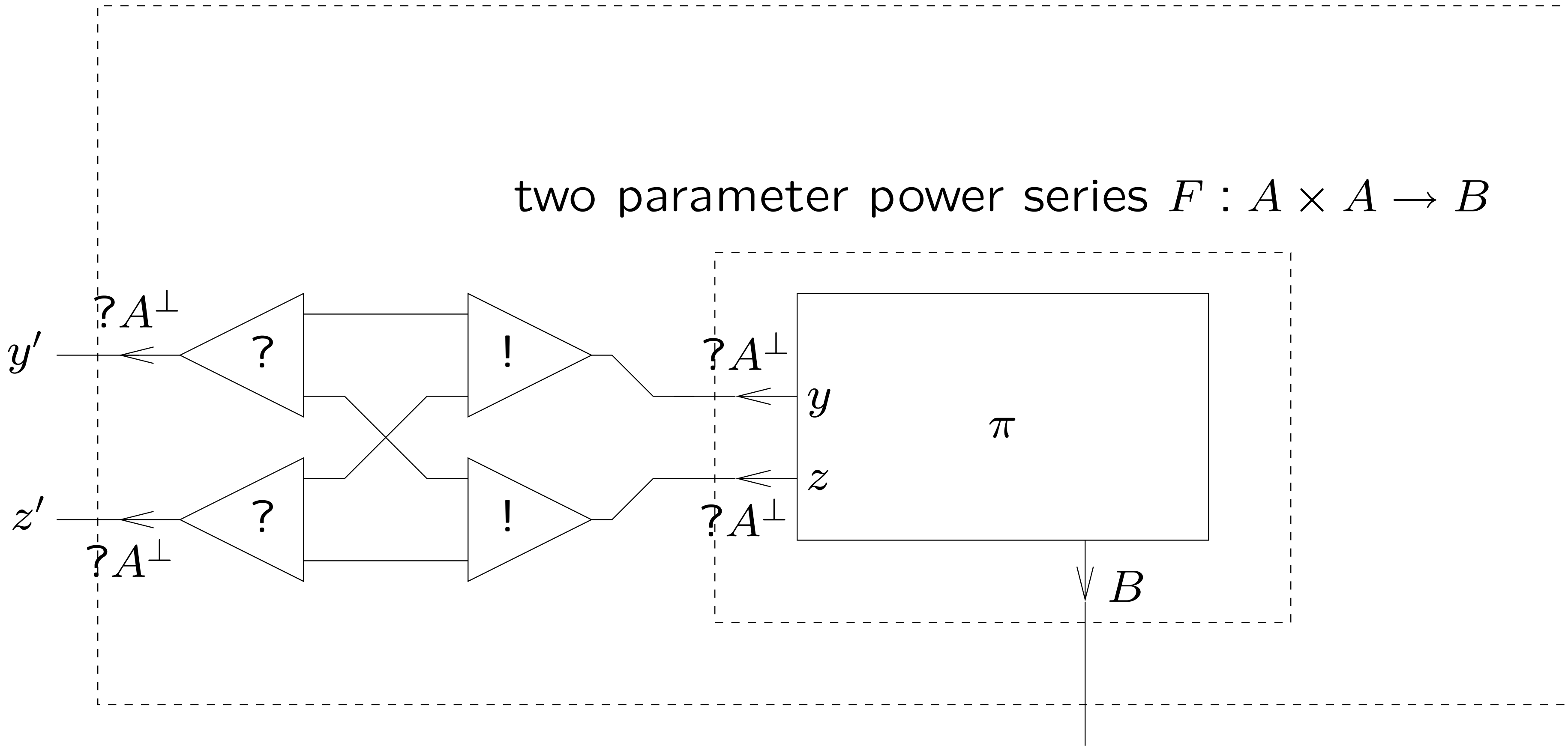
$\rightsquigarrow$



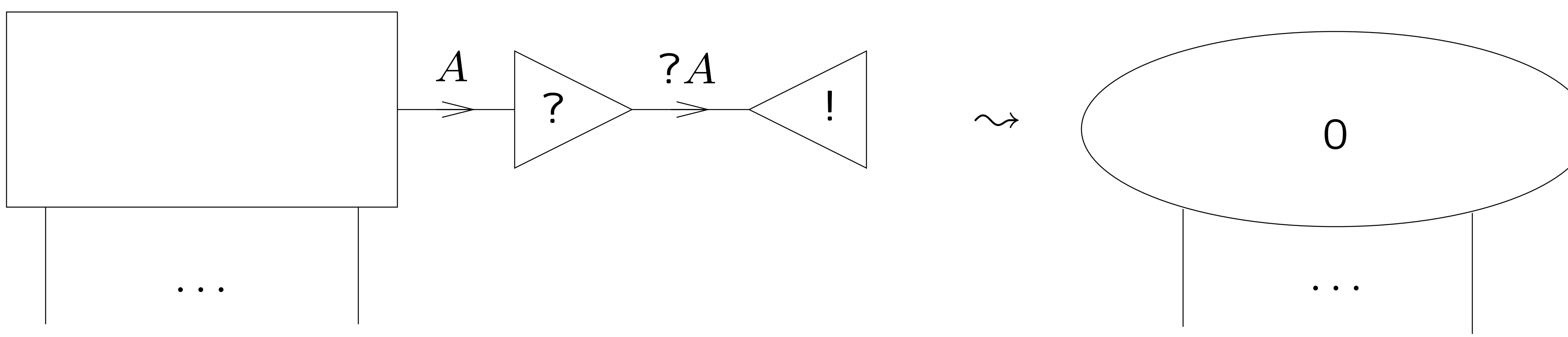

two parameter power series $F(y' + z', y' + z')$
one parameter power series $F(x, x)$
two parameter power series $F : A \times A \to B$
$?A^{\perp}$
y'
z'
$?A^{\perp}$
!
$?A^{\perp}$
?
