Not enough points is enough

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Abstract. Models of the untyped λ-calculus may be defined either as applicative structures satisfying a bunch of first order axioms, known as "λ-models", or as (structures arising from) *any* reflexive object in a cartesian closed category (ccc, for brevity). These notions are tightly linked in the sense that: given a λ-model $\mathcal{A}$, one may define a ccc in which A (the carrier set) is a reflexive object; conversely, if U is a reflexive object in a ccc $\mathbf{C}$, *having enough points*, then $\mathbf{C}(\mathbb{1}, U)$ may be turned into a λ-model.
It is well known that, if $\mathbf{C}$ does not have enough points, then the applicative structure $\mathbf{C}(\mathbb{1}, U)$ is not a λ-model in general.
This paper:
(i) shows that this mismatch can be avoided by choosing appropriately the carrier set of the λ-model associated with U;
(ii) provides an example of an extensional reflexive object $\mathcal{D}$ in a ccc without enough points: the Kleisli-category of the comonad "finite multisets" on **Rel**;
(iii) presents some algebraic properties of the λ-model associated with $\mathcal{D}$ by (i) which make it suitable for dealing with non-deterministic extensions of the untyped λ-calculus.
Keywords: λ-calculus, cartesian closed categories, λ-models, relational model.

1 Introduction

The following citation from [4, Pag. 107] may be used to introduce this paper:

"In this section it will be shown that in arbitrary cartesian closed categories reflexive objects give rise to λ-algebras and to all of them. The λ-models are then those λ-algebras that come from categories "with enough points". The method is due to Koymans [...] and is based on work of Scott."

The point of the present work, in its first part, is to argue that the "enough points" condition can be relaxed, thus obtaining a λ-model from *any* reflexive object in a cartesian closed category (ccc, for short), via a definition of the carrier set of this λ-model which is somehow "more generous" than the canonical one.

Let us recall briefly what λ-algebras and λ-models are, taking for granted the notion of *combinatory algebra* $(A, \cdot, \mathbf{k}, \mathbf{s})$:

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– A λ-*algebra* is a combinatory algebra satisfying the five combinatory axioms of Curry [4, Thm. 5.2.5].
– A λ-*model* is a λ-algebra satisfying the Meyer-Scott (or *weak extensionality*) axiom: $\forall x\ (a \cdot x = b \cdot x) \Rightarrow \boldsymbol{\varepsilon} \cdot a = \boldsymbol{\varepsilon} \cdot b$ where $\boldsymbol{\varepsilon}$ is the combinator $\mathbf{s} \cdot (\mathbf{k} \cdot ((\mathbf{s} \cdot \mathbf{k}) \cdot \mathbf{k}))$.

Of course, weak extensionality is subsumed by extensionality, expressed by the axiom $\forall x\ (a \cdot x = b \cdot x) \Rightarrow a = b$, so the notions of extensional λ-algebra and extensional λ-model coincide.

We claim that any reflexive object in an arbitrary ccc gives rise to a λ-model, by an appropriate choice of the underlying combinatory algebra.

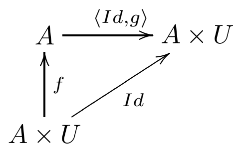
Before going further, let us remark that our construction does not give anything new for the categories of domains generally used to solve the domain inequality $U \Rightarrow U \lhd U$ (see, e.g., [20, 19, 18]), which do have enough points.

In order to illustrate our claim, let us recall the classic construction of the λ-algebra associated with a reflexive object, and point out where the "enough points" hypothesis comes into play. We recall [4, Pag. 108] that an object U has enough points if for all $f, g \in \mathbf{C}(U, U)$, whenever $f \neq g$ there exists a morphism $h \in \mathbf{C}(\mathbb{1}, U)$ such that $f \circ h \neq g \circ h$.

If $\mathcal{U} = (U, \mathrm{Ap}, \lambda)$ is a reflexive object in a small ccc $\mathbf{C}$, and A is an object of $\mathbf{C}$, then $A_{\mathcal{U}} = \mathbf{C}(A, U)$, may be equipped with the following application operator: $a \bullet b = ev \circ \langle \mathrm{Ap} \circ a, b \rangle$. The applicative structure $(A_{\mathcal{U}}, \bullet)$ is canonically endowed with constants $\mathbf{k}, \mathbf{s}$ in such a way that $(A_{\mathcal{U}}, \bullet, \mathbf{k}, \mathbf{s})$ is a λ-algebra, and this algebra is a λ-model if U has enough points.

Hence, the choice $A = \mathbb{1}$ appears as canonical (and it is actually adopted for instance in [4, 1]) if U has enough points, since in that case the Meyer-Scott axiom holds independently from the choice of A.

In the general case, keeping the above definition of application, we can prove weak extensionality if A is chosen in such a way that the following diagram is "quasi-commutative", in the sense expressed by Lemma 1, for some f, g:



The terminal object is no longer a candidate, neither are the finite products U^n, despite the fact that U^n is a retract of U, since the use of an encoding in the definition of f, forces to pair g with the correspondent decoding instead of Id.

We will show in Section 3 that a suitable choice is:

– $A = U^{\mathrm{Var}}$: the countable product of U indexed by the variables of the λ-calculus, whose elements may intuitively be thought of as environments,
– $g = \pi_z$: the projection corresponding to a variable z,
– $f = \eta_z$: an "updating" morphism which leaves unchanged the values of all the variables, but z whose new value is determined by applying π_2.

This approach asks for countable products in $\mathbf{C}$. In practice, this hypothesis does not seem to be very restrictive. Nevertheless, we do claim full generality for this construction. The price to pay is having a quotient over $\bigcup_{n\in\mathbb{N}} \mathbf{C}(U^n, U)$ as carrier set of the λ-model (this approach is sketched in Section 3.2).

Having set up the framework allowing to associate a reflexive object (without enough points) of a ccc with a λ-model, we discuss in Section 5 a paradigmatic example to which it can be applied.

In denotational semantics, ccc's without enough points arise naturally when morphisms are not simply functions, but carry some "intensional" information, like for instance sequential algorithms or strategies in various categories of games [6, 2, 16]. The original motivation for these constructions was the semantic characterization of sequentiality, in the simply typed case. As far as we know, most often the study of reflexive objects in the corresponding ccc's has not been undertaken. Notable exceptions are [11] and [17], where reflexive objects in categories of games yielding the λ-theories $\mathcal{H}^*$ and $\mathcal{B}$, respectively, are defined.

This deserves probably a short digression, from the perspective of the present work: there is of course no absolute need of considering the combinatory algebra associated with a reflexive object, in order to study the λ-theory thereof; it is often a matter of taste whether to use categorical or algebraic notations. What we are proposing here is simply an algebraic counterpart of any categorical model which satisfies weak extensionality.

A framework simpler than game semantics, where reflexive objects cannot have enough points is the following: given the category **Rel** of sets and relations, consider the comonad $\mathcal{M}_f(-)$ of "finite multisets". **MRel**, the Kleisli category of $\mathcal{M}_f(-)$, is a ccc which has been studied in particular as a semantic framework for linear logic [12, 3, 7].

