Decision issues on functions realized by finite automata

May 7, 1999

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1 Introduction

Let D be some finite alphabet of symbols, (a set of "digits"). A numeration system is a function π that associates with each sequence of symbols of D, a number of which it is a representation. Consider further a mapping f from sequences to sequences performed by some device. We say this device computes a numerical function or (in Eilenberg's terminology [4], p. 370) that this device is *consistent* relative to the numeration system, whenever for two input sequences representing the same number, f associates two output sequences representing the same number. In [3] consistent "sequential" machines relative to the usual m-adic representations of reals are considered and their properties of continuity, monotonicity and invertibility are studied.

These questions are not only theoretical but also practical since the use of certain non-standard representations of numbers can sometimes increase the speed of computation in at least two ways. Indeed, Avizienis introduced in 1961 a system that allows addition of integers to be performed "locally", i. e., without propagation of the carry, [1]. It can also be interesting to perform operations "highest digits first" by processing the representations from left to right in a sequential manner and by allowing pipe-lining. This approach was proposed by Ercegovac and Trivedi who named it "on-line computation", [14, 7]. J. -M. Muller [12, 13] continued along this line by studying functions that are on-line computable by finite automata using Avizienis system.

Here, by "numbers" we mean either integers or reals. Our purpose is to keep the assumption of computability via some kind of finite automaton while both relaxing the condition of sequentiality and considering more general numeration systems than the m-adic ones. In other words the scope of our results covers properly all on-line functions on reals or integers computed by finite automata and using standard representations. We show that provided a function can be performed by a finite "synchronous" two-tape automaton and provided the representation of numbers is based on a Pisot number (such as the golden ratio whose definition is given below) several questions can be decided in polynomial time such as e. g.: given a function on sequences whether or not it defines a function on numbers and in that case whether or not this function is monotone, injective, continuous (when the numbers are real).

2 Consistency

We start with recalling elementary set-theoretic notions. Let X and Y be two sets and let $f : X \to Y$ be a mapping. The *graph* of f is the subset

$$\hat{f} = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

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Given two relations $R \subseteq X \times X$ and $S \subseteq Y \times Y$, f is *consistent* (relative to R and S) if

$$\text{for all } x, y, (x, y) \in R \text{ implies } (f(x), f(y)) \in S$$

Let $f \times f : X \times X \to Y \times Y$ be defined by $(f \times f)(x, x') : (f(x), f(x'))$. Then consistency is expressed by the inclusion

$$\widehat{f \times f} \cap R \times Y \times Y \subseteq R \times S \tag{1}$$

2.1 Finite automata

Given a finite *alphabet* A whose elements are *symbols*, we denote by A^* (resp. A^ω) the set of finite (resp. infinite) sequences over A. The different kinds of finite automata are probably among the most commonly used devices for specifying subsets of sequences. Though very well-known, for the reader's convenience, we recall some definitions here.

A *finite automaton* (on finite sequences) is a quadruple $\mathcal{A} = (Q, I, F, T)$ where Q is the finite set of *states*, $I \subseteq Q$ is the set of *initial* states, $F \subseteq Q$ is the set of *final* states and $T \subseteq Q \times A \times Q$ the set of *transitions*, e. g., [4]. The subset of A^* recognized by $\mathcal{A}$ consists of those finite sequences that are the labels of a path starting in an initial state and ending in a final state.

In a similar way, a *Büchi automaton* (on infinite sequences) is a quadruple $\mathcal{A} = (Q, I, R, T)$ where Q, I, T are as in the previous definition and $R \subseteq Q$ is the set of *repeated* states. The subset of A^ω recognized by $\mathcal{A}$ consists of those infinite sequences that are the labels of a path starting in an initial state and visiting infinitely often some repeated state.

An automaton, whether on finite or infinite sequences is *deterministic* the set I is a singleton and the transitions associated with every symbol $a \in A$ defines a mapping of Q into itself. Determinism has the same expressive power than non-determinism when applied to finite sequences. For infinite sequences this no longer holds.

A *semi-automaton* whether on finite or infinite sequences, is an automaton with no specified sets of initial and final (resp. repeated) states. In other words, the transitions only matter.

