Determinants and Möbius functions in trace monoids

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Abstract

We show that the independence relation defining a trace monoid M admits a transitive orientation if and only if the characteristic series ξ of a lexicographic cross section of M satisfies an equation of the form $\xi \cdot \mathrm{Det}(\mathrm{Id} - X) = 1$, where X is a matrix representing the minimum finite automaton recognizing ξ and Id is the identity matrix. This implies that, if the independence relation of a trace monoid M admits a transitive orientation, then any unambiguous lifting of the Möbius function of M is the determinant of a matrix defined by the smallest acceptor of the corresponding cross section.

Keywords: matrices, Möbius function, rational formal series, trace monoids.

1 Introduction

The famous Möbius inversion formula on integer functions has been extended in various ways. This work deals with its extension to trace monoids which are obtained by taking the quotient of the free monoids by a congruence defined by some commutations of the generators. Those were introduced in the late sixties in [3] in order to give a completely combinatorial proof of Mac Mahon's Master theorem (cf. [9], Chapter XI). They were studied further as a possible model of parallelism by interpreting generators of the monoid as actions and their commutation as independence of actions, cf. [10]. A theory of trace languages has been then developped which extends the classical properties of formal languages. Nowadays, it has grown to a vast autonomous field of research providing rich conjectures and open problems [7].

Let us illustrate the importance of the Möbius function in the theory of traces with an example taken from an enumeration problem. Intuitively, if M is a trace monoid, its Möbius function μ_M is the (Cauchy) inverse of the characteristic function ξ_M of M in the monoid algebra over the ring of integers Z. It just happens that the Möbius function is a polynomial whose monomials can be identified with the cliques of the graph of commutations. Now, if we consider the enumerative image of ξ_M obtained by mapping all indeterminates over one unique indeterminate z, we obtain a rational function $f_M(z)$ over the variable z. The coefficient $t_n \in Z$ associated with the power z^n in its Taylor development is the number of traces of a length n in M. Then, from the equality $\xi_M = (\mu_M)^{-1}$ we can derive an explicit expression for $f_M(z)$ which allows to give exact and asymptotic expressions for t_n or an estimate of the average number of prefixes of a trace of length n, quantities which are related to the complexity of algorithms for the recognition of rational trace languages [2].

Since a trace monoid M is defined as the image of a free monoid Σ^* in the canonical morphism, a general technique used in different contexts when solving some problem on trace monoids is to "lift" its properties in the free monoid, to solve the problem in the free monoid, and then to map it back into the trace monoid. The present work is in the vein of [5] where it was settled the following natural question concerning the lifting of the equality $(\mu_M)^{-1} = \xi_M$: choose for each monomial (i.e., each trace) occurring in μ_M a representative in the free monoid (i.e., lift it) and consider the formal inverse in Σ^* of the polynomial thus obtained; can we define some way of selecting the monomials so that this inverse be the characteristic function of a cross-section of M in Σ^*? The answer is yes if and only if the relations of commutations of the trace monoid can be transitively oriented. This was further refined in [6].

Here we present an interpretation of the lifting of the Möbius function which gives another characterization of the trace monoids whose independence relation admits a transitive orientation.

In order to state our result more precisely, some specific remarks are necessary. First we recall that, given a finite alphabet Σ and an independence relation $I \subseteq \Sigma \times \Sigma$, the corresponding trace monoid M is the quotient of the free monoid Σ^* by the congruence $\equiv_I$ generated by the equations $ab = ba$ such that $(a, b) \in I$. Second, it is customary to represent any finite automaton with a matrix whose rows and columns are indexed by the states Q of the automaton and whose entry in position $(p, q) \in Q \times Q$ consists of all the letters that take p to q (cf. [8]). We say that such a matrix is a *minimal acceptor matrix* of a language L if it represents the smallest finite automaton recognizing L. For a matrix of this kind, with entries in a noncommutative ring, a natural notion of determinant can be given as a sum of terms obtained by choosing in the order one entry for each column so that every row is taken exactly once.

Now, given an independence relation I over a finite alphabet Σ and a linear ordering $<$ over Σ, consider the cross section $L \subseteq \Sigma^*$ given by the set of all maximal representatives in each class of the congruence $\equiv_I$. We prove that the restriction of $<$ to the pairs in I is transitive *if and only if* the characteristic series ξ of L satisfies an equation of the form

$$\xi \cdot \mathrm{Det}(\mathrm{Id} - X) = 1,$$

where X is a minimal acceptor matrix of L and Id is the corresponding identity matrix.

