On the maximum coefficients of rational formal series in commuting variables

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Abstract

We study the growth function of a $\mathbb{R}_+$-rational formal series in two commuting variables which assigns to every integer $n \in \mathbb{N}$, the maximum coefficient of the monomials of degree n. We give a rather general sufficient condition under which the growth function of such a series is of the order $\Theta(n^{k/2}\lambda^n)$ for some integer $k \geq -1$ and some positive real λ. Our analysis is related to the study of limit distributions in pattern statistics. In particular, we prove a general criterion for establishing Gaussian local limit laws for sequences of discrete positive random variables.

Keywords: Automata and Formal Languages, Local Limit Theorems, Pattern Statistics, Rational Formal Series.

1 Introduction

The general observation motivating this paper is the following. Consider a rational fraction $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are two polynomials with coefficients in the field of real numbers (with $q(0) \neq 0$). It is well-known that the coefficient of the term x^n of its Taylor expansion is asymptotically equivalent to a linear combination of expressions of the form $n^{k-1}\lambda^n$ where λ is a root of $q(x)$ and k its multiplicity, cf. [9, Theorem 6.8] or [14, Lemma II.9.7].

The purpose of this work is to pose the problem in more general terms. Indeed, assume we are given a semiring $\mathbb{K}$ whose underlying set is contained in the reals, e.g., $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}_+$, etc. Assume further we are given a trace monoid, i.e., a monoid $\mathcal{M}$ whose presentation is of the form $\langle \Sigma; R\rangle$ where Σ is a finite alphabet and R is a family of relations of the form $ab = ba$ with $a, b \in \Sigma$. Observe that for these monoids, the notion of length makes sense. Consider next a rational formal series r over $\mathcal{M}$ with coefficients in $\mathbb{K}$ and denote by (r, x) the corresponding *coefficient* of the element $x \in \mathcal{M}$. Denoting by $|x|$ the length of x, we want to investigate the (*growth*) function

$$g_r(n) = \max_{|x|=n} |(r, x)|$$

where $|(r, x)|$ denotes the absolute value of (r, x).

The problem for free monoids and the semirings $\mathbb{Z}$ and $\mathbb{N}$ was completely solved in [16] where it is shown that if the growth function is not exponential then there exists an integer k such that the growth function is of the order $\Theta(n^k)$. In that case, the series is the sum of terms each of which is the product of at most $k + 1$ rational series that are the characteristic series of rational subsets of a free monoid, see also [2, Corollary 2.11.]. When the semiring of coefficients is $\mathbb{N}$ the problem is related to the analysis of ambiguity of formal grammar (or finite automata) generating (recognizing, resp.) the support of the series; a wide literature has been devoted to this problem (see for instance [11, 20, 23]). It is worth noting that, in the algebraic case, rather surprising results have been obtained concerning the asymptotic of growth functions [21, 22].

Another case can be found in the literature. It is concerned with the tropical semiring $\mathcal{T}$ whose support is the set $\mathbb{N} \cup \{\infty\}$ and whose operations are the $\min$ for the addition and the $+$ for the multiplication. In [18], Imre Simon proves that for all $\mathcal{T}$-rational series s over the free monoid $\{a, b\}^*$, there exists an integer k such that $(s, w) = O(|w|^{1/k})$ holds for all $w \in \{a, b\}^*$. Moreover it is proven that for each positive integer k, there exist a $\mathcal{T}$-rational series s_k such that $g_{s_k}(n) = \Theta(n^{1/k})$. Thus, the hierachy is strict though it is not proven that all series have an asymptotic growth of this kind.

In the present work, we study the growth function of rational formal series in two commuting variables a, b with coefficients in $\mathbb{R}_+$. As far as we know, the general problem of characterizing the order of magnitude of $g_s(n)$ for such a series s is still open, though some contributions can be found in the literature [3]. We prove that for each integer $k \geq 1$ if s is the commutative image of r^k, for certain $\mathbb{R}_+$-rational formal series r over the free monoid $\{a, b\}^*$, then $g_s(n) = \Theta(n^{k-(3/2)}\lambda^n)$ for some positive real λ.

This result is obtained by studying the limit distribution of the discrete random variables $\{X_n\}$ naturally associated with the series s: each X_n takes values only in $\{0, 1, \dots, n\}$ and $\Pr\{X_n = i\}$ is proportional to the coefficient $(s, a^i b^{n-i})$. We prove that if s is the commutative image of r^k, for some positive $k \in \mathbb{N}$ and some (nondegenerate) primitive $\mathbb{R}_+$-rational formal series r over $\{a, b\}^*$, then X_n has a Gaussian limit distribution; if further r satisfies a simple condition on the eigenvalues of certain matrices associated with its primitive representation, then a local limit theorem holds for X_n (in the sense of de Moivre-Laplace [8]). This is obtained by proving a general criterion for local limit laws which could be useful in other contexts. These probabilistic results are of interest also in pattern statistics, where the main goal is the analysis of the number of occurrences of patterns in a word of length n generated at random according with suitable stochastic models [12, 4]. In particular, they extend some results obtained for stochastic models defined by the product of two primitive rational formal series [6] to models defined by powers of series of the same kind.