$?A^{\perp}$
y
z
$?A^{\perp}$
π
B

two parameter power series $F(y' + z', y' + z')$

two parameter power series $F : A \times A \to B$

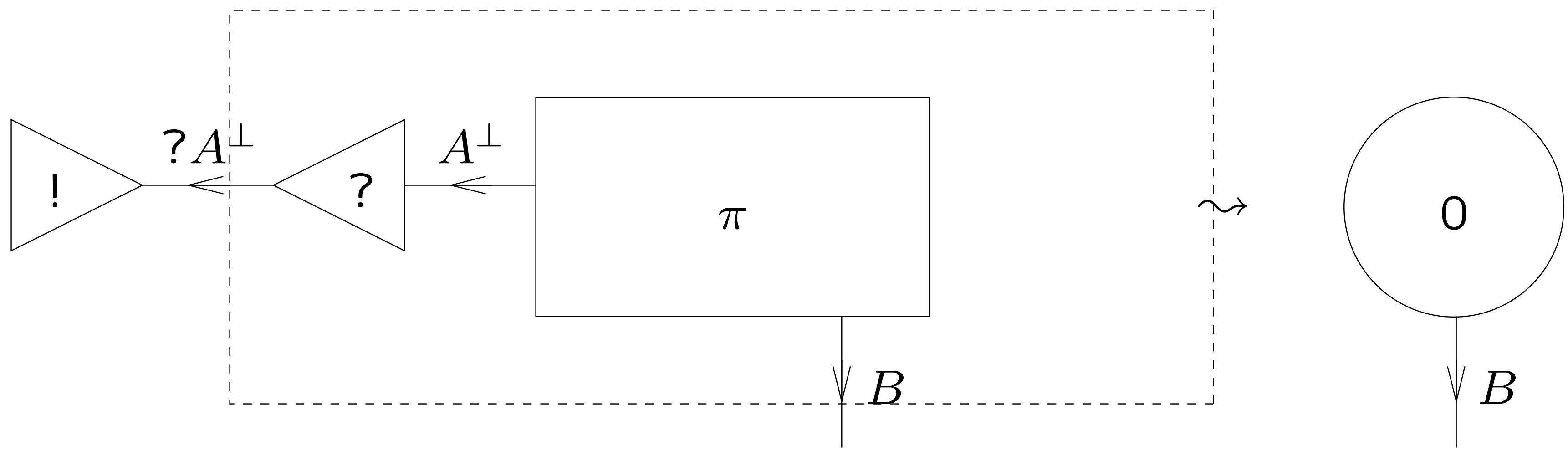


Dereliction/Co-weakening

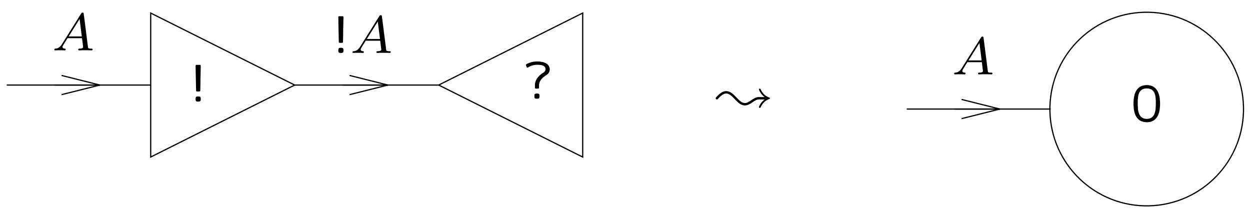


Linear power series applied to 0: the result is 0.