Finite automata have been adapted in the mid sixties to operate on n-tuples of sequences yielding thus *n-tape automata*. Let A_i, $i = 1, \ldots, n$, be n alphabets. Partition the set of states into n different (possibly empty) subsets in one-to-one correspondence with the alphabets A_i. Given an n-tuple $(u_1, \ldots, u_n) \in A_1^* \times \ldots A_n^*$, a reading head is assigned to each of the n components and moves one-way from left to right. The current state determines which component to read and which next state to (possibly non deterministically) go to, [6]. The n-tuple is recognized whenever the automaton enters a final state after having read all the components. The subsets of $\prod_{1 \leq i \leq n} A_i^*$ recognized in this manner are called *rational relations*. A *rational function* of A^* into B^* is a function whose graph is a rational relation of $A^* \times B^*$.

2.2 Synchronous relations

There is an important subfamily of the rational relations which enjoys nice closure properties. Consider a fresh symbol $\sharp$ not belonging to the A_i's. With each n-tuple $(u_1, \ldots, u_n) \in$

$\prod_{1 \le i \le n} A_i^*$ associate the n-tuple of sequences of the same length defined as

$$(u_1, \ldots, u_n)^\sharp = (u_1 \sharp^{\ell - |u_1|}, \ldots, u_n \sharp^{\ell - |u_n|}) \text{ with } \ell = \max_i |u_i| \tag{2}$$

Extending the notation to subsets $R \subseteq \prod_{1 \le i \le n} A_i^*$ in the natural way, we identify $R^\sharp$ with a subset of sequences over the alphabet $\Sigma = \prod_{1 \le i \le n} (A_i \cup \{\sharp\})$. Then the relation R is *left synchronous* if the subset $R^\sharp$ is recognized by a finite automaton over the alphabet Σ. Dually, instead of "padding" the n-tuples of R to the right as in (2) we can pad them to the left and obtain the subset ${}^\sharp R$. The relation R is *right synchronous* whenever the mirror image of ${}^\sharp R$ is recognized by a finite automaton over the alphabet Σ. Unless otherwise stated, the term left synchronous will be abbreviated to synchronous. It is not difficult to verify that the synchronous relations form a subfamily of the rational relations that is closed under the Boolean operations, composition of relations, direct products and projections. These relations were called FAD-relations in [6] and were logically characterized in [5]. Finally a function $f : A^* \to B^*$ is *synchronous* if its graph $\hat{f}$ is a synchronous relation in $A^* \times B^*$. Observe that because a non deterministic automata (as a device for recognizing subsets of a free monoid) is always equivalent to some deterministic automaton, the graph of a given synchronous relation in $A^* \times B^*$ can be recognized by some deterministic automaton on the alphabet $A \times B$.

We now turn to relations on infinite sequences. Given two alphabets A and B we make the convention of viewing the direct product $A^\omega \times B^\omega$ as the set of infinite sequences over the alphabet $A \times B$ by assigning to each pair of sequences $(u_1 u_2 \ldots u_n \ldots, v_1 v_2 \ldots v_n \ldots)$, the sequence of pairs $(u_1, v_1)(u_2, v_2) \ldots (u_n, v_n) \ldots$. In particular, a subset of $A^\omega \times B^\omega$ is a *Büchi relation* if it is recognized by a Büchi automaton over the alphabet $A \times B$. A *Büchi function* $f : A^\omega \to B^\omega$ is a function whose graph is a Büchi relation.

A relation $R \subseteq A^\omega \times B^\omega = (A \times B)^\omega$ is a *deterministic Büchi* relation if there exists a deterministic Büchi finite automaton which recognizes it.

2.3 Decision issues

From now on, when a left synchronous (resp. a right synchronous, a Büchi, a deterministic Büchi) function is given it is understood that its graph is actually given by a left synchronous (resp. a right synchronous, a Büchi, a deterministic Büchi) automaton. The purpose of this paragraph is to observe that consistency for synchronous and Büchi functions is decidable.

PROPOSITION **1** *Let $R \subseteq A^* \times A^*$ and $S \subseteq B^* \times B^*$ be two synchronous relations. The problem of deciding whether or not a synchronous function $f : A^* \to B^*$ is consistent relative to R and S can be solved in polynomial time relative to the number of states of an automaton recognizing $\hat{f}^\sharp$.*

Proof. Because of (1) it suffices to verify that $\widehat{f \times f} \cap R \times Y \times Y$ has an empty intersection with the complement of $R \times S$. Since R and S are not inputs of the problem, the constructions involving them do not affect the overall complexity. The problem reduces to an accessibility problem in a graph with $O(N^2)$ nodes where N is the number of states of the automaton recognizing $\hat{f}^\sharp$. ■

All synchronous relations may be defined by a deterministic automaton. Thus, if we assume the relations R and S are effectively given by deterministic automata, taking the complement of $R \times S$ can be done in linear time. Thus R and S may be inputs of the problem without affecting the complexity of the problem. The proof of the following is straightforward (observe that the graph of f need not be given by a deterministic automaton).