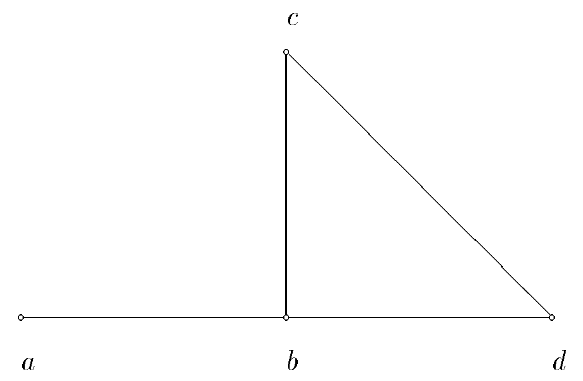
Intuitively, this means that a lifting μ of the Möbius function of a trace monoid M is the inverse of a cross section L of M if and only if μ is the determinant of $(\mathrm{Id} - X)$, for a matrix X defining the smallest automaton that recognizes L.

A further consequence concerns the enumerative image $f_M(z)$ of the characteristic series ξ_M. If the independence relation of a trace monoid M admits a transitive orientation, then $f_M(z)$ is the inverse of the determinant of $(\mathrm{Id} - Y)$, where Y is the enumerative image of *any* minimal acceptor matrix X of a cross section of M (i.e. Y is obtained from X by replacing in each entry every element of the alphabet Σ by the unique symbol z).

2 Preliminaries

Let Σ be a finite alphabet, let Σ^* be the free monoid it generates, i.e., the set of all *words* or *strings* over Σ and let 1 be its identity, the *empty word*. A trace monoid M is defined as the quotient $\Sigma^*/\equiv_I$ where $I \subseteq \Sigma \times \Sigma$ is a symmetric and irreflexive relation, and $\equiv_I$ is the smallest congruence extending the set of equations $\{ab = ba \mid (a,b) \in I\}$. The elements of M are called *traces*. In order to simplify notations, a string denotes either an actual string in the free monoid or a trace in the quotient monoid of traces, whichever is meant should be clear from the context. In particular, 1 denotes the identity of Σ^* or of M. The pair $\langle \Sigma, I\rangle$ is called an *independence alphabet* and can be represented by an undirected graph, Σ being the set of nodes and I the set of edges.

Example: with the following independence alphabet $\langle \Sigma, I\rangle$ defined by its graph, the strings $bacd$ and $abdc$ represent the same trace.



□

We are concerned with properties of the ring $Z\langle\langle M\rangle\rangle$ of all *formal power series* σ over the monoid M with integer coefficients, i.e., of all mappings of M into the ring of integers. Given a power series σ and an element t of M, we denote by $\sigma(t)$ the coefficient of the monomial t in σ. We can write

$$\sigma = \sum_{t\in M} \sigma(t)t.$$

The addition and multiplication of two arbitrary power series $\sigma_1, \sigma_2 \in Z\langle\langle M\rangle\rangle$ are respectively defined by

$$(\sigma_1 + \sigma_2)(t) = \sigma_1(t) + \sigma_2(t),$$
$$(\sigma_1\sigma_2)(t) = \sum_{t=t_1t_2} \sigma_1(t_1)\sigma_2(t_2),$$

for all $t \in M$.

The *support* of a power series $\sigma \in Z\langle\langle M\rangle\rangle$ is the subset consisting of all the elements $t \in M$ such that $\sigma(t) \neq 0$. The subring $Z\langle M\rangle$ of *polynomials* is the subset of all series having finite support.

We recall that the Möbius function of a trace monoid M is defined as the formal power series μ_M such that

$$\mu_M(t) = \begin{cases} (-1)^n & \text{if } t = a_1 a_2 \cdots a_n, \text{ where all } a_i \in \Sigma \text{ are different and} \\ & (a_i, a_j) \in I \text{ holds for all } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

(we adopt the usual convention that $a_1 a_2 \cdots a_n = 1$ whenever $n = 0$).