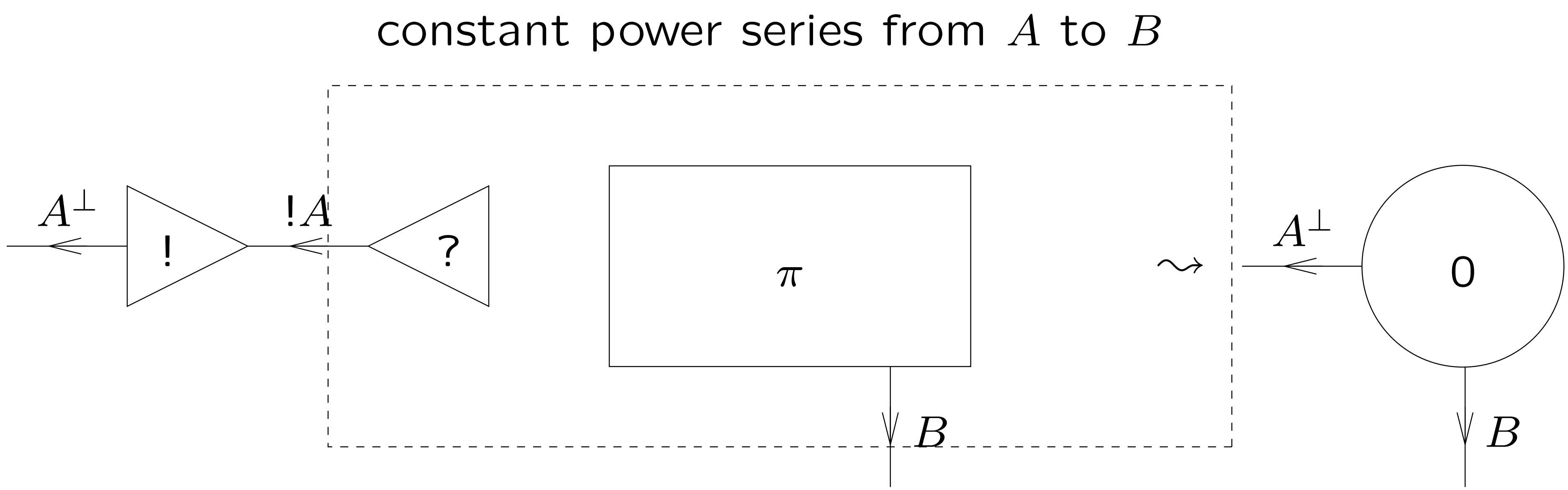
linear power series from A to B



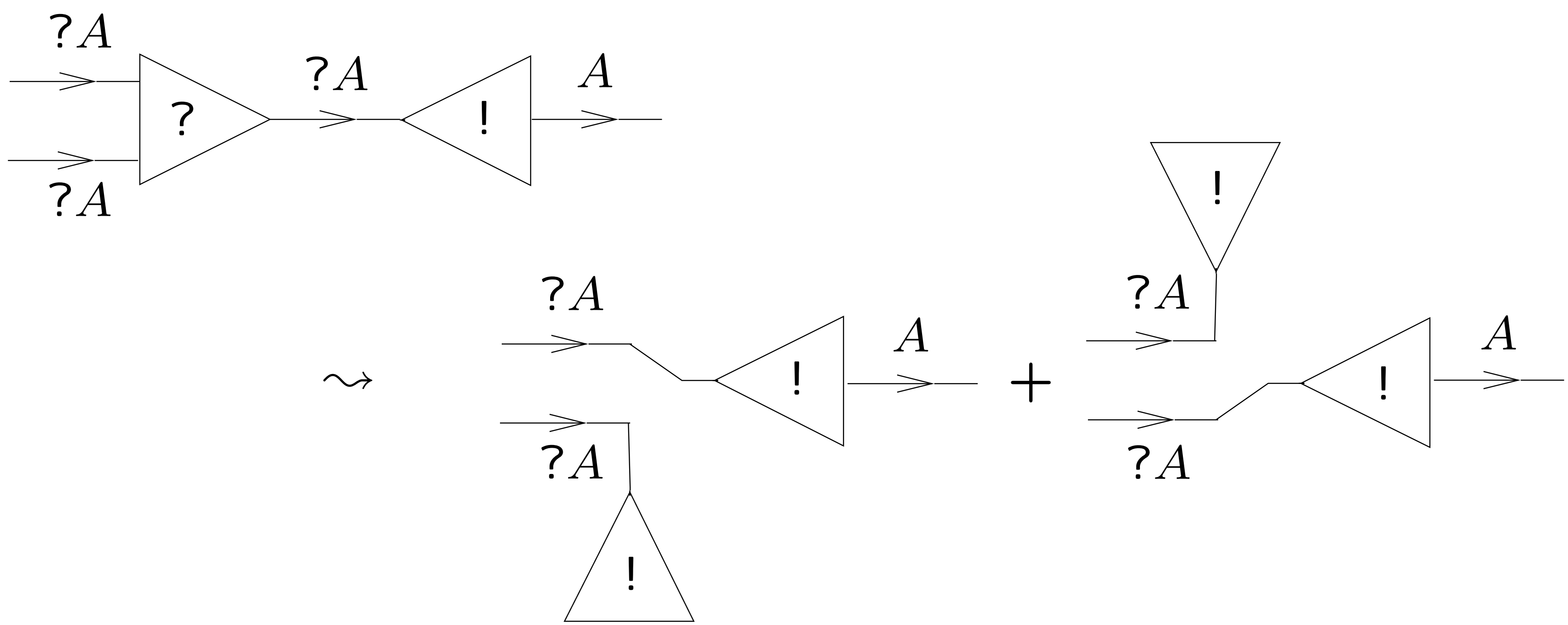