An even simpler framework, based on **Rel**, would be provided by the functor "finite sets" instead of "finite multisets". The point is that the former is not a comonad. Nevertheless, a ccc may eventually be obtained in this case too, via a "quasi Kleisli" construction [15]. Interestingly, from the perspective of the present work, these Kleisli categories over **Rel** are advocated in [15] as the "natural" framework in which standard models of the λ-calculus like Engeler's model, and graph models [5] in general, *should* live.

As a matter of fact, in Section 5 we define a relational version, in **MRel**, of another classical model: Scott's $\mathcal{D}_\infty$. Instead of the inverse limit construction, we get our reflexive object $\mathcal{D}$ by an iterated completion operation similar to the canonical completion of graph models. In this case D is isomorphic to $D \Rightarrow D$ by construction.

Finally, in Section 6 we show that the λ-model $\mathcal{M}_\mathcal{D}$ associated with $\mathcal{D}$ by the construction described above has a rich algebraic structure. In particular, we define two operations of sum and product making the carrier set of $\mathcal{M}_\mathcal{D}$ a commutative semiring, which are left distributive with respect to the application. This opens the way to the interpretation of conjunctive-disjunctive λ-calculi [9] in the relational framework.

2 Preliminaries

To keep this article self-contained, we summarize some definitions and results used in the paper. With regard to the λ-calculus we follow the notation and terminology of [4]. Our main reference for category theory is [1].

2.1 Generalities

Let S be a set. We denote by $\mathcal{P}(S)$ the collection of all subsets of S and we write $A \subset_{\mathrm{f}} S$ if A is a finite subset of S. A *multiset* m over S can be defined as an unordered list $m = [a_1, a_2, \ldots]$ with repetitions such that $a_i \in S$ for all i. For each $a \in S$ the *multiplicity of* a *in* m is the number of occurrences of a in m. Given a multiset m over S, its *support* is the set of elements of S belonging to m. A multiset m is called *finite* if it is a finite list. We write $[]$ for the empty multiset and $m_1 \uplus m_2$ for the union of the multisets m_1 and m_2. The set of all finite multisets over S will be denoted by $\mathcal{M}_f(S)$.

We denote by $\mathbb{N}$ the set of natural numbers. A $\mathbb{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all but a finite number of indices i; σ_i denotes the i-th element of σ. If S is a set, we denote by $\mathcal{M}_f(S)^{(\omega)}$ the set of all quasi-finite $\mathbb{N}$-indexed sequences of multisets over S. We write $*$ for the $\mathbb{N}$-indexed family of empty multisets, i.e., $*$ is the only inhabitant of $\mathcal{M}_f(\emptyset)^{(\omega)}$.

2.2 Cartesian closed categories

Throughout the paper, $\mathbf{C}$ is a small *cartesian closed category* (ccc, for short). Let A, B, C be arbitrary objects of $\mathbf{C}$. We denote by $A \& B$ the *product*[3] of A and B, by $\pi_1 \in \mathbf{C}(A\&B, A)$, $\pi_2 \in \mathbf{C}(A\&B, B)$ the associated *projections* and, given a pair of arrows $f \in \mathbf{C}(C, A)$ and $g \in \mathbf{C}(C, B)$, by $\langle f, g\rangle \in \mathbf{C}(C, A\&B)$ the unique arrow such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. We will write $A \Rightarrow B$ for the *exponential object* and $ev_{AB} \in \mathbf{C}((A\Rightarrow B) \,\&\, A, B)$ for the *evaluation morphism*[4]. Moreover, for any object C and arrow $f \in \mathbf{C}(C \,\&\, A, B)$ we write $\Lambda(f) \in \mathbf{C}(C, A\Rightarrow B)$ for the (unique) morphism such that $ev_{AB} \circ (\Lambda(f) \& Id_A) = f$. Finally, $\mathbb{1}$ denotes the terminal object and $!_A$ the only morphism in $\mathbf{C}(A, \mathbb{1})$. We recall that in every ccc the following equalities hold:

(pair)	$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$	$\Lambda(f) \circ g = \Lambda(f \circ (g \times Id))$	(Curry)
(beta)	$ev \circ \langle \Lambda(f), g\rangle = f \circ \langle Id, g\rangle$	$\Lambda(ev) = Id$	(Id-Curry)

We say that $\mathbf{C}$ has *enough points* if, for all $f, g \in \mathbf{C}(A, B)$, whenever $f \neq g$, there exists a morphism $h \in \mathbf{C}(\mathbb{1}, A)$ such that $f \circ h \neq g \circ h$.

[3] We use the symbol & instead of $\times$ because, in the example we are interested in, the categorical product is the disjoint union. The usual notation is kept to denote the set-theoretical product.

[4] We simply write ev when A and B are clear from the context.

2.3 The pure λ-calculus and its models

The set Λ of λ-terms over a countable set Var of variables is constructed as usual: every variable is a λ-term; if P and Q are λ-terms, then also are PQ and $\lambda z.P$ for each variable z.

It is well known [4, Ch. 5] that there are, essentially, two tightly linked notions of model of λ-calculus. The former is connected with category theory (*categorical models*) and the latter is related to combinatory algebras (λ-*models*).

Categorical models. A *categorical model* of λ-calculus is a *reflexive object* in a ccc $\mathbf{C}$, that is, a triple $\mathcal{U} = (U, \mathrm{Ap}, \lambda)$ such that U is an object of $\mathbf{C}$, and $\lambda \in \mathbf{C}(U \Rightarrow U, U)$ and $\mathrm{Ap} \in \mathbf{C}(U, U \Rightarrow U)$ satisfy $\mathrm{Ap} \circ \lambda = Id_{U \Rightarrow U}$. In this case we write $U \Rightarrow U \lhd U$. When moreover $\lambda \circ \mathrm{Ap} = Id_U$, the model $\mathcal{U}$ is called *extensional*.

In the sequel we always suppose that $\overline{x} = (x_1, \ldots, x_n)$ is a finite ordered sequence of variables without repetitions of length n. Given an arbitrary λ-term M and a sequence $\overline{x}$, we say that $\overline{x}$ is *adequate for* M if $\overline{x}$ contains all the free variables of M. We simply say that $\overline{x}$ is adequate whenever M is clear from the context.

Given a categorical model $\mathcal{U} = (U, \mathrm{Ap}, \lambda)$, for all $M \in \Lambda$ and for all adequate $\overline{x}$, the *interpretation of* M (in $\overline{x}$) is a morphism $|M|_{\overline{x}} \in \mathbf{C}(U^n, U)$ defined by structural induction on M as follows:

- If $M \equiv z$, then $|z|_{\overline{x}} = \pi_i$, if z occurs in i-th position in the sequence $\overline{x}$.
- If $M \equiv PQ$, then by inductive hypothesis we have defined $|P|_{\overline{x}}, |Q|_{\overline{x}} \in \mathbf{C}(U^n, U)$. So we set $|PQ|_{\overline{x}} = ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{x}}, |Q|_{\overline{x}} \rangle \in \mathbf{C}(U^n, U)$.
- If $M \equiv \lambda z.P$, by inductive hypothesis we have defined $|P|_{\overline{x},z} \in \mathbf{C}(U^n \& U, U)$ and so we set $|\lambda z.P|_{\overline{x}} = \lambda \circ \Lambda(|P|_{\overline{x},z})$.