PROPOSITION **2** *The problem of deciding whether or not a synchronous function $f : A^* \to B^*$ is consistent relative to two synchronous relations $R, S \subseteq A^* \times A^*$ given by two deterministic automata, can be solved in polynomial time relative to the sum of the numbers of states of the automata recognizing $\hat{f}^\sharp$, R and S.*

Observe that the hypothesis that the function f is synchronous is crucial. If we drop it and assume more generally that the function is rational then the result no longer holds.

PROPOSITION **3** *There exist two synchronous relations $R \subseteq A^* \times A^*$, $S \subseteq B^* \times B^*$ for which determining whether or not a given rational function $f : A^* \to B^*$ is consistent relative to R and S is undecidable.*

Proof. Let $(u_1, v_1), \ldots, (u_n, v_n) \in B^* \times B^*$ be an instance of Post Correspondence Problem. We recall that determining whether or not there exist an integer $k > 0$ and a sequence $0 < i_1, \ldots, i_k \leq n$ such that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$ holds is undecidable. Let $A = \{a, b, c, d\}$ be an alphabet of new symbols and let $f : A^* \to B^*$ be the rational function whose graph equals

$$(\bigcup_{1 \leq i \leq n} (a^i b, u_i))^* \cup (\bigcup_{1 \leq i \leq n} (c^i d, v_i))^*$$

The sets $R = ((a, c)+(b, d))^* \subseteq A^* \times A^*$ and $S = \{(x, y) \in B^* \times B^* \mid x \neq y\}$ are synchronous but determining whether or not the instance has a solution reduces to determining whether or not f is consistent relative to R and S. ■

The previous discussion extends to infinite sequences. The proof of the following goes along the same line as for finite sequences

PROPOSITION **4** *Let $R \subseteq A^\omega \times A^\omega$ and $S \subseteq B^\omega \times B^\omega$ be Büchi relations. The problem of deciding whether or not a Büchi function $f : A^\omega \to B^\omega$ is consistent relative to R and S can be solved in polynomial time relative to the number of states of a Büchi automaton recognizing $\hat{f}^\sharp \subseteq (A \times B)^\omega$.*

Contrarily to automata on finite sequences, not all Büchi relations are deterministic. However, the complement of a deterministic Büchi relation can be constructed in polynomial time, [11]. We thus have the following.

PROPOSITION **5** *The problem of deciding whether or not a synchronous function $f : A^* \to B^*$ is consistent relative to two relations $R, S \subseteq A^* \times A^*$ given by two deterministic Büchi automata, can be solved in polynomial time relative to the sum of the numbers of states of the automata recognizing $\hat{f}^\sharp$, R and S.*

3 Application

3.1 Numeration systems for integers: the hypotheses

In order to specify a numeration system for integers with reasonable properties some hypotheses are generally assumed in the literature. A strictly increasing sequence of integers $U = (U_n)_{n\geq 0}$ satisfying

$$\begin{cases} U_0 = 1 \\ U_n = a_{k-1}U_{n-1} + \ldots a_0 U_{n-k} \text{ for all } n \geq k \end{cases} \tag{3}$$

is given where the coefficients $a_{k-1} \ldots a_0$ are fixed integers. Furthermore, it is supposed that the characteristic polynomial

$$x^k - a_{k-1}x^{k-1} - \ldots - a_0 \tag{4}$$

associated with the recurrence is irreducible over the integers, that it has a single dominant root $\theta > 1$ and that all other roots have modulus less than 1, in other words that θ is a *Pisot* number. One of the most typical example is the famous Fibonacci system defined by the linear recurrence $U_n = U_{n-1} + U_{n-2}$. Its characteristic polynomial is $x^2 - x - 1$ and its root of maximal modulus is the golden ratio $\frac{1+\sqrt{5}}{2}$. Given U and a finite subset $D \subseteq \mathbb{Z}$, we define a function $\pi_{U,D} : D^* \to \mathbb{Z}$ by setting

$$\pi_{U,D}(d_n \ldots d_0) = d_n U_n + \ldots + d_0 U_0 \text{ where } d_i \in D \text{ for all } 0 \leq i \leq n \tag{5}$$