Example (continued): the Möbius function is given by the polynomial

$$\mu_M = 1 - a - b - c - d + ab + bc + bd + cd - bcd$$

□

The Möbius function enjoys the fundamental property that for any power series $\sigma_1, \sigma_2 \in Z\langle\langle M \rangle\rangle$, the following conditions are equivalent:

$$\begin{aligned} \sum_{t_1 t_2 = t} \sigma_1(t_2) &= \sigma_2(t) \quad \text{for all } t \in M, \\ \sum_{t_1 t_2 = t} \mu_M(t_1)\sigma_2(t_2) &= \sigma_1(t) \quad \text{for all } t \in M. \end{aligned}$$

Equivalently, if $\xi_M = \sum_{t \in M} t$ is the characteristic series of M in the ring $Z\langle\langle M \rangle\rangle$, then the following holds

$$\mu_M \cdot \xi_M = \xi_M \cdot \mu_M = 1. \tag{1}$$

As mentioned previously, this property is at the basis of the algebraic proof of Mac Mahon's Master Theorem given by Cartier and Foata and helps solving several enumeration problems [2]. For instance, taking the morphism φ that maps each generator of M into the unique symbol z, equation (1) reduces to

$$\sum_{n \in N} t_n z^n = \{\sum_j (-1)^j C_j z^j\}^{-1},$$

where t_n is the number of traces of length n in M and C_j is the number of cliques of size j in $\langle \Sigma, I \rangle$.

Example (continued): the commutative image of the Möbius function is

$$\varphi(\mu_M) = 1 - 4z + 4z^2 - z^3$$

which turns out to be the inverse of the generating function of $\{t_n\}$. Hence, applying standard calculations, we obtain

$$t_n = \left(1 + \frac{2}{\sqrt{5}}\right)\left(\frac{2}{3 - \sqrt{5}}\right)^n + \left(1 - \frac{2}{\sqrt{5}}\right)\left(\frac{2}{3 + \sqrt{5}}\right)^n - 1$$

□

Our present contribution deals with the non-commuting interpretation of equation (1). This requires some comments. The problem consists in asking whether it is possible to choose a representative in each monomial of μ_M and to interpret the resulting polynomial μ as a polynomial in $Z\langle\langle \Sigma \rangle\rangle$ in such a way that its formal inverse μ^{-1} in $Z\langle\langle \Sigma \rangle\rangle$ be the characteristic series of a set of representatives of M in Σ^* [4]. As an easy example, consider the free abelian monoid generated by a, b. Then its Möbius function is $\mu_M = 1 - a - b + ab$. Considered as a polynomial in $Z\langle\langle\Sigma\rangle\rangle$, its inverse is the power series

$$\sum_{m \geq 0, n \geq 0} b^n a^m$$

which is the characteristic series of the maximal strings in each congruence class when the strings are lexicographically ordered relative to $a < b$.

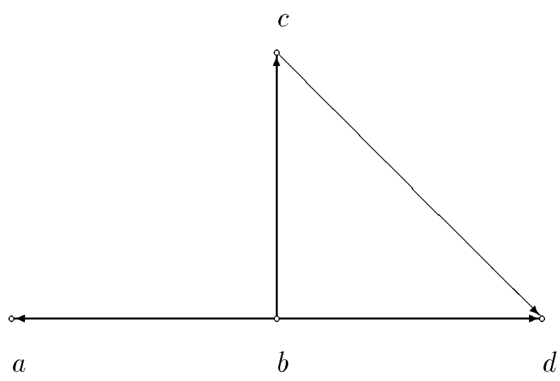
More precisely, consider the canonical morphism $\psi : Z\langle\langle\Sigma\rangle\rangle \longrightarrow Z\langle\langle M\rangle\rangle$. Let us say that a polynomial $\mu \in Z\langle\langle\Sigma\rangle\rangle$ is a *non-commuting lifting* of the Möbius function for M if $\psi(\mu) = \mu_M$ and the restriction $\psi : \Sigma^* \longrightarrow M$ is injective on the support of μ. Such a function is called *unambiguous* if its inverse ξ in $Z\langle\langle\Sigma\rangle\rangle$ is the characteristic series of a cross-section of M, i. e., $\psi(\xi) = \sum_{t \in M} t$.