It is routine to check that, if M and N are β-equivalent, then $|M|_{\overline{x}} = |N|_{\overline{x}}$ for all $\overline{x}$ adequate for M and N. If the reflexive object $\mathcal{U}$ is extensional, then $|M|_{\overline{x}} = |N|_{\overline{x}}$ holds as soon as M and N are $\beta\eta$-equivalent.

Combinatory algebras and λ-models An *applicative structure* $\mathcal{A} = (A, \cdot)$ is an algebra where $\cdot$ is a binary operation on A called *application*. We may write it infix as $s \cdot t$, or even drop it and write st. Application associates to the left.

A *combinatory algebra* $\mathcal{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ is an applicative structure for a signature with two constants $\mathbf{k}$ and $\mathbf{s}$, such that $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ for all x, y, and z. See, e.g., [8] for a full treatment.

We call $\mathbf{k}$ and $\mathbf{s}$ the *basic combinators*. In the equational language of combinatory algebras the derived combinators $\mathbf{i}$ and $\boldsymbol{\varepsilon}$ are defined as $\mathbf{i} \equiv \mathbf{skk}$ and $\boldsymbol{\varepsilon} \equiv \mathbf{s}(\mathbf{ki})$. It is not hard to verify that every combinatory algebra satisfies the identities $\mathbf{i}x = x$ and $\boldsymbol{\varepsilon}xy = xy$.

We say that $c \in C$ *represents* a function $f : C \to C$ (and that f is *representable*) if $cz = f(z)$ for all $z \in C$. Two elements $c, d \in C$ are *extensionally* *equal* when they represent the same function in $\mathcal{C}$. For example, c and εc are always extensionally equal.

The axioms of an elementary subclass of combinatory algebras, called *λ-models*, were expressly chosen to make coherent the definition of interpretation of λ-terms (see [4, Def. 5.2.1]). The *Meyer-Scott axiom* is the most important axiom in the definition of a λ-model. In the first-order language of combinatory algebras it becomes: $\forall x \forall y (\forall z (xz = yz) \Rightarrow \varepsilon x = \varepsilon y)$.

The combinator ε becomes an inner choice operator, that makes coherent the interpretation of an abstraction λ-term. A λ-model is said *extensional* if, moreover, we have that $\forall x \forall y (\forall z (xz = yz) \Rightarrow x = y)$.

3 From reflexive objects to λ-models.

In the common belief, probably coming from [4, Prop. 5.5.7], a reflexive object $\mathcal{U}$ in a ccc $\mathbf{C}$ may be turned in a λ-model if, and only if, U has *enough points*, i.e., for all $f, g \in \mathbf{C}(U, U)$, whenever $f \neq g$ there exists a morphism $h \in \mathbf{C}(\mathbb{1}, U)$ such that $f \circ h \neq g \circ h$. This trivially holds if $\mathbf{C}$ has enough points.

In the main result of this section we show that this hypothesis is unnecessary if we choose appropriately the associated λ-model and in Section 5 we will also provide a concrete example.

3.1 Syntactical λ-models

We give now the definition of "syntactical λ-models" [13]. Recall that, by [4, Thm. 5.3.6], λ-models are equal to syntactical λ-models, up to isomorphism.

Given an applicative structure $\mathcal{A}$, we let $Env_{\mathcal{A}}$ be the set of environments ρ mapping the set Var of variables of λ-calculus into A. For every $x \in \mathrm{Var}$ and $a \in A$ we denote by $\rho[x := a]$ the environment ρ' which coincides with ρ, except on x, where ρ' takes the value a.

Definition 1. *A* syntactical λ-model *is a pair* $(\mathcal{A}, [\![-]\!])$ *where,* $\mathcal{A}$ *is an applicative structure and* $[\![-]\!] : \Lambda \times Env_{\mathcal{A}} \to A$ *satisfies the following conditions:*

(i) $[\![z]\!]_\rho = \rho(z)$,
(ii) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \cdot [\![Q]\!]_\rho$,
(iii) $[\![\lambda z.P]\!]_\rho \cdot a = [\![P]\!]_{\rho[z:=a]}$,
(iv) $\rho\restriction_{FV(M)} = \rho'\restriction_{FV(M)} \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$,
(v) $\forall a \in A,\ [\![M]\!]_{\rho[z:=a]} = [\![N]\!]_{\rho[z:=a]} \Rightarrow [\![\lambda z.M]\!]_\rho = [\![\lambda z.N]\!]_\rho$

A syntactical λ-model is extensional *if, moreover,* $\forall a \forall b (\forall x (a \cdot x = b \cdot x) \Rightarrow a = b)$.

Let us fix a reflexive object $\mathcal{U} = (U, \mathrm{Ap}, \lambda)$ in a ccc $\mathbf{C}$ *having countable products*[5]. The set $\mathbf{C}(U^{\mathrm{Var}}, U)$, where Var is the set of the variables of λ-calculus,

[5] Note that this hypothesis is not so restrictive. All the underlying categories of the models present in the literature, e.g., the Scott continuous semantics [20] and its refinements, satisfy this requirement.

can be naturally seen as an applicative structure whose application is defined by $u \bullet v = ev \circ \langle \mathrm{Ap} \circ u, v \rangle$. Moreover, the categorical interpretation $|M|_{\overline{x}}$ of a λ-term M, can be intuitively viewed as a morphism in $\mathbf{C}(U^{\mathrm{Var}}, U)$ only depending from a finite number of variables.

In order to capture this informal idea, we now focus our attention on the set $A_{\mathcal{U}}$ whose elements are the "finitary" morphisms in $\mathbf{C}(U^{\mathrm{Var}}, U)$.

A morphism $f \in \mathbf{C}(U^{\mathrm{Var}}, U)$ is *finitary* if there exist a finite set J of variables, and a morphism $f_J \in \mathbf{C}(U^J, U)$ such that $f = f_J \circ \pi_J$, where π_J denotes the canonical projection of U^{Var} onto U^J. In this case we say that the pair (f_J, J) is *adequate* for f, and we write $(f_J, J) \in \mathrm{Ad}(f)$.

Given two finitary morphisms f, g it is easy to see that if $(f_J, J) \in \mathrm{Ad}(f)$ and $(g_I, I) \in \mathrm{Ad}(g)$, then also $f \bullet g$ is finitary and $((f_J \circ \pi_J) \bullet (g_I \circ \pi_I), J \cup I) \in \mathrm{Ad}(f \bullet g)$.

We are going to show that the applicative structure $\mathcal{A}_{\mathcal{U}} = (A_{\mathcal{U}}, \bullet)$, associated with the reflexive object $\mathcal{U}$, gives rise to a syntactical λ-model $\mathcal{M}_{\mathcal{U}}$ which is extensional if, and only if, $\lambda \circ \mathrm{Ap} = Id_U$. To begin with, let us define this applicative structure.