2 Preliminaries

Given a monoid $\mathcal{M}$ and a semiring $\mathbb{K}$ we call *formal series* over $\mathcal{M}$ any application $r : \mathcal{M} \longrightarrow \mathbb{K}$, usually denoted as an infinite formal sum $r = \sum_{w\in\mathcal{M}}(r, w)\, w$, which associates each $w \in \mathcal{M}$ with its *coefficient* $(r, w) \in \mathbb{K}$. The set of all series over $\mathcal{M}$ with coefficients in $\mathbb{K}$ is a monoid algebra, provided with the usual operations of sum, product and star restricted to the elements r such that $(r, 1_{\mathcal{M}}) = 0_{\mathbb{K}}$. These operations are called rational operations. A series r is called *rational* if it belongs to the smallest set closed under rational operations, containing the series 0 and all the series χ_w, for $w \in \mathcal{M}$, such that $(\chi_w, w) = 1$ and $(\chi_w, x) = 0$ for each $x \neq w$.

In this work we study the formal series over free monoids or free commutative monoids with coefficients in the semiring $\mathbb{R}_+$ of nonnegative real numbers. If $\mathcal{M}$ is the free monoid generated by a set symbols $\{\sigma_1, \sigma_2, \cdots, \sigma_\ell\}$, we denote by $\mathbb{R}_+\langle\langle\sigma_1, \cdots, \sigma_\ell\rangle\rangle$ the family of all formal series over $\mathcal{M}$ with coefficients in $\mathbb{R}_+$ and by $\mathbb{R}_+^{Rat}\langle\langle\sigma_1, \cdots, \sigma_\ell\rangle\rangle$ the subset of rational series. By Kleene's Theorem [14, 13], we know that every $r \in \mathbb{R}_+^{Rat}\langle\langle\sigma_1, \cdots, \sigma_\ell\rangle\rangle$ admits a *linear representation* over the alphabet $\{\sigma_1, \sigma_2, \cdots, \sigma_\ell\}$, i.e. a triple (ξ, μ, η) such that, for some integer $m > 0$, ξ and η are (nonnull column) vectors in $\mathbb{R}_+^m$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}_+^{m\times m}$ is a monoid morphism, satisfing $(r, w) = \xi^T \mu(w)\ \eta$ for each $w \in \mathcal{M}$. We say that m is the *size* of the representation. Observe that considering such a triple is equivalent to defining a (weighted) non–deterministic finite automaton over the alphabet $\{\sigma_1, \sigma_2, \cdots, \sigma_\ell\}$, where the state set is given by $\{1, 2, \ldots, m\}$ and the transitions, the initial and the final states are assigned weights in $\mathbb{R}_+$ by μ, ξ and η respectively.

Analogously, if $\mathcal{M}$ is the free commutative monoid with set of generators $\{\sigma_1, \sigma_2, \cdots, \sigma_\ell\}$, we denote by $\mathbb{R}_+[[\sigma_1, \cdots, \sigma_\ell]]$ the family of all formal series over $\mathcal{M}$ with coefficients in $\mathbb{R}_+$. In this case, any element $\sigma_1^{i_1}\cdots\sigma_\ell^{i_\ell}$ of $\mathcal{M}$ is represented in the form $\underline{\sigma}^{\underline{i}}$, where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_\ell)$ and $\underline{i} = (i_1, \ldots, i_\ell) \in \mathbb{N}^\ell$. The canonical morphism $\varphi : \{\sigma_1, \cdots, \sigma_\ell\}^* \to \mathcal{M}$, associating each $w \in \{\sigma_1, \cdots, \sigma_\ell\}^*$ to the monomial $\underline{\sigma}^{\underline{i}}$ (where $i_j = |w|_{\sigma_j}$ for every j), extends to the semiring of formal series: for every $r \in \mathbb{R}_+\langle\langle\sigma_1, \cdots, \sigma_\ell\rangle\rangle$ and every $\underline{\sigma}^{\underline{i}} \in \mathcal{M}$,

$$(\varphi(r), \underline{\sigma}^{\underline{i}}) = \sum_{|x|_{\sigma_j}=i_j} (r, x) .$$

Moreover, we denote by $\mathbb{R}_+^{Rat}[[\sigma_1, \cdots, \sigma_\ell]]$ the family of all rational formal series in $\mathbb{R}_+[[\sigma_1, \cdots, \sigma_\ell]]$. Since φ is a morphism, we know that for every $r \in \mathbb{R}_+^{Rat}\langle\langle\sigma_1, \cdots, \sigma_\ell\rangle\rangle$, $\varphi(r)$ belongs to $\mathbb{R}_+^{Rat}[[\sigma_1, \cdots, \sigma_\ell]]$.