It was shown in [5] that an unambiguous non-commuting lifting μ of the Möbius function for M exists if and only if the graph $\langle\Sigma, I\rangle$ admits a transitive orientation. Actually, if $<_I$ is a transitive orientation of $\langle\Sigma, I\rangle$, then the unambiguous Möbius function in question μ is given by

$$\mu(x) = \begin{cases} (-1)^n & \text{if } x = x_1 x_2 \cdots x_n,\ \{x_1, x_2, \ldots, x_n\} \text{ is a clique of } \langle\Sigma, I\rangle \text{ and} \\ & x \text{ is the lexicographically smallest string in } [x] \text{ with respect to } <_I, \\ 0 & \text{otherwise.} \end{cases}$$

We say that μ is the unambiguous Möbius function corresponding to $<_I$.

Example (continued): $\mu = 1 - a - b - c - d + ba + bc + bd + cd - cbd$, $\mu = 1 - a - b - c - d + ba + bc + bd + cd - dcb, \ldots$ are possible non-commuting liftings of μ_M. Orient the independence graph as follows



and assume that an edge from a node r to a node s means $s <_I r$. Then, the corresponding non-commuting lifting

$$\mu = 1 - a - b - c - d + ab + cb + db + dc - dcb$$

is unambiguous. □

In this work we show another characterization of the independence alphabets that admit a transitive orientation. This result also proves that any unambiguous Möbius function μ is the determinant of a matrix $(\mathrm{Id} - X)$, where X is a minimal acceptor matrix of the support of the series $\xi = \mu^{-1}$ and Id is the identity matrix.

In order to present this result we give some further definitions. Given a total order relation $<$ on Σ, the *restriction* of $<$ to I is the relation $<_I$ such that, for every $a, b \in \Sigma$, $a <_I b$ if and only if $(a, b) \in I$ and $a < b$. Moreover, the *lexicographic cross section* of M associated with $<$ is defined as the language $L \subseteq \Sigma^*$ obtained by taking, for each $t \in M$, the maximum representative string $x \in t$ with respect to the lexicographic ordering on Σ^* induced by $<$. It can be proved [5] that L is given by the following equation:

$$L = \Sigma^* - \bigcup_{a <_I b} \Sigma^* a I_b^* b \Sigma^*, \tag{2}$$

where $I_b = \{c \in \Sigma^* \mid (b, c) \in I\}$ for every $b \in \Sigma$. This proves that the cross section L only depends on the restriction $<_I$. For this reason we say that L is the lexicocraphic cross section *defined* by $<_I$.

Equation (2) also shows that L is regular and hence we can consider the smallest finite automaton recognizing L. Given a linear order over the set of its states we can represent the automaton by a matrix $X \in Z\langle\Sigma\rangle^{n\times n}$ such that each entry $X_{i,j}$ is the characteristic polynomial of the set of letters a that take state i to state j. For brevity, we say that X is a *minimal acceptor matrix* of the language L. Such a matrix is defined over a ring of polynomials in noncommutative variables and here we consider the following notion of determinant for matrices of this kind: for every $n \times n$ matrix $Y = [Y_{ij}]$ with entries in $Z\langle\Sigma\rangle$, the determinant of Y is an element of $Z\langle\Sigma\rangle$ defined by

$$\mathrm{Det}(Y) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma) Y_{\sigma(1),1} Y_{\sigma(2),2} \dots Y_{\sigma(n),n}.$$

Note that the terms of the sum are obtained by choosing, in the order, one entry for each column so that every row is taken just once. Observe that $\mathrm{Det}(Y)$ changes sign whenever we exchange two rows in Y and this proves that $\mathrm{Det}(Y) = 0$ if there are two equal rows in Y.

Example (continued): it is easy to verify that the lexicographic cross section L defined by $<_I$ is the set of all words in Σ^* which do not contain the factors ab, cb, db, dc. Then, a minimal acceptor matrix of L is given by

$$X = \begin{pmatrix} d & a & 0 \\ d & a + c & 0 \\ d & a + c & b \end{pmatrix}$$

Here X represents a finite automaton of three states, all of them are accepting, and the third state is the initial one. □

Using the previous notation our result can be stated as follows:

Theorem 1 *Given an independence alphabet $\langle\Sigma, I\rangle$ defining a trace monoid M, let $<$ be a linear ordering over Σ and let L be the lexicographic cross section defined by $<_I$. Then $<_I$ is transitive if and only if an equation of the form*

$$\xi \cdot \mathrm{Det}(\mathrm{Id} - X) = 1$$

holds, where ξ is the characteristic series of L, X is a minimal acceptor matrix of L and Id *is the corresponding identity matrix.*

A further consequence of this result is that if $\langle \Sigma, I \rangle$ admits a transitive orientation, then the polynomial $p = \sum_j (-1)^j C_j z^j$ is the determinant of the enumerative image of $\mathrm{Id} - X$:

$$p = \mathrm{Det}(\mathrm{Id} - U), \text{ where } U = \varphi(\psi(X)).$$

Example (continued): it is easy to verify that $\mathrm{Det}(\mathrm{Id} - X)$ equals the unambiguous lifting $\mu = 1 - a - b - c - d + ab + cb + db + dc - dcb$. Moreover, in this case we have

$$U = \begin{pmatrix} z & z & 0 \\ z & 2z & 0 \\ z & 2z & z \end{pmatrix}$$

and another simple computation shows that $\mathrm{Det}(\mathrm{Id} - U) = 1 - 4z + 4z^2 - z^3$. □

3 From transitivity to the Möbius function

In this section we prove the result in one direction. We show that, if an independence alphabet $\langle \Sigma, I \rangle$ admits a transitive orientation $<_I$, then the unambiguous Möbius function corresponding to $<_I$ is the polynomial $\mu = \mathrm{Det}(\mathrm{Id} - X)$, where X is a minimal acceptor matrix of the lexicographic cross section L defined by $<_I$. We note that here the language L is the support of the series $\xi = \mu^{-1}$ [5].

The proof is based on the properties of the minimum finite automaton recognizing the language L; in particular the states of the automaton are represented by disjoint subsets of Σ which can be ordered by extending the given orientation.

We first observe that, since $<_I$ is transitive, Equation (2) can be simplified as follows:

$$L = \Sigma^* - \bigcup_{a <_I b} \Sigma^* ab \Sigma^*. \tag{3}$$

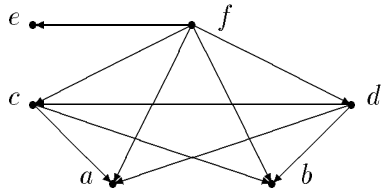
Then, we define the equivalence relation $\equiv$ on Σ by writing $a \equiv b$ *if and only if* the following holds

$$\text{for all } c \in \Sigma, a <_I c \text{ if and only if } b <_I c \tag{4}$$

This induces a partition of Σ into equivalence classes $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$. Observe that if $a, b \in \Sigma_i$ holds for some $i = 1, \ldots, n$, then $(a, b) \notin I$.

Next, we extend the relation $<_I$ to the equivalence classes of $\equiv$ by setting $\Sigma_i <_I \Sigma_j$ *if and only if* there exist two letters $a \in \Sigma_i, b \in \Sigma_j$ such that $a <_I b$. This relation is transitive since $a <_I b, b \equiv c$ and $c <_I d$ implies $b <_I d$, and hence $a <_I d$. Consider a linear extension $\prec$ of the relation $<_I$ on the equivalence classes. This again extends to a linear ordering of the set Σ itself, which we still denote by $\prec$.

Example. Consider the oriented independence alphabet $\langle \Sigma, I \rangle$ defined by the following graph (again, an edge from r to s means $s <_I r$):



In this case the orientation $<_I$ is transitive and the equivalence classes of $\equiv$, considered in the order, are $\{a,b\}$, $\{c\}$, $\{d,e\}$, $\{f\}$. Moreover, the lexicographic cross section defined by $<_I$ is given by

$$L = \Sigma^* - \Sigma^*\{ac, ad, af, bc, bd, bf, cd, cf, df, ef\}\Sigma^*.$$

$\square$

Now, consider the synctactic right invariant $\sim_L$ of L. It is easy to see that, for every nonempty words $x, y \in \Sigma^*$, $x \sim_L y$ *if and only if* either x and y do not belong to L, or x and y are in L and they both end with a symbol in the same class Σ_i, for some i. This implies that the minimum finite automaton $\mathcal{A}$ recognizing L is defined by the set of states

$$Q = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$$

where, without loss of generality, we assume that $\Sigma_1 \prec \Sigma_2 \prec \cdots \prec \Sigma_n$ holds and that the initial state is the set Σ_n containing the elements $a \in \Sigma$ which are maximal with respect to $<_I$. In $\mathcal{A}$ all states are accepting and the transition function is defined by

$$\Sigma_i \cdot b = \begin{cases} \emptyset & \text{if } a <_I b \text{ for some } a \in \Sigma_i \\ \Sigma_j & \text{otherwise, where } b \in \Sigma_j \end{cases} \tag{5}$$

for any $b \in \Sigma$ and any $i = 1, 2, \ldots, n$.