Traditional numeration systems fall into this category, e. g., the binary representation of integers with $U_n = 2U_{n-1}$ and $D = \{0, 1\}$, and the above mentioned Fibonacci system. Of particular practical significance are Avizienis numeration systems that are given by an equation $U_n = rU_{n-1}$ for some $r \in \mathbb{N}$ and by a set of digits $D = \{-a, \ldots, -1, 0, 1, \ldots, a\}$ satifying $\frac{r}{2} \leq a \leq r-1$. For instance, when $r = 2$ and $a = 1$, then all sequences of the form $1\bar{1}\bar{1}\ldots\bar{1}$ with $\bar{1} = -1$ represent the integer 1. The benefit of this system of numeration lies in the fact that in adding two integers the carry does not propagate. The operation can thus be performed in parallel by use of local operations (i. e., circuits of depth one).

3.2 Decidable properties

Since we are dealing with consistency and monotony, the reader will certainly agree that it is natural to consider the following relations where $\pi_{U,D}$ is abbreviated as π.

$$\begin{cases} E = \{(u,v) \in D^* \times D^* \mid \pi(u) = \pi(v)\} \\ G = \{(u,v) \in D^* \times D^* \mid \pi(u) > \pi(v)\} \end{cases} \tag{6}$$

Given a function $f : D^* \to D^*$ we are interested in the following properties

P1 FUNCTIONALITY: $\forall u, \forall v, \pi(u) = \pi(v) \Longrightarrow \pi(f(u)) = \pi(f(v))$

P2 MONOTONY: $\forall u, \forall v, \pi(u) \leq \pi(v) \Longrightarrow \pi(f(u)) \leq \pi(f(v))$ (or $\forall u, \forall v, \pi(u) \geq \pi(v) \Longrightarrow \pi(f(u)) \leq \pi(f(v))$)

P3 STRICT MONOTONY: $\forall u, \forall v, \pi(u) < \pi(v) \Longrightarrow \pi(f(u)) < \pi(f(v))$ (or $\forall u, \forall v, \pi(u) < \pi(v) \Longrightarrow \pi(f(u)) > \pi(f(v))$)

P4 INJECTIVITY: $\forall u, \forall v, \pi(f(u)) = \pi(f(v)) \Longrightarrow \pi(u) = \pi(v)$.

Given two functions $f, g : D^* \to D^*$ we also ask the question

P5 EQUALITY: $\forall u, \forall v, \pi(u) = \pi(v) \Longrightarrow \pi(f(u)) = \pi(g(v))$

It was proven in [9, Thm. 2] that the set E is right synchronous. An easy elaboration of the arguments, e.g., along the line of [2, Thm. 14] shows more generally that the following holds

PROPOSITION **6** *With the above hypotheses, the sets E and G defined in (6) are right synchronous. Furthermore, if the set of digits consists of positive integers, then E is left and right synchronous.*

Given a numeration system based on the previous hypotheses and with set of digits D we obtain

PROPOSITION **7** *The properties* FUNCTIONALITY, MONOTONY, STRICT MONOTONY, INJECTIVITY *(resp.* EQUALITY*) are decidable in polynomial time for an arbitrary right synchronous function (resp. two arbitrary right synchronous functions).*

If the digits of the set D are non negative, FUNCTIONALITY, INJECTIVITY *and* EQUALITY *are also decidable in polynomial time for an arbitrary (whether right or left) synchronous function.*

The complexity is relative to the size of an automaton recognizing $\hat{f}^\sharp$.

Proof. FUNCTIONALITY and INJECTIVITY can be expressed as consistency properties envolving the relation E or its complement. These relations are right synchronous (resp. left and right synchronous when the digits are positive). Thus the assertions concerning FUNCTIONALITY, MONOTONY, STRICT MONOTONY and INJECTIVITY are immediate consequences of Proposition 1 and Proposition 6.