Co-dereliction/Weakening



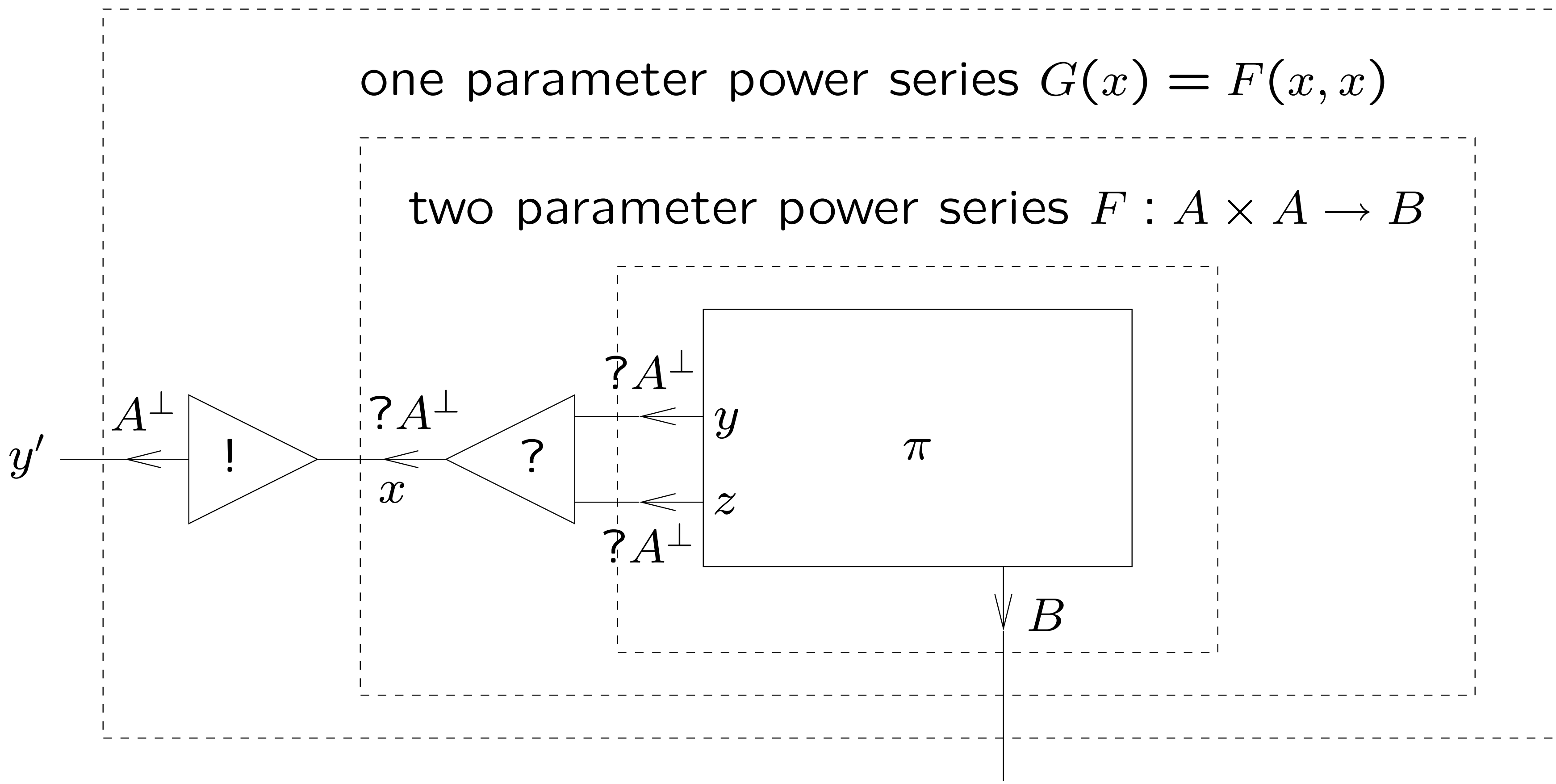
Derivation (at 0) of a *constant* power series: the result is 0.