If we order the set of states Q according to $\prec$, the finite automaton $\mathcal{A}$ can be represented by the matrix $X \in \mathrm{Z}\langle\Sigma\rangle^{n \times n}$ such that each entry X_{ij} is given by

$$X_{ij} = \{b \in \Sigma \mid \Sigma_i \cdot b = \Sigma_j\} = \{b \in \Sigma_j \mid \forall a \in \Sigma_i \ \neg(a <_I b)\}.$$

A key property of this matrix is that $X_{ij} \subseteq \Sigma_j$ for all $i, j = 1, 2, \ldots, n$.

Example: in the previous example, we have $\Sigma_1 = \{a,b\}$, $\Sigma_2 = \{c\}$, $\Sigma_3 = \{d,e\}$, $\Sigma_4 = \{f\}$ and the matrix X is as follows

$$\begin{array}{c c} & \begin{array}{cccc} \{a,b\} & \{c\} & \{d,e\} & \{f\} \end{array} \\ \begin{array}{c} \{a,b\} \\ \{c\} \\ \{d,e\} \\ \{f\} \end{array} & \begin{pmatrix} a+b & 0 & e & 0 \\ a+b & c & e & 0 \\ a+b & c & d+e & 0 \\ a+b & c & d+e & f \end{pmatrix} \end{array}$$

$\square$

Observe that under the diagonal each entry contains all symbols in the corresponding state. This is actually a general property that is used in the proof of the following result.

Proposition 2 *Let $\langle\Sigma, I\rangle$ be an independence alphabet that admits a transitive orientation $<_I$. Then the unambiguous Möbius function μ corresponding to $<_I$ is given by*

$$\mu = \mathrm{Det}(\mathrm{Id} - X), \tag{6}$$

where X is the matrix defined above and Id *is the identity matrix of size n.*

Proof. We recall that the determinant of a matrix Y is defined as

$$\mathrm{Det}(Y) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma) Y_{\sigma(1),1} Y_{\sigma(2),2} \dots Y_{\sigma(n),n}$$

In our case this means that

$$\mathrm{Det}(\mathrm{Id} - X) = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} (\delta_{\sigma(i)i} - X_{\sigma(i)i}) \tag{7}$$

where δ_{ij} is the Kronecker symbol. Observe that each product $\prod_{i=1}^{n}(\delta_{\sigma(i)i} - X_{\sigma(i)i})$ is the sum of monomials of the form

$$(-1)^k X_{\sigma(i_1)i_1} X_{\sigma(i_2)i_2} \cdots X_{\sigma(i_k)i_k}$$

where $J = \{i_1, i_2, \dots, i_k\}$ is a subset of $\{1, 2, \dots, n\}$ and the permutation σ is the identity on the complement J^c of J and hence a permutation on J. Then, for a given subset $J = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$, we can group the monomials in the right hand side of (7) for all permutations σ that are identity on J^c. Taking the sum of these monomials, each multiplied by the corresponding $\mathrm{sign}(\sigma)$, we get $(-1)^k \mathrm{Det}(X^J)$, where X^J is the matrix obtained from X by erasing all rows and columns with index in J^c. Hence, letting J range over the family $\mathcal{J}$ of all subsets of $\{1, 2, \dots, n\}$, we obtain

$$\mathrm{Det}(\mathrm{Id} - X) = \sum_{J \in \mathcal{J}} (-1)^{|J|} \mathrm{Det}(X^J),$$

where by convention $\mathrm{Det}(X^{\emptyset}) = 1$.