Concerning EQUALITY, let $f, g : D^* \to D^*$ be two functions defining numerical functions and let $\overline{D}$ be a disjoint copy of D. To each sequence $u \in D^*$ is naturally associated the sequence $\bar{u} \in \overline{D}^*$. Consider the function $h : (D \cup \overline{D})^* \to (D \cup \overline{D})^*$ defined by $h(u) = f(u)$, $h(\bar{u}) = g(\bar{u})$ for all $u \in D^*$ and where $h(u)$ is arbitrary otherwise. Set $\Delta = (\bigcup_{d \in D}(d, \bar{d}))^*$ and let F be as the relation E in (6) where $D \cup \overline{D}$ is substituted for D. Then f and g define equal numerical functions if and only if h is consistent relative to the left and right synchronous relations $R = \Delta \cap F$ and $S = \Delta \cap F$ respectively. ■

3.3 The case of the reals

Numeration systems for reals is easier to deal with. The hypotheses on the numeration system remain the same, i. e., $\theta > 1$ is a Pisot number and $D \subseteq \mathbb{Z}$ is finite. The function π assigns to each finite sequence $u = u_1 \ldots u_n \in D^*$ the real $\pi(u) = \sum_{1 \leq i \leq n} \frac{u_i}{\theta^i}$ and to each infinite sequence $u = u_1 \ldots u_n \ldots \in D^\omega$ the real $\pi(u) = \sum_{i \geq 1} \frac{u_i}{\theta^i}$. Also, for all $0 < i \leq j < \infty$ and for all $u = u_1 \ldots u_n \ldots \in D^\omega$, we write $u_{[i,j]} = u_i \ldots u_j$ with the convention $u_{[i,\infty]} = u_i u_{i+1} \ldots$.

As for finite sequences, we define the following relations on infinite sequences.

$$\begin{cases} E = \{(u,v) \in D^\omega \times D^\omega \mid \pi(u) = \pi(v)\} \\ G = \{(u,v) \in D^\omega \times D^\omega \mid \pi(u) > \pi(v)\} \end{cases} \tag{7}$$

It was proven in [8, Cor. 3.4] that E is a Büchi relation. It can easily be extended to G

Proposition **8** *The sets E and G defined in (7) are Büchi deterministic.*

The proof of the following proceeds as in the finite case.

Proposition **9** *The properties* FUNCTIONALITY, MONOTONY, STRICT MONOTONY, INJECTIVITY *(resp.* EQUALITY*) are decidable in polynomial time for an arbitrary Büchi function (resp. two arbitray Büchi functions).*

The complexity is relative to the size of an automaton recognizing the graph $\hat{f}$.

To the set of questions of paragraph 3.2 we add the following

P6 CONTINUITY: for all $u, \lim \pi(v) = \pi(u)$ implies $\lim \pi(f(v)) = \pi(f(u))$

Because the two spaces D^ω and $[0,1]$ are compact and the function $u \to \pi(u)$ is continuous it follows from a general theorem in elementary functional analysis that if the function $u \to f(u)$ is continuous then so is the function $\pi(u) \to \pi(f(u))$. The converse does not hold. E. g. the normalization in base 2 is the identity for all sequences except for those of the form $u01^\omega$ whose image is $u10^\omega$ and $\pi(u) = u$ otherwise. More specifically, it is not true in general that whenever the function $\pi(u) \in [0,1] \to \pi(f(u)) \in [0,1]$ is continuous then there exists a continuous function $g : D^\omega \to D^\omega$ such that $\pi(f(u)) = \pi(g(u))$ holds for all $u \in D^\omega$. Indeed, consider $F(x) = \frac{1}{3} + \frac{1}{2}x$. It is clear that there exists a Büchi function $f : D^\omega \to D^\omega$ such that $F(\pi(u)) = \pi(f(u))$ where π is the usual binary interpretation of sequences of $\{0,1\}$. Let us verify that this function cannot be continuous. Indeed, assume first $f((01)^\omega) = 10^\omega$. Then for all integers $n > 0$ we have $f((01)^n0^\omega) = 01^{2n+1}(01)^\omega$, i. e, $\lim_{n\to\infty} f((01)^n0^\omega) = 01^\omega$. Now assume $f((01)^\omega) = 01^\omega$. Then for all integers $n > 0$ we have $f((01)^n10^\omega) = 10^{2n}(01)^\omega$, i. e, $\lim_{n\to\infty} f((01)^n10^\omega) = 10^\omega$.

Thus the question of continuity requires a special treatment.