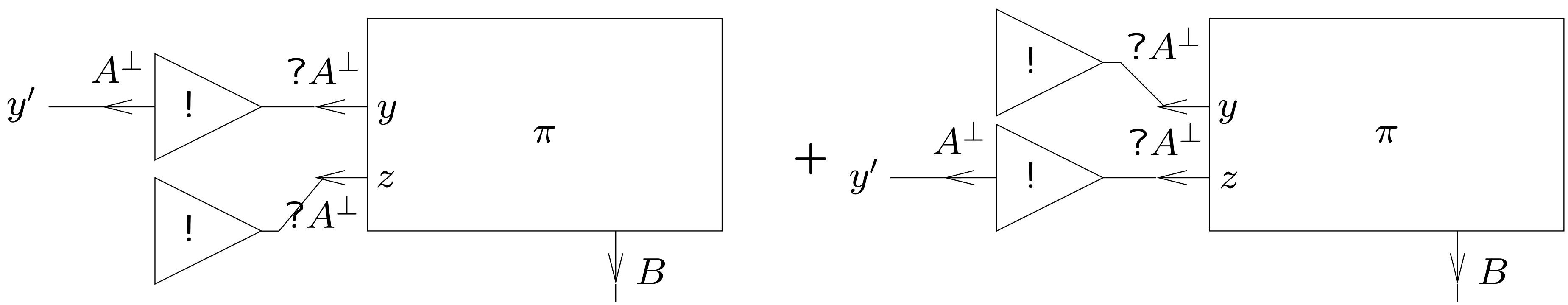
Contraction/Co-dereliction



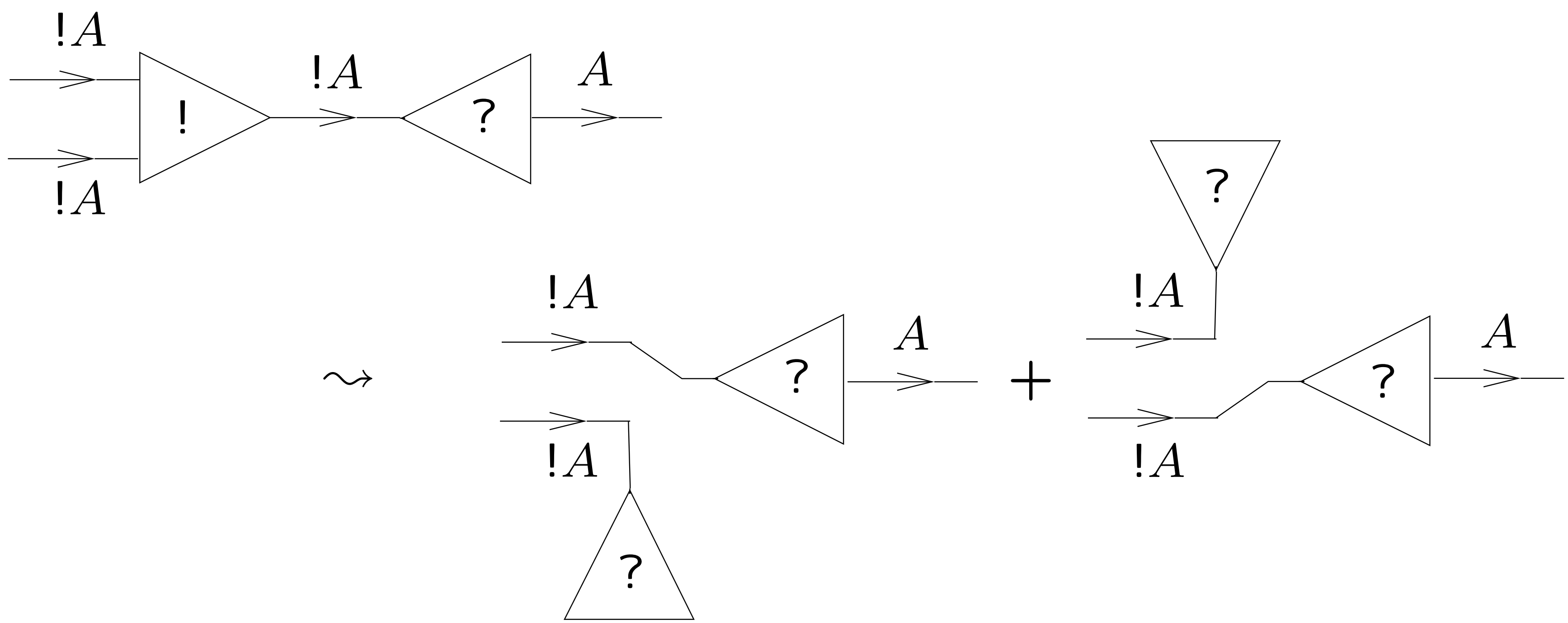
derivative at 0 of G, linear function of y': $G'(0).y'$