Definition 2. *Let $\mathcal{U}$ be a reflexive object in a ccc $\mathbf{C}$. The applicative structure associated with $\mathcal{U}$ is defined by $\mathcal{A}_{\mathcal{U}} = (A_{\mathcal{U}}, \bullet)$, where:*

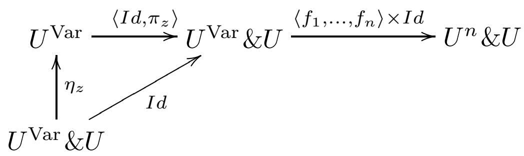
- $A_{\mathcal{U}} = \{f \in \mathbf{C}(U^{\mathrm{Var}}, U) : \exists J \subset_{\mathrm{f}} \mathrm{Var}, \exists f_J \in \mathbf{C}(U^J, U)$ *such that* $f = f_J \circ \pi_J\}$,
- $a \bullet b = ev \circ \langle \mathrm{Ap} \circ a, b \rangle$.

The following technical lemma will be used for defining the syntactical λ-model $\mathcal{M}_{\mathcal{U}}$ associated with $\mathcal{A}_{\mathcal{U}}$.

Lemma 1. *Let $f_1, \ldots, f_n \in A_{\mathcal{U}}$ and $(f'_k, J_k) \in \mathrm{Ad}(f_k)$ for all $1 \leq k \leq n$. Given $z \in \mathrm{Var}$ such that $z \notin \bigcup_{k \leq n} J_k$, and $\eta_z \in \mathbf{C}(U^{\mathrm{Var}} \& U, U^{\mathrm{Var}})$ defined by:*

$$\eta_z^x = \begin{cases} \pi_2 & \text{if } x = z, \\ \pi_x \circ \pi_1 & \text{otherwise,} \end{cases}$$

the following diagram commutes:



Proof. Starting by $(\langle f_1, \ldots, f_n \rangle \times Id) \circ \langle Id, \pi_z \rangle \circ \eta_z$, we get $\langle \langle f_1, \ldots, f_n \rangle \circ \eta_z, \pi_2 \rangle$ via easy calculations. Hence, it is sufficient to prove that $\langle f_1, \ldots, f_n \rangle \circ \eta_z = \langle f_1, \ldots, f_n \rangle \circ \pi_1$. We show that this equality holds componentwise. By hypothesis, we have that, for all $1 \leq k \leq n$, $f_k \circ \eta_z = f'_k \circ \pi_{J_k} \circ \eta_z$. Since $z \notin J_k$, we have that $\pi_{J_k} \circ \eta_z = \pi_{J_k} \circ \pi_1$ (computing componentwise in π_{J_k} and applying the definition of η_z). To conclude, we note that $f'_k \circ \pi_{J_k} \circ \eta_z = f'_k \circ \pi_{J_k} \circ \pi_1 = f_k \circ \pi_1$. $\square$

As a matter of notation, given a sequence $\overline{x}$ of variables and an environment $\rho \in Env_{\mathcal{A}_{\mathcal{U}}}$, we denote by $\rho(\overline{x})$ the morphism $\langle \rho(x_1), \ldots, \rho(x_n) \rangle \in \mathbf{C}(U^{\mathrm{Var}}, U^n)$.

Lemma 2. *For all λ-terms M, environments ρ and sequences $\overline{x}, \overline{y}$ adequate for M, we have that $|M|_{\overline{x}} \circ \rho(\overline{x}) = |M|_{\overline{y}} \circ \rho(\overline{y})$.*

Proof. The proof is by structural induction on M.
If $M \equiv z$, then z occurs in, say, i-th position in $\overline{x}$ and j-th position in $\overline{y}$. Then $|z|_{\overline{x}} \circ \rho(\overline{x}) = \pi_i \circ \rho(\overline{x}) = \rho(z) = \pi_j \circ \rho(\overline{y}) = |z|_{\overline{y}} \circ \rho(\overline{y})$.
If $M \equiv PQ$, then $|PQ|_{\overline{x}} \circ \rho(\overline{x}) = ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{x}}, |Q|_{\overline{x}} \rangle \circ \rho(\overline{x})$. By (pair), this is equal to $ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{x}} \circ \rho(\overline{x}), |Q|_{\overline{x}} \circ \rho(\overline{x}) \rangle$ which is, by inductive hypothesis, $ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{y}} \circ \rho(\overline{y}), |Q|_{\overline{y}} \circ \rho(\overline{y}) \rangle = |PQ|_{\overline{y}} \circ \rho(\overline{y})$.
If $M \equiv \lambda z.N$, then $|\lambda z.N|_{\overline{x}} \circ \rho(\overline{x}) = \lambda \circ \Lambda(|N|_{\overline{x},z}) \circ \rho(\overline{x})$ and by (Curry), we obtain $\lambda \circ \Lambda(|N|_{\overline{x},z} \circ (\rho(\overline{x}) \times Id))$. Let $(\rho_1, J_1) \in \mathrm{Ad}(\rho(x_1)), \dots, (\rho_n, J_n) \in \mathrm{Ad}(\rho(x_n))$. By α-conversion we can suppose that $z \notin \bigcup_{k \le n} J_k$, hence by Lemma 1 we obtain $\lambda \circ \Lambda(|N|_{\overline{x},z} \circ (\rho(\overline{x}) \times Id) \circ \langle Id, \pi_z \rangle \circ \eta_z) = \lambda \circ \Lambda(|N|_{\overline{x},z} \circ \rho[z := \pi_z](\overline{x}, z) \circ \eta_z)$. This is equal, by inductive hypothesis, to $\lambda \circ \Lambda(|N|_{\overline{y},z} \circ \rho[z := \pi_z](\overline{y}, z) \circ \eta_z) = |\lambda z.N|_{\overline{y}}$. □

As a consequence of Lemma 2 we have that the following definition is sound.

Definition 3. $\mathcal{M}_{\mathcal{U}} = (\mathcal{A}_{\mathcal{U}}, [\![-]\!])$, *where* $[\![M]\!]_\rho = |M|_{\overline{x}} \circ \rho(\overline{x})$ *for some adequate sequence* $\overline{x}$.

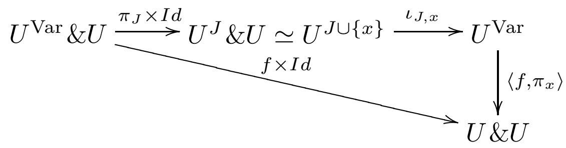
We are going to prove that $\mathcal{M}_{\mathcal{U}}$ is a syntactical λ-model, which is extensional if, and only if, $\mathcal{U}$ is extensional.

For this second property we need another categorical lemma. Remark that the morphism $\iota_{J,x} \in \mathbf{C}(U^{J \cup \{x\}}, U^{\mathrm{Var}})$ defined below is a sort of canonical injection. In particular, the morphism $\lambda \circ \Lambda(Id_U) \circ !_{U^{J \cup \{x\}}}$ does not play any role in the rest of the argument.