Proposition **10** *The* CONTINUITY *problem is decidable in polynomial time for a function $f : D^\omega \to D^\omega$ specified by a graph which is a Büchi relation.*

Proof. Assume $D \subseteq [-d, d]$ holds for some positive integer $d > 0$. Set $C = [-2d, 2d]$ and $\lambda = \frac{2d}{\theta - 1}$. For $k > 0$ put

$$\epsilon_k = \min\{(x - \lambda) \mid x = u_1\theta^{k-1} + u_2\theta^{k-2} + \ldots + u_k > \lambda, |u_i| \leq 2d, 0 < i \leq k\}$$

Because θ is a Pisot number, by a result of [10] the set

$$\{u_1\theta^{n-1} + u_2\theta^{n-2} + \ldots + u_n \mid n > 0, |u_i| \leq 2d\} \cap [-\lambda, \lambda] \tag{8}$$

is finite. Let P be the union of the two element set $\{-\infty, \infty\}$ and of the set defined in (8). To every $c \in C$ assign an action $p \to p.c$ on P by putting

$$p.c = \begin{cases} p\theta + c & \text{if } -\lambda \leq p\theta + c \leq \lambda \\ \infty & \text{if } p\theta + c > \lambda \\ -\infty & \text{if } p\theta + c < -\lambda \end{cases} \tag{9}$$

(with the usual convention $-\infty \times \theta = -\infty + c = -\infty$ and $\infty \times \theta = \infty + c = \infty$).

Let $\mathcal{A} = (Q, I, R, T)$ be a Büchi automaton that recognizes the graph of f. The idea is to define a Büchi automaton that performs simultaneously the computations over two different inputs. To that purpose, we define a set of transitions $T' \subseteq Q' \times D' \times Q'$ where $Q' = Q \times Q \times P \times P$ and $D' = D^4$ by setting for all $(x_1, y_1), (x_2, y_2) \in D^2$

$$\begin{cases} ((q_1, q_2, p_1, p_2), x_1, x_2, y_1, y_2, (q'_1, q'_2, p'_1, p'_2)) \in T' \\ \text{where } (q_1, x_1, y_1, q'_1), (q_2, x_2, y_2, q'_2) \in T, \; p'_1 = p_1.(x_1 - x_2) \\ \text{and } p'_2 = p_2.(y_1 - y_2) \end{cases} \tag{10}$$

Intuitively, the third factor P of the direct product Q' records the difference (conveniently normalized by the multiplicative factor) between the two inputs and the fourth factor records the difference (with the same normalization) between the two corresponding outputs.

Let S be the set of states in Q' that are accessible from $I \times I \times \{0\} \times \{0\}$ and coaccessible to $R \times R \times P \times P$. Let $\mathcal{B}$ be the semi-automaton whose transitions are equal to $T' \cap (S \times D' \times S)$. We claim that the function $g : \pi(u) \to \pi(f(u))$ is continuous if and only if the following condition on $\mathcal{B}$ holds:

$$\begin{array}{l} \text{for all } (q_1, q_2, p_1, p_2), (q_1, q_2, p_1, p'_2) \in S, p_1 \notin \{-\infty, \infty\} \\ \text{if } (q_1, q_2, p_1, p'_2) \text{is accessible from } (q_1, q_2, p_1, p_2) \\ \text{by some path of non zero length, then } p'_2 \notin \{-\infty, \infty\} \end{array} \tag{11}$$

Assume by contradiction that it is not satisfied. Then there exist $u, v, w = f(u), z = f(v) \in D^\omega$, two integers $n > 0, m > 0$ and two states $i_1, i_2 \in I$ such that the paths labelled by $(u, f(u))$ and $(v, f(v))$ respectively start as follows in $\mathcal{A}$

$$\begin{array}{l} i_1 \xrightarrow{(u_{[1,n]}, w_{[1,n]})} q_1 \xrightarrow{(u_{[n+1,n+m]}, w_{[n+1,n+m]})} q_1 \\ i_2 \xrightarrow{(v_{[1,n]}, z_{[1,n]})} q_2 \xrightarrow{(v_{[n+1,n+m]}, z_{[n+1,n+m]})} q_2 \end{array}$$