Now, given a nonempty set $J \subseteq \{1, 2, \dots, n\}$, let $i_1, i_2, \dots, i_k$ be the elements of J in increasing order; we define D_J as the family of subsets

$$D_J = \{\Delta \subseteq \Sigma \mid \ |\Delta \cap \Sigma_i| = 1\ \forall i \in J, \text{ and } \Delta \cap \Sigma_i = \emptyset\ \forall i \notin J\};$$

then we have

$$\mathrm{Det}(X^J) = \sum_{\Delta \in D_J} \mathrm{Det}(X^{J,\Delta}),$$

where $X^{J,\Delta}_{rs} = X_{i_r i_s} \cap \Delta$, for all $i_r, i_s \in J$.

As a consequence $\mathrm{Det}(\mathrm{Id} - X)$ can be written as

$$\mathrm{Det}(\mathrm{Id} - X) = 1 + \sum_{J \in \mathcal{J} \setminus \emptyset} \ \sum_{\Delta \in D_J} (-1)^{|J|} \mathrm{Det}(X^{J,\Delta}) \tag{8}$$

Then, we have to prove that, for every $\Delta \in D_J$, if $a_1, a_2, \dots, a_k$ are the elements of Δ in increasing order then

$$\mathrm{Det}(X^{J,\Delta}) = \begin{cases} 0 & \text{if } \Delta \text{ contains two distinct symbols } a_i, a_j \text{ such that } (a_i, a_j) \notin I, \\ a_1 \dots a_k & \text{if } \Delta \text{ is a clique of } \langle \Sigma, I \rangle. \end{cases}$$

To this end, let $i_1, i_2, \dots, i_k$ be the elements of J in increasing order. Clearly a_j belongs to Σ_{i_j} for each $a_j \in \Delta$. Now assume that Δ is not a clique of $\langle \Sigma, I \rangle$ and consider the last two indices $i_r, i_s \in J$ such that $(a_r, a_s) \notin I$. Then, we can prove the following

Claim *For every $j = 1, 2, \ldots, k$, $a_r <_I a_j$ if and only if $a_s <_I a_j$.*

Indeed, assume $i_r < i_s$. If $a_s <_I a_j$ then $(a_r, a_j) \in I$, otherwise i_j and i_r would be the last two indices of dependent symbols in Δ; then, since $i_r < i_j$, we obtain $a_r <_I a_j$. For the same reason, if $a_r <_I a_j$ then $(a_s, a_j) \in I$; however, we cannot have $(a_j <_I a_s)$ because $<_I$ is transitive and $(a_r, a_s) \notin I$; hence, the only possibility is $a_s <_I a_j$ and this concludes the proof of the claim.

This claim shows that, for each $a_j \in \Delta$,

$$\Sigma_{i_r} \cdot a_j = \Sigma_{i_s} \cdot a_j$$

and hence the rows i_r and i_s of the matrix $X^{J,\Delta}$ are equal proving that the determinant is zero. Moreover, if all letters of Δ commute then, by equation (5), the submatrix $X^{J,\Delta}$ is lower triangular

$$\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2 & 0 & \cdots & 0 \\ & & \cdots & & \\ a_1 & a_2 & \cdots & a_{k-1} & 0 \\ a_1 & a_2 & \cdots & a_{k-1} & a_k \end{pmatrix}$$

Hence, we have proved that the right hand side of (8) is the sum of minimum strings (with respect to the lexicographic order induced by $\prec$) that represent cliques of $\langle \Sigma, I \rangle$. On the other hand, since independent symbols in Σ do not belong to the same class Σ_i, each clique of $\langle \Sigma, I \rangle$ appears as a set Δ in the right hand side of (8). This means that Det(Id-X) is the unambiguous Möbius function corresponding to $<_I$. $\square$

4 From the Möbius function to transitivity

In this section we prove Theorem 1 in the other direction. Given an independence alphabet $\langle \Sigma, I \rangle$ defining a trace monoid M, let $<$ be a total order relation on Σ and let $L \subseteq \Sigma^*$ be the lexicographic cross section of M associated with $<$. Also in this case we denote by $X \in Z\langle\langle \Sigma \rangle\rangle^{n \times n}$ a minimal acceptor matrix of L. Our goal is to prove that if the characteristic series of L is the inverse in $Z\langle\langle \Sigma \rangle\rangle$ of the polynomial $\mathrm{Det}(\mathrm{Id} - X)$, then the restriction of $<$ to I (denoted by $<_I$) is transitive.