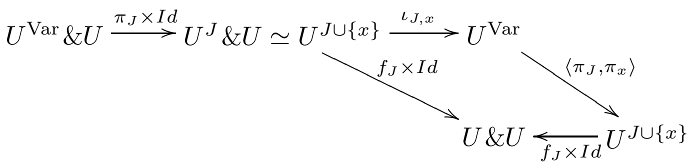
Lemma 3. *Let $f \in A_{\mathcal{U}}$, $(f_J, J) \in \mathrm{Ad}(f)$, $x \notin J$ and $\iota_{J,x}$ defined as follows:*

$$\iota^z_{J,x} = \begin{cases} \pi_z & \text{if } z \in J \cup \{x\}, \\ \lambda \circ \Lambda(Id_U) \circ !_{U^{J \cup \{x\}}} & \text{otherwise.} \end{cases}$$

Then the following diagram commutes:



Proof. Since by hypothesis $f = f_J \circ \pi_J$, this is equivalent to ask that the following diagram commutes, and this is obvious from the definition of $\iota_{J,x}$.



□

Theorem 1. *Let $\mathcal{U}$ be a reflexive object in a ccc* **C**. *Then:*

1) $\mathcal{M}_{\mathcal{U}}$ is a syntactical λ-model,
2) $\mathcal{M}_{\mathcal{U}}$ is extensional if, and only if, $\mathcal{U}$ is.

Proof. 1) In the following $\overline{x}$ is any adequate sequence and the items correspond to those in Definition 1.
(i) $[\![z]\!]_\rho = |z|_{\overline{x}} \circ \rho(\overline{x}) = \pi_z \circ \rho(\overline{x}) = \rho(z)$.
(ii) $[\![PQ]\!]_\rho = |PQ|_{\overline{x}} \circ \rho(\overline{x}) = (|P|_{\overline{x}} \bullet |Q|_{\overline{x}}) \circ \rho(\overline{x}) = ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{x}}, |Q|_{\overline{x}} \rangle \circ \rho(\overline{x})$. By (pair) this is equal to $ev \circ \langle \mathrm{Ap} \circ |P|_{\overline{x}} \circ \rho(\overline{x}), |Q|_{\overline{x}} \circ \rho(\overline{x}) \rangle = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$.
(iii) $[\![\lambda z.P]\!]_\rho \bullet a = (|\lambda z.P|_{\overline{x}} \circ \rho(\overline{x})) \bullet a = ev \circ \langle \mathrm{Ap} \circ \lambda \circ \Lambda(|P|_{\overline{x},z}) \circ \rho(\overline{x}), a \rangle$. Since $\mathrm{Ap} \circ \lambda = Id_{U \Rightarrow U}$ and by applying the rules (Curry) and (beta) we obtain $|P|_{\overline{x},z} \circ (\rho(\overline{x}) \times Id) \circ \langle Id, a \rangle$. Finally, by (pair) we get $|P|_{\overline{x},z} \circ \langle \rho(\overline{x}), a \rangle = [\![P]\!]_{\rho[z:=a]}$.
(iv) Obvious since, by Lemma 2, $[\![M]\!]_\rho = |M|_{\overline{x}} \circ \rho(\overline{x})$ where $\overline{x}$ are exactly the free variables of M.
(v) $[\![\lambda z.M]\!]_\rho = |\lambda z.M|_{\overline{x}} \circ \rho(\overline{x}) = \lambda \circ \Lambda(|M|_{\overline{x},z} \circ (\rho(\overline{x}) \times Id))$. Let $(\rho_1, J_1) \in \mathrm{Ad}(\rho(x_1)), \ldots, (\rho_n, J_n) \in \mathrm{Ad}(\rho(x_n))$. Without loss of generality we can suppose that $z \notin \bigcup_{k \leq n} J_k$. Hence, by Lemma 1 we obtain $\lambda \circ \Lambda(|M|_{\overline{x},z} \circ (\rho(\overline{x}) \times Id) \circ \langle Id, \pi_z \rangle \circ \eta_z)$. By (pair), this is $\lambda \circ \Lambda(|M|_{\overline{x},z} \circ \langle \rho(\overline{x}), \pi_z \rangle \circ \eta_z) = \lambda \circ \Lambda([\![M]\!]_{\rho[z:=\pi_z]} \circ \eta_z)$ which is equal to $\lambda \circ \Lambda([\![N]\!]_{\rho[z:=\pi_z]} \circ \eta_z)$ since, by hypothesis, $[\![M]\!]_{\rho[z:=a]} = [\![N]\!]_{\rho[z:=a]}$ for all $a \in A_{\mathcal{U}}$. It is, now, routine to check that $\lambda \circ \Lambda([\![N]\!]_{\rho[z:=\pi_z]} \circ \eta_z) = [\![\lambda z.N]\!]_\rho$.
2) ($\Rightarrow$) Let $x \in \mathrm{Var}$ and $\pi_x \in \mathbf{C}(U^{\mathrm{Var}}, U)$. For all $a \in A_{\mathcal{U}}$ we have $(\lambda \circ \mathrm{Ap} \circ \pi_x) \bullet a = ev \circ \langle \mathrm{Ap} \circ \lambda \circ \mathrm{Ap} \circ \pi_x, a \rangle = ev \circ \langle \mathrm{Ap} \circ \pi_x, a \rangle = \pi_x \bullet a$. If $\mathcal{M}_{\mathcal{U}}$ is extensional, this implies $\lambda \circ \mathrm{Ap} \circ \pi_x = \pi_x$. Since π_x is an epimorphism, we get $\lambda \circ \mathrm{Ap} = Id_U$.
($\Leftarrow$) Let $a, b \in A_{\mathcal{U}}$, then there exist $(a_J, J) \in \mathrm{Ad}(a)$ and $(b_I, I) \in \mathrm{Ad}(b)$ such that $I = J$. Let us set $\varphi = \iota_{J,x} \circ (\pi_J \times Id)$ where $x \notin J$ and $\iota_{J,x}$ is defined in Lemma 3. Suppose that for all $c \in A_{\mathcal{U}}$ we have $(a \bullet c = b \bullet c)$ then, in particular, $ev \circ \langle \mathrm{Ap} \circ a, \pi_x \rangle = ev \circ \langle \mathrm{Ap} \circ b, \pi_x \rangle$ and this implies that $\langle \mathrm{Ap} \circ a, \pi_x \rangle \circ \varphi = \langle \mathrm{Ap} \circ b, \pi_x \rangle \circ \varphi$. By applying Lemma 3, we get $\langle \mathrm{Ap} \circ a, \pi_x \rangle \circ \varphi = (\mathrm{Ap} \circ a) \times Id$ and $\langle \mathrm{Ap} \circ b, \pi_x \rangle \circ \varphi = (\mathrm{Ap} \circ b) \times Id$. Then $\mathrm{Ap} \circ a = \mathrm{Ap} \circ b$ which implies $\lambda \circ \mathrm{Ap} \circ a = \lambda \circ \mathrm{Ap} \circ b$. We conclude since $\lambda \circ \mathrm{Ap} = Id_U$. □

Note that, by using a particular environment $\hat{\rho}$, it is possible to "recover" the categorical interpretation $|M|_{\overline{x}}$ from the interpretation $[\![M]\!]_\rho$ in the syntactical λ-model. Let us fix the environment $\hat{\rho}(x) = \pi_x$ for all $x \in \mathrm{Var}$. Then $[\![M]\!]_{\hat{\rho}} = |M|_{\overline{x}} \circ \langle \pi_{x_1}, \ldots, \pi_{x_n} \rangle$, i.e., it is the morphism $|M|_{\overline{x}}$ "viewed" as element of $\mathbf{C}(U^{\mathrm{Var}}, U)$.