Furthermore, $-\lambda \leq \theta^n(\pi(u_{[1,n]}) - \pi(v_{[1,n]})) = \theta^{n+m}(\pi(u_{[1,n+m]}) - \pi(v_{[1,n+m]})) \leq \lambda$ and $|\theta^{n+m}(\pi(w_{[1,n+m]}) - \pi(z_{[1,n+m]}))| \geq \lambda + \epsilon_{n+m}$. For all $k > 0$ we set

$$u^{(k)} = u_{[1,n]}(u_{[n+1,n+m]})^k u_{[n+m+1,\infty]} \text{ and } v^{(k)} = v_{[1,n]}(v_{[n+1,n+m]})^k v_{[n+m+1,\infty]}$$

and

$$w^{(k)} = w_{[1,n]}(w_{[n+1,n+m]})^k w_{[n+m+1,\infty]} \text{ and } z^{(k)} = z_{[1,n]}(z_{[n+1,n+m]})^k z_{[n+m+1,\infty]}$$

and we observe that $w^{(k)} = f(u^{(k)})$ and $z^{(k)} = f(v^{(k)})$. Thus we get

$$0 \leq |\pi(u^{(k)}) - \pi(v^{(k)})| \leq \frac{\lambda}{\theta^{n+km}} \text{ and } |\pi(f(u^{(k)})) - \pi(f(v^{(k)}))| > \frac{\epsilon_{n+m}}{\theta^{n+m}}$$

Conversely, assume the condition is satisfied. Set $K = \sharp Q' + 1$ and for some arbitrary $k \geq K$ set $\epsilon = \frac{\epsilon_k}{\theta^k}$. Consider $u, v \in D^\omega$ satisfying $|\pi(u) - \pi(v)| \leq \epsilon$ and set $w = f(u)$, $z = f(v)$. For the paths labelled by the pairs (u, w) and (v, z) respectively, there exist $n < n + m \leq k$, $k - n \leq K$ such that the following four conditions hold

$$i_1 \xrightarrow{(u_{[1,n]}, w_{[1,n]})} q_1 \xrightarrow{(u_{[n+1,n+m]}, w_{[n+1,n+m]})} q_1$$
$$i_2 \xrightarrow{(v_{[1,n]}, z_{[1,n]})} q_2 \xrightarrow{(v_{[n+1,n+m]}, z_{[n+1,n+m]})} q_2$$

$$\theta^n(\pi(u_{[1,n]}) - \pi(v_{[1,n]})) = \theta^{n+m}(\pi(u_{[1,n+m]}) - \pi(v_{[1,n+m]}))$$

and

$$\theta^n(\pi(w_{[1,n]}) - \pi(z_{[1,n]})) = \theta^{n+m}(\pi(w_{[1,n+m]}) - \pi(z_{[1,n+m]}))$$

By definition of ϵ_k, we have $|\theta^k(\pi(u_{[1,k]}) - \pi(v_{[1,k]}))| < \lambda + \epsilon_k$ which implies

$$|\theta^{n+m}(\pi(w_{[1,n+m]}) - \pi(z_{[1,n+m]}))| < \lambda$$

Now inequality

$$|\pi(f(u)) - \pi(f(v))| \leq \frac{2\lambda}{\theta^{n+m}} \leq \frac{2\lambda}{\theta^{k-K}}$$

shows the continuity.

In order to verify the complexity claim, observe that if the initial automaton realizing f possesses $O(N)$ states then automaton $\mathcal{B}$ possesses $O(N^2)$ states. Condition (11) can be verified in cubic time in the size of $\mathcal{B}$. Indeed, consider the finite semiring consisting of the 3 elements $0, 1, -\infty$ subject to the addition and multiplication respectively defined by $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$. Furthermore consider the matrix M indexed with the states of $\mathcal{B}$ whose generic entry in position $(q_1, q_2, p_1, p_2), (q_3, q_4, p_3, p_4)$ equals 0 (resp. 1) if there is a transition from (q_1, q_2, p_1, p_2) to (q_3, q_4, p_3, p_4) and $p_4 \neq \pm\infty$ (resp. $p_4 = \pm\infty$) and $-\infty$ otherwise.

Its transitive closure can be computed in cubic time. Then the condition is satisfied if and only if the entry in position $(q_1, q_2, p_1, p_2), (q_1, q_2, p_1, p'_2)$ with $p_2 \neq \pm\infty$ and $p_2 \neq \pm\infty$ has a value equal to 1.

■

Acknowledgement: the authors wish to thank Christiane Frougny for useful discussions.

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