The proof is based on the properties of the local language L. Let us say that a language $R \subseteq \Sigma^*$ is *local* if, for every pair of words $x, y \in \Sigma^*$ having the same factors of length 2, $x \in R$ implies $y \in R$ (this is a slightly less general class of languages than usually considered, cf. [1], p.18 or [8], p.27). We first prove the following

Proposition 3 *The lexicographic cross-section L is local if and only if $<_I$ is transitive.*

Proof. If $<_I$ is transitive we know from the previous section that Equation (3) holds and this proves that L is local. On the other hand, assume that $<_I$ is not transitive. Then, for some $a, b, c \in \Sigma$, we have $a <_I b <_I c$ while $\neg a <_I c$. Hence, the words $u = cbaca$ and $v = cacba$ represent the same trace. Moreover, since $<_I$ is the restriction of a linear order on Σ, we cannot have $c <_I a$ and hence $(a, c) \notin I$. This means that u is the largest string in $[u]$ and hence it belongs to L while v does not. However u and v have the same factors of length 2 and this proves that L is not local. $\square$

Another general property of the language L concerns the relation $\sim_L$ denoting the syntactic right invariant of L.

Lemma *For every $u \in \Sigma^*$ and every $a \in \Sigma$ we have $ua \sim_L uaa$.*

Proof. Indeed, from Equation (2) it is clear that, for every $v \in \Sigma^*$, $uaav \in L$ implies $uav \in L$. Moreover, if $uaav \notin L$ and $ua \in L$, then either $v \notin L$, or $v = lbv'$ for some $b \in \Sigma$ and some $l, v' \in \Sigma^*$ such that $a <_I b$ and all symbols occurring in l belong to I_b. In both cases this means that $uav \notin L$ and hence the lemma is proved. □

Now, let us consider the smallest finite automaton $\mathcal{A}$ recognizing L and let X be a matrix representing $\mathcal{A}$. We denote by q_0 the initial state of this automaton. By Equation (2), the language L contains all the prefixes of its words; this implies that all states of $\mathcal{A}$ are final and hence any word $x \in \Sigma^*$ belongs to L if and only if $q_0 \cdot x$ is well-defined. Then, denoting by ξ the characteristic series of L, we can prove the following

Proposition 4 *If* $\xi \cdot \mathrm{Det}(\mathrm{Id} - X) = 1$ *then L is a local language.*

Proof. Since each entry X_{ij} is a subset of Σ, it is clear that the monomials of first degree in $\mathrm{Det}(\mathrm{Id} - X)$ appear on the diagonal of X. By the hypothesis this implies

$$\sum_{a \in \Sigma} \xi(a) = -\sum_{i=1}^{n} X_{ii}.$$

Hence, every $a \in \Sigma$ can be associated just with one state, denoted by $q(a)$, such that

$$q(a) \cdot a = q(a). \tag{9}$$

Together with the previous Lemma this equation shows that for every state p and every $a \in \Sigma$ either $p \cdot a$ is not defined, or $p \cdot a = q(a)$.
Then, consider two words $x, y \in \Sigma^*$ that have the same factors of length 2. In order to prove the proposition we have to show that $x \in L$ implies $y \in L$. Indeed, let $x = a_1 a_2 \cdots a_h$ and $y = b_1 b_2 \cdots b_k$, where $h, k > 1$ and $a_i, b_j \in \Sigma$ for every i, j. Reasoning by induction, we prove that $q_0 \cdot b_1 b_2 ... b_j$ is well-defined for every $j = 1, 2, \ldots, k$. Clearly, this is true for $j = 1$. Assume that the property is true for some $j < k$. Then we have $q_0 \cdot b_1 b_2 ... b_j = q(b_j)$. Let i be an index such that $a_i a_{i+1} = b_j b_{j+1}$; since $x \in L$, we have $q(a_{i-1}) \cdot a_i a_{i+1} = q(a_{i+1})$ and hence

$$q(b_j) \cdot b_{j+1} = q(a_i) \cdot a_{i+1} = q(a_{i+1}) = q(b_{j+1})$$

proving that $q_0 \cdot b_1 b_2 ... b_{j+1}$ is well-defined. □

It is clear that Propositions 4 and 3 complete the proof of Proposition 1.

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