3.2 Working without countable products

The construction provided in the previous section works if the underlying category **C** has countable products. We remark, once again, that this hypothesis is not really restrictive since all the categories used in the literature in order to obtain models of λ-calculus satisfy this requirement. Nevertheless, there exists an alternative, but less simple and natural, construction to turn a reflexive object $\mathcal{U}$

into a syntactical λ-model $\mathcal{M}'_{\mathcal{U}}$, which does not need this additional hypothesis. We give here the basic ideas of this approach.

Let us consider the set $A = \bigcup_{I \subset_{\mathrm{f}} \mathrm{Var}} \mathbf{C}(U^I, U)$ and the equivalence relation $\sim$ on A defined as follows: if $f \in \mathbf{C}(U^J, U)$ and $g \in \mathbf{C}(U^I, U)$, then $f \sim g$ if, and only if, $f \circ \pi_J = g \circ \pi_I$ where $\pi_J \in \mathbf{C}(U^{I \cup J}, U^J)$ and $\pi_I \in \mathbf{C}(U^{I \cup J}, U^I)$. The candidate for the applicative structure $\mathcal{A}'_{\mathcal{U}}$ associated with $\mathcal{U}$ is the set $A/_{\sim}$ together with a suitable application operator.

We claim that $\mathcal{M}'_{\mathcal{U}} = (\mathcal{A}'_{\mathcal{U}}, [\![-]\!])$, where $[\![-]\!]$ is an appropriate interpretation map, is a syntactical λ-model.

4 A cartesian closed category of sets and relations

It is quite well known [12, 3, 15, 7] that, by endowing the monoidal closed category **Rel** with a suitable comonad, one gets a ccc via the co-Kleisli construction. In this section we present the ccc obtained by using the comonad $\mathcal{M}_f(-)$, without explicitly going through the monoidal structure of **Rel**.

Hence we define directly the category **MRel** as follows:

- The objects of **MRel** are all the sets.
- Given two sets S and T, a morphism from S to T is a relation from $\mathcal{M}_f(S)$ to T, in other words, $\mathbf{MRel}(S,T) = \mathcal{P}(\mathcal{M}_f(S) \times T)$.
- The identity morphism of S is the relation:
$$Id_S = \{([a], a) : a \in S\} \in \mathbf{MRel}(S, S)\,.$$
- Given two morphisms $s \in \mathbf{MRel}(S,T)$ and $t \in \mathbf{MRel}(T,U)$, we define:
$t \circ s = \{(m, c) : \exists (m_1, b_1), \ldots, (m_k, b_k) \in s$ such that
$m = m_1 \uplus \ldots \uplus m_k$ and $([b_1, \ldots, b_k], c) \in t\}$.

It is easy to check that this composition law is associative, and that the identity morphisms defined above are neutral for this composition.

Theorem 2. *The category* **MRel** *is cartesian closed.*

Proof. The terminal object $\mathbb{1}$ is the empty set $\emptyset$, and the unique element of $\mathbf{MRel}(S, \emptyset)$ is the empty relation.

Given two sets S_1 and S_2, their cartesian product $S_1 \& S_2$ in **MRel** is their disjoint union:
$$S_1 \& S_2 = (\{1\} \times S_1) \cup (\{2\} \times S_2)$$
and the projections π_1, π_2 are given by:
$$\pi_i = \{([(i, a)], a) : a \in S_i\} \in \mathbf{MRel}(S_1 \& S_2, S_i), \text{ for } i = 1, 2.$$
It is easy to check that this is actually the cartesian product of S_1 and S_2 in **MRel**; given $s \in \mathbf{MRel}(U, S_1)$ and $t \in \mathbf{MRel}(U, S_2)$, the corresponding morphism $\langle s, t \rangle \in \mathbf{MRel}(U, S_1 \& S_2)$ is given by:
$$\langle s, t \rangle = \{(m, (1, a)) : (m, a) \in s\} \cup \{(m, (2, b)) : (m, b) \in t\}\,.$$

We will consider the canonical bijection between $\mathcal{M}_f(S_1) \times \mathcal{M}_f(S_2)$ and $\mathcal{M}_f(S_1 \,\&\, S_2)$ as an equality, hence we will still denote by (m_1, m_2) the corresponding element of $\mathcal{M}_f(S_1 \& S_2)$.

Given two objects S and T the exponential object $S \Rightarrow T$ is $\mathcal{M}_f(S) \times T$ and the evaluation morphism is given by:

$$ev_{ST} = \{((([(m,b)],m),b) : m \in \mathcal{M}_f(S) \text{ and } b \in T\} \in \mathbf{MRel}((S \Rightarrow T) \& S, T)\,.$$

Again, it is easy to check that in this way we defined an exponentiation. Indeed, given any set U and any morphism $s \in \mathbf{MRel}(U \,\&\, S, T)$, there is exactly one morphism $\Lambda(s) \in \mathbf{MRel}(U, S \Rightarrow T)$ such that:

$$ev_{ST} \circ \langle \Lambda(s), Id_S \rangle = s.$$

where $\Lambda(s) = \{(p,(m,b)) : ((p,m),b) \in s\}$. $\square$

Here, the points of an object S, i.e., the elements of $\mathbf{MRel}(\mathbb{1}, S)$, are relations between $\mathcal{M}_f(\emptyset)$ and S. These are, up to isomorphism, the subsets of S.

In the next section we will present an extensional model of λ-calculus living in **MRel** which is a strongly non extensional ccc in the following sense. It is, in fact, possible to prove not only that **MRel** has not enough points but that there exists no object $U \neq \mathbb{1}$ of **MRel** having enough points.

In fact we can always find $t_1, t_2 \in \mathbf{MRel}(U, U)$ such that $t_1 \neq t_2$ and, for all $s \in \mathbf{MRel}(\mathbb{1}, U)$, $t_1 \circ s = t_2 \circ s$. Recall that, by definition of composition, $t_1 \circ s = \{([], b) : \exists a_1, \dots, a_n \in U \; ([], a_i) \in s \quad ([a_1, \dots, a_n], b) \in t_1\} \in \mathbf{MRel}(\mathbb{1}, U)$, and similarly for $t_2 \circ s$. Hence it is sufficient to choose $t_1 = \{(m_1, b)\}$ and $t_2 = \{(m_2, b)\}$ such that m_1, m_2 are different multisets with the same support.

5 An extensional relational model of λ-calculus

In this section we build a reflexive object in **MRel**, which is extensional by construction.

5.1 Constructing an extensional reflexive object.

We build a family of sets $(D_n)_{n \in \mathbb{N}}$ as follows[6]:

- $D_0 = \emptyset$,
- $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)}$.