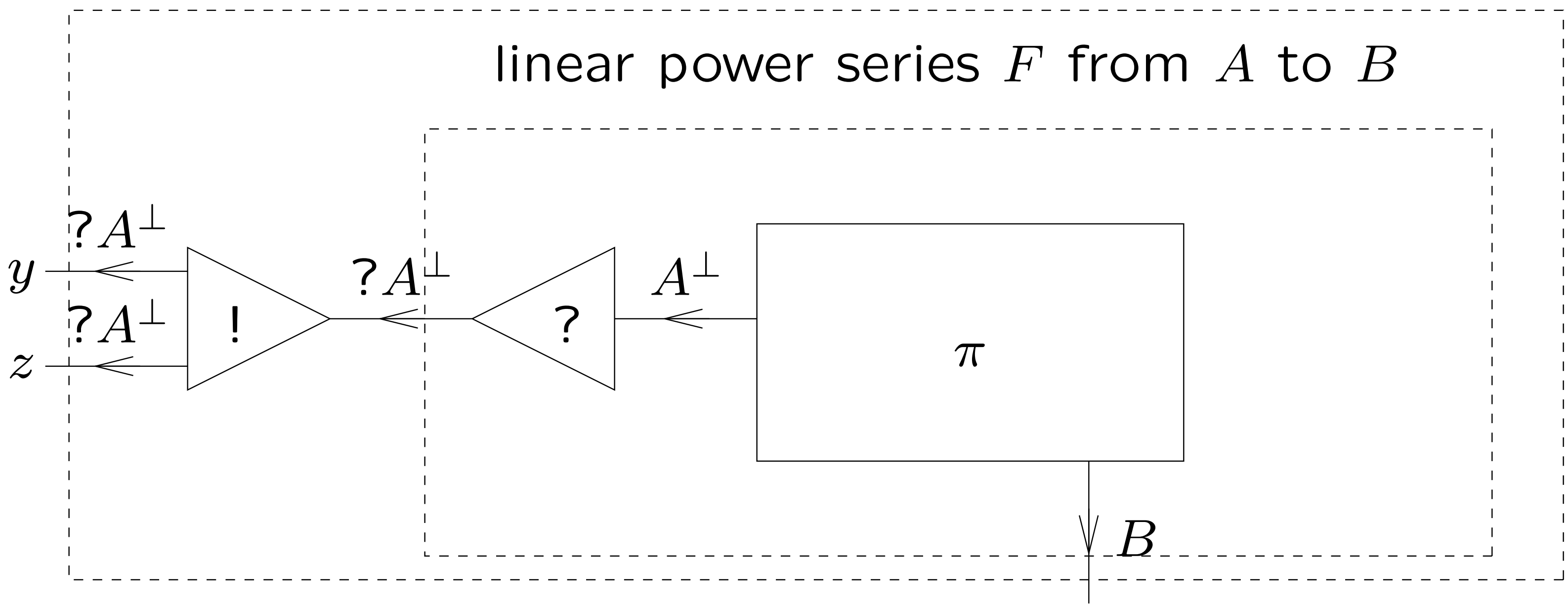
By Leibniz formula, this should be equal to $F'_y(0,0).y' + F'_z(0,0).y'$ (sum of the two partial derivatives), which corresponds to the following sum of simple nets