Since the operation $S \mapsto \mathcal{M}_f(S)^{(\omega)}$ is monotonic on sets, and since $D_0 \subseteq D_1$, we have $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$. Finally, we set $D = \bigcup_{n \in \mathbb{N}} D_n$.

So we have $D_0 = \emptyset$ and $D_1 = \{*\} = \{([], [], \dots)\}$. The elements of D_2 are quasi-finite sequences of multisets over a singleton, i.e., quasi-finite sequences of natural numbers. More generally, an element of D can be represented as a finite tree which alternates two kinds of layers:

[6] Note that, in greater generality, we can start from a set A of "atoms" and take: $D_0 = \emptyset$, $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)} \times A$. Nevertheless the set of atoms A is not essential to produce a non-trivial model of λ-calculus.

- ordered nodes (the quasi-finite sequences), where immediate subtrees are indexed by a possibly empty finite set of natural numbers,
- unordered nodes where subtrees are organised in a *non-empty* multiset.

In order to define an isomorphism in **MRel** between D and $D \Rightarrow D$ (which is equal to $\mathcal{M}_f(D) \times D$) just remark that every element $\sigma \in D$ stands for the pair $(\sigma_0, (\sigma_1, \sigma_2 ...))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_f(D)$, we write $m \cdot \sigma$ for the element $\tau \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_f(D) \times D$ and D, and hence an isomorphism in **MRel** as follows:

Proposition 1. *The triple* $\mathcal{D} = (D, \mathrm{Ap}, \lambda)$ *where:*
- $\lambda = \{([(m, \sigma)], m \cdot \sigma) : m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D \Rightarrow D, D)$,
- $\mathrm{Ap} = \{([m \cdot \sigma], (m, \sigma)) : m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D, D \Rightarrow D)$,

is an extensional categorical model of λ*-calculus.*

Proof. It is easy to check that $\lambda \circ \mathrm{Ap} = Id_D$ and $\mathrm{Ap} \circ \lambda = Id_{D \Rightarrow D}$. □

5.2 Interpreting the untyped λ-calculus in $\mathcal{D}$

In Section 2.3, we have recalled how a λ-term is interpreted when a reflexive object is given, in any ccc. We provide the result of the corresponding computation, when it is performed in the present structure $\mathcal{D}$.

Given a λ-term M and a sequence $\overline{x}$ of length n, which is adequate for M, the interpretation $|M|_{\overline{x}}$ is an element of $\mathbf{MRel}(D^n, D)$, where $D^n = D \,\&\, ... \,\&\, D$, i.e., a subset of $\mathcal{M}_f(D)^n \times D$. This set is defined by structural induction on M.

- $|x_i|_{\overline{x}} = \{(([], \dots, [], [\sigma], [], \dots, []), \sigma) : \sigma \in D\}$, where the only non-empty multiset stands in i-th position.
- $|NP|_{\overline{x}} = \{((m_1, \dots, m_n), \sigma) : \exists k \in \mathbb{N}$
  $\exists (m_1^j, \dots, m_n^j) \in \mathcal{M}_f(D)^n$ for $j = 0...k$
  $\exists \sigma_1, \dots, \sigma_k \in D$ such that
  $m_i = m_i^0 \uplus \dots \uplus m_i^k$ for $i = 1...n$
  $((m_1^0, \dots, m_n^0), [\sigma_1, \dots, \sigma_k] \cdot \sigma) \in |N|_{\overline{x}}$
  $((m_1^j, \dots, m_n^j), \sigma_j) \in |P|_{\overline{x}}$ for $j = 1...k\}$
- $|\lambda z.P|_{\overline{x}} = \{((m_1, \dots, m_n), m \cdot \sigma) : ((m, m_1, \dots, m_n), \sigma) \in |P|_{\overline{x}, z}\}$, where we assume that z does not occur in $\overline{x}$.

Since $\mathcal{D}$ is extensional, if $M =_{\beta\eta} N$ then M and N have the same interpretation in the model. Note that if M is a closed λ-terms then it is simply interpreted, in the empty sequence, by a subset of D. If M is moreover a solvable term, i.e., if it is β-convertible to a term of the shape $\lambda x_1 \dots x_n . x_i M_1 \cdots M_k$ $(n, k \geq 0)$, then its interpretation is non-empty. It is quite clear, in fact, that $[] \cdots [] \cdot [*] \cdot * \in |M|$ (where $[*]$ stands in i-th position).

6 Modelling non-determinism

Since **MRel** has countable products, the construction given in Section 3.1 provides an applicative structure $\mathcal{A}_{\mathcal{D}} = (A_{\mathcal{D}}, \bullet)$, whose elements are the finitary morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$, and the associated λ-model $\mathcal{M}_{\mathcal{D}} = (\mathcal{A}_{\mathcal{D}}, [\![-]\!])$. This λ-model is extensional by Theorem 1(2).

We are going to define two operations of sum and product on $A_{\mathcal{D}}$; in order to show easily that these operations are well defined, we provide a characterization of the finitary elements of $\mathbf{MRel}(D^{\mathrm{Var}}, D)$.

Proposition 2. *Let $f \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$ and $J \subset_{\mathrm{f}} \mathrm{Var}$. Then there exists f_J such that $(f_J, J) \in \mathrm{Ad}(f)$ if, and only if, for all $(m, \sigma) \in f$ and for all $x \notin J$, $\pi_x(m) = []$.*

Proof. Straightforward.

Hence, the union of finitary elements is still a finitary element. As a matter of notation, we will write $a \oplus b$ for $a \cup b$.

We now recall the definition of semilinear applicative structure given in [10].

Definition 4. *A* semilinear applicative structure *is a pair $((A, \cdot), +)$ such that:*

(i) $(A, \cdot)$ *is an applicative structure.*
(ii) $+ : A^2 \to A$ *is an idempotent, commutative and associative operation.*
(iii) $\forall x, y, z \in A \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

Straightforwardly, the union operation makes $\mathcal{A}_{\mathcal{D}}$ semilinear.

Proposition 3. $(\mathcal{A}_{\mathcal{D}}, \oplus)$ *is a semilinear applicative structure.*

Moreover, the syntactic interpretation of Definition 1 may be extended to the non-deterministic λ-calculus $\Lambda_{\oplus}$ of [10], by stipulating that $[\![M \oplus N]\!]_{\rho} = [\![M]\!]_{\rho} \oplus [\![N]\!]_{\rho}$. Hence, we get that $(\mathcal{A}_{\mathcal{D}}, \oplus, [\![-]\!])$ is an extensional syntactical model of $\Lambda_{\oplus}$ in the sense of [10]. The operation $\oplus$ can be seen intuitively as a non-deterministic choice.

We define another binary operation on $A_{\mathcal{D}}$, which can be thought of as parallel composition.

Definition 5.

- *Given $\sigma, \tau \in D$, we set $\sigma \odot \tau = (\sigma_1 \uplus \tau_1, \ldots, \sigma_n \uplus \tau_n, \ldots)$.*
- *Given $a, b \in A_{\mathcal{D}}$, we set $a \odot b = \{(m_1 \uplus m_2, \sigma \odot \tau) : (m_1, \sigma) \in a, (m_2, \tau) \in b\}$.*

Once again, it is easy to see that $\odot$ produces finitary elements when applied to finitary elements.

Note that $\mathcal{A}_{\mathcal{D}}$, equipped with $\odot$, is *not* a semilinear applicative structure, simply because the operator $\odot$ is not idempotent. Nevertheless, the left distributivity with respect to the application is satisfied.

Proposition 4. *For all $a, b, c \in A_{\mathcal{D}}$, $(a \odot b) \bullet c = (a \bullet c) \odot (b \bullet c)$.*

Proof. Straightforward.

The units of the operations $\oplus$ and $\odot$ are $0 = \emptyset$ and $1 = \{([], *)\}$, respectively; $(A_{\mathcal{D}}, \oplus, 0)$ and $(A_{\mathcal{D}}, \odot, 1)$ are commutative monoids. Moreover 0 annihilates $\odot$, and multiplication distributes over addition. Summing up, the following proposition holds.

Proposition 5.

- $(A_{\mathcal{D}}, \oplus, \odot, 0, 1)$ *is a commutative semiring.*
- $\oplus$ *and* $\odot$ *are left distributive over* $\bullet$.
- $\oplus$ *is idempotent.*

7 Conclusions and Further works

We have proposed a general method for getting a λ-model out of a reflexive object of a ccc, which does not rely on the fact that the object has enough points. We have applied this construction to an extensional reflexive object $\mathcal{D}$ of **MRel**, the Kleisli category of the comonad "finite multisets" on **Rel**, and showed some algebraic properties of the resulting λ-model $\mathcal{M}_{\mathcal{D}}$. A first natural question about $\mathcal{M}_{\mathcal{D}}$ concerns its theory. We know that it is extensional, and that $\mathcal{M}_{\mathcal{D}}$ can be "stratified" following the construction of $D = \bigcup_{n\in\mathbb{N}} D_n$ given in Section 5.1. Not surprisingly, the theory of $\mathcal{M}_{\mathcal{D}}$ turns out to be $\mathcal{H}^*$, the maximal consistent sensible λ-theory. In a forthcoming paper, we show how the proof method based on the approximation theorem, due to Hyland [14], can be adapted to all suitably defined "stratified λ-models" in order to prove that their theory is $\mathcal{H}^*$.

Proposition 5 shows that $\mathcal{M}_{\mathcal{D}}$ has a quite rich algebraic structure. In order to interpret conjunctive-disjunctive λ-calculi, endowed with both "non-deterministic choice" and "parallel composition", a notion of λ-lattice have been introduced in [9]. It is interesting to notice that our structure $(A_{\mathcal{D}}, \subseteq, \bullet, \oplus, \odot)$ does not gives rise to a real λ-lattice essentially because $\odot$ is not idempotent. Roughly speaking, this means that in the model $\mathcal{M}_{\mathcal{D}}$ of the conjunctive-disjunctive calculus $[\![M||M]\!] \neq [\![M]\!]$, i.e., that the model is "resource sensible". We aim to investigate full abstraction results for must/may semantics in $\mathcal{M}_{\mathcal{D}}$.

A concluding remark: for historical reasons, most of the work on models of untyped λ-calculus, and its extensions, has been carried out in subcategories of **CPO**. *A posteriori*, we can propose two motivations:

(i) because of the seminal work of Scott, the Scott-continuity of morphisms has been seen as *the* canonical way of getting $U \Rightarrow U \lhd U$.
(ii) the classic result relating algebraic and categorical models of pure λ-calculus asks for reflexive objects with enough points.

Our proposal allows to overcome (ii). It remains to be proved that, working in categories like **MRel** allows to get new interesting classes of models.

References

1. A. Asperti and G. Longo. Categories, types and structures. Category theory for the working computer scientist. *M.I.T. Press*, 1991
2. S. Abramsky, R. Jagadeesan and P. Malacaria. Full abstraction for PCF. *Inf. Comput.*, 163(2): 409-470, 2000.
3. R.M. Amadio and P.-L. Curien, Domains and lambda-calculi, *Cambridge University Press*, New York, NY, 1998.
4. H.P. Barendregt. The lambda calculus: Its syntax and semantics. *North-Holland Publishing Co.*, Amsterdam, 1984.
5. C. Berline. From computation to foundations via functions and application: The λ-calculus and its webbed models. *Theoretical Computer Science*, 249:81–161, 2000.
6. G. Berry and P.-L. Curien, Sequential algorithms on concrete data structures, *Theoretical Computer Science* 20, 265-321, 1985.
7. A. Bucciarelli and T. Ehrhard. On phase semantics and denotational semantics: the exponentials. *Ann. Pure Appl. Logic* 109(3): 205-241, 2001.
8. H.B. Curry and R. Feys. *Combinatory Logic, Vol. I.* North-Holland, Amsterdam, 1958.
9. M. Dezani Ciancaglini, U. de'Liguoro and A. Piperno. A filter model for concurrent lambda-calculus. *SIAM Journal of Computing* 27(5): 1376-1419, 1998.
10. U. de'Liguoro and A. Piperno. Non deterministic extensions of untyped λ-calculus. *Information and Computation*, 109:149-177, 1955.
11. P. Di Gianantonio, G. Franco and F. Honsell. Game semantics for untyped lambda calculus. *Proc. of the conference Typed Lambda Calculus and Applications* LNCS 1591, pp 114-128, 1999.
12. J.-Y. Girard. Normal functors, power series and the λ-calculus. *Annals of pure and applied logic*, 37:129-177, 1988.
13. J.R. Hindley and G. Longo. Lambda calculus models and extensionality. *Z. Math. Logik Grundlag. Math.* 26:289-310, 1980.
14. M. Hyland. A syntactic characterization of the equality in some models for the lambda calculus. *J. London Math. Soc.* (2), 12(3):361-370, 1975/76.
15. M. Hyland, M. Nagayama, J. Power and G. Rosolini. A category theoretic formulation for Engeler-style models of the untyped lambda. *Electronic notes in theoretical computer science.* 161: 43-57, 2006.
16. M. Hyland and C.-H.L. Ong. On full abstraction for PCF: I. Models, observables and the full abstraction problem, II. Dialogue games and innocent strategies, III. A fully abstract and universal game model. *Information and Computation*, 163:285-408, 2000.
17. A.D. Ker, H. Nickau and C.-H.L. Ong. A universal innocent game model for the Boehm tree lambda theory. *In Computer Science Logic: Proceedings of the 8th Annual Conference on the EACSL* (CSL 1999) Madrid, Spain, September 1999, LNCS Volume 1683, Springer-Verlag, pp. 405-419, 1999.
18. R. Kerth. On the construction of stable models of λ-calculus. *Theoretical Computer Science* 269:23–46, 2001.
19. G.D. Plotkin. T_ω as a Universal Domain. *J. Comput. System Sci.* 17:209–236, 1978.
20. D.S. Scott. Continuous lattices. In *Toposes, Algebraic geometry and Logic.* LNM 274, Springer-Verlag, Berlin, 1972.