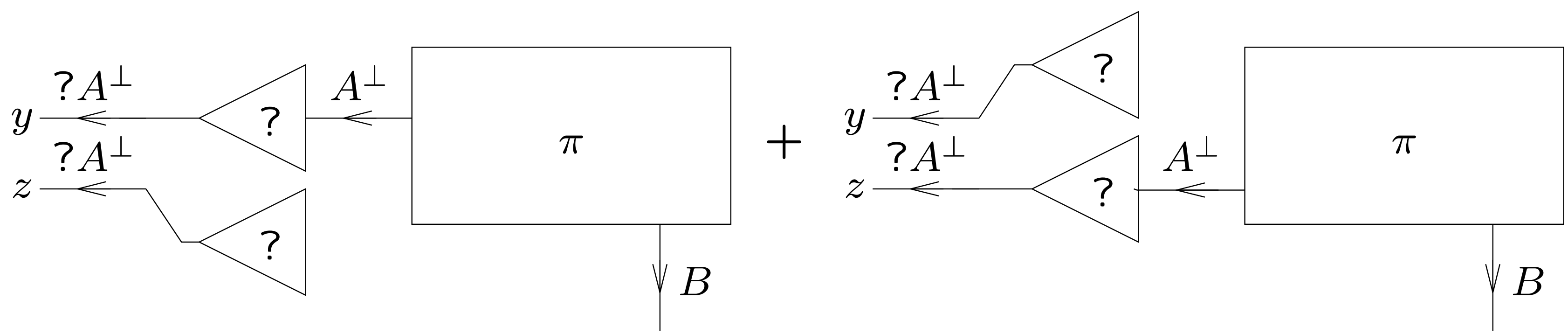
Dereliction/Co-contraction



power series $G(y, z) = F(y + z)$ from $A \times A$ to B



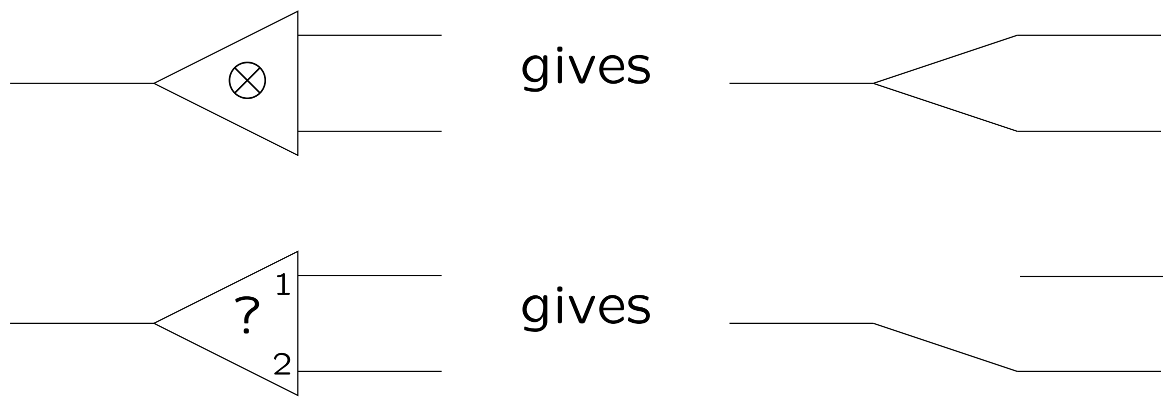
By linearity of F, this should be equal to $F(y) + F(z)$ which corresponds to the following sum of simple nets (observe the role of weakening: y does not appear in $F(z)$ and symmetrically).



Danos-Regnier correctness criterion

A *switching* assigns to each ⅋ cell and each contraction cell a position, 1 or 2.

Given a simple net and a switching we define a graph, e.g.



if this contraction cell is assigned the value 2.

A simple net is *correct* if the graphs induced by all switchings are acyclic (we cannot ask for connectedness). A net is correct if it is a sum of correct simple nets.

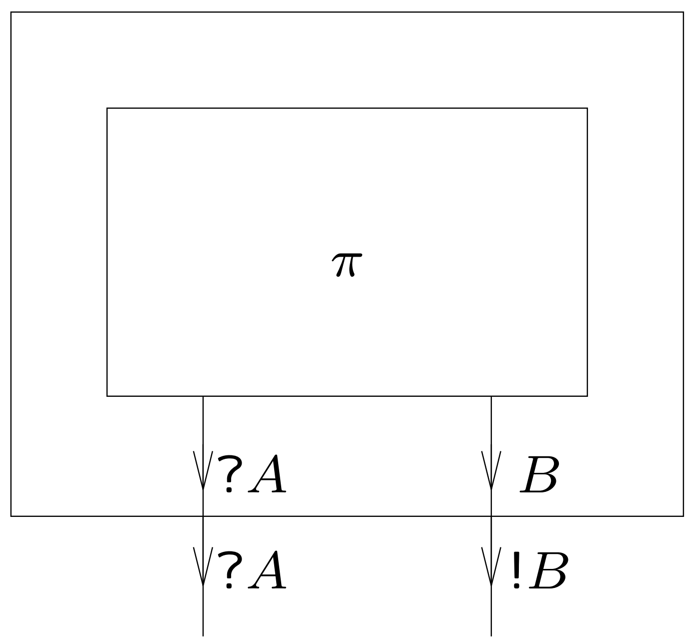
Correctness is preserved under reduction, and correct nets strongly normalize (even in the untyped setting).

What about promotion?

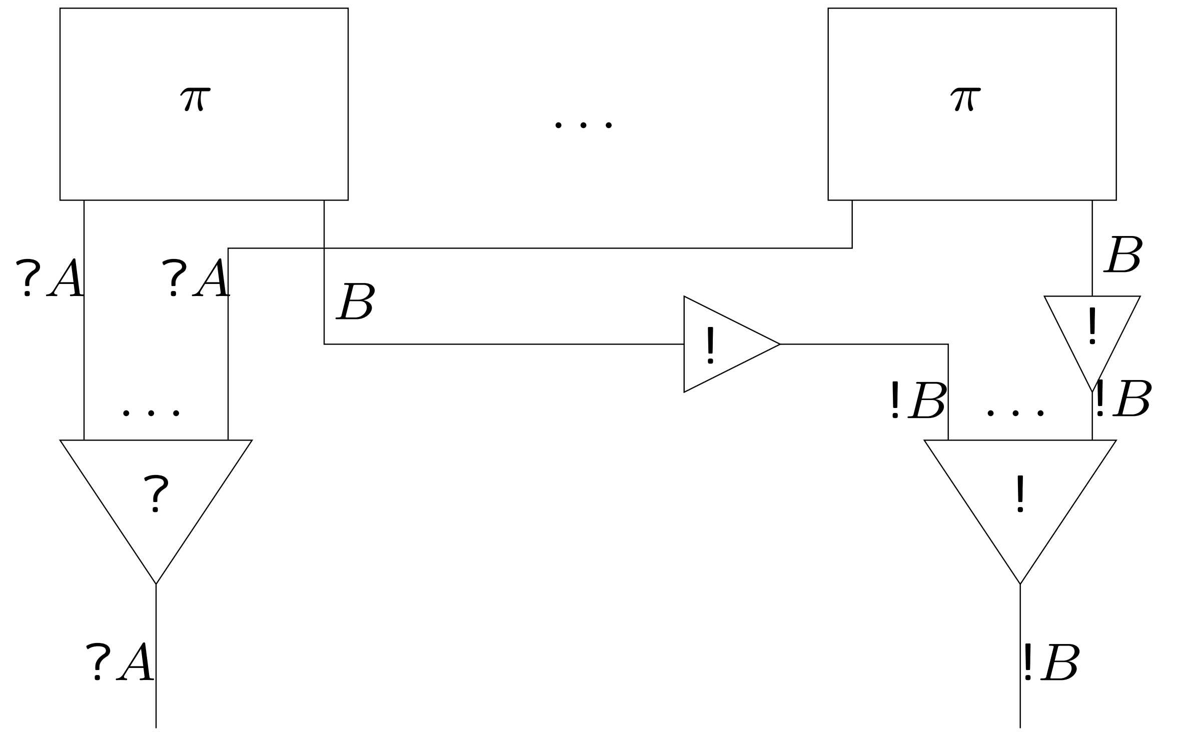
In multiplicative-exponential linear logic, it is the most complicated operation and requires boxes. For instance, if π is a net with conclusions $?A$ and B, it can be promoted producing a net of conclusions $?A$ and $!B$:

as many copies of π as required become available.

Promotion of π:



Can be seen as the infinite sum, with $1/n!$ coefficients, of



[Big triangles are contraction and co-contraction trees.]