For our purpose, a subset of rational series of particular interest is defined by the so-called primitive linear representations. We recall that a matrix $M \in \mathbb{R}_+^{m\times m}$ is *primitive* if for some $k \in \mathbb{N}$ all entries of M^k are strictly positive. The main property of these matrices is given by the Perron–Froblenius Theorem, stating that any primitive matrix M has a unique eigenvalue λ of largest modulus, which is real positive (see for instance [17, Chapter 1]). Such a λ is called the Perron-Frobenius eigenvalue of M.

Thus, a linear representation (ξ, μ, η) defined over a set of generators Σ is called primitive if $\sum_{\sigma\in\Sigma}\mu(\sigma)$ is a primitive matrix. In this case we say that also the series r defined by (ξ, μ, η) is primitive. Moreover, if Σ is a binary alphabet $\{a, b\}$ we say that (ξ, μ, η) is *degenerate* if $\mu(\sigma) = 0$ for some $\sigma \in \Sigma$. Analogously, we say that $r \in R_+\langle\langle a, b\rangle\rangle$ is degenerate if, for some $\sigma \in \{a, b\}$, $(r, w) \neq 0$ implies $w \in \{\sigma\}^*$.

Given a monoid $\mathcal{M}$, let us assume that a length function $|\cdot| : \mathcal{M} \longrightarrow \mathbb{N}$ is well-defined for $\mathcal{M}$ such that the set $\{x \in \mathcal{M} : |x| = n\}$ is finite for each $n \in \mathbb{N}$. Then, for any formal series $s : \mathcal{M} \longrightarrow \mathbb{C}$ we define the *growth* function $g_s : \mathbb{N} \longrightarrow \mathbb{R}_+$ as

$$g_s(n) = \max\{|(s, x)| \ :\ x \in \mathcal{M}, |x| = n\} \qquad \text{(for every } n \in \mathbb{N}).$$

In the following $|\cdot|$ will denote both the modulus of a complex number and the length of a word: the meaning will be clear from the context [1].

[1] On the contrary, $|x|_a$ will always represent the number of occurrences of a symbol a in a word x.

Our main results concern the order of magnitude of $g_r(n)$ for some $r \in \mathbb{R}_+[[\sigma_1, \cdots, \sigma_\ell]]$. To state them precisely we use the symbol Θ with the following meaning: given two sequences $\{f_n\}, \{g_n\} \subseteq \mathbb{R}_+$, the equality $g_n = \Theta(f_n)$ means that for some pair of positive constant c_1, c_2, the relation

$$c_1 f_n \leq g_n \leq c_2 f_n$$

holds for any n large enough.

2.1 Convergence in distribution

Let X be a random variable (r.v.) with values in a set $\{x_0, x_1, \ldots, x_k, \ldots\}$ of real numbers and set $p_k = \Pr\{X = x_k\}$, for every $k \in \mathbb{N}$. We denote by F_X its distribution function, i.e. $F_X(\tau) = \Pr\{X \leq \tau\}$ for every $\tau \in \mathbb{R}$. If the set of indices $\{k \mid p_k \neq 0\}$ is finite we can consider the moment generating function of X, given by

$$\Psi_X(z) = \mathbb{E}(e^{zX}) = \sum_{k \in \mathbb{N}} p_k e^{z x_k} ,$$

which is well-defined for every $z \in \mathbb{C}$. This function can be used to compute the first two moments of X,

$$\mathbb{E}(X) = \Psi'_X(0) , \qquad \mathbb{E}(X^2) = \Psi''_X(0) , \tag{1}$$

and to prove convergence in distribution. We recall that, given a sequence of random variables $\{X_n\}_n$ and a random variable X, X_n converges to X *in distribution* (or *in law*) if $\lim_{n\to\infty} F_{X_n}(\tau) = F_X(\tau)$ for every point $\tau \in \mathbb{R}$ of continuity for F_X. It is well-known that if Ψ_{X_n} and Ψ_X are defined all over $\mathbb{C}$ and $\Psi_{X_n}(z)$ tends to $\Psi_X(z)$ for every $z \in \mathbb{C}$, then X_n converges to X in distribution [8].

A convenient approach to prove the convergence in law to a Gaussian random variable relies on the so called "quasi-power" theorems introduced in [10] and implicitely used in the previous literature [1] (see also [7]). For our purpose we present the following simple variant of such a theorem, the proof of which is here given in detail because some steps will be used in subsequent sections.

Theorem 1 *Let $\{X_n\}$ be a sequence of random variables, where each X_n takes values in $\{0, 1, \ldots, n\}$ and let us assume the following conditions:*

C1 *There exist two functions $r(z)$, $y(z)$, both analytic at $z = 0$ where they take the value $r(0) = y(0) = 1$, and a positive constant c, such that for every $|z| < c$*

$$\Psi_{X_n}(z) = r(z) \cdot y(z)^n \left(1 + O(n^{-1})\right) ; \tag{2}$$

C2 *The constant σ given by*

$$\sigma = y''(0) - (y'(0))^2$$

is strictly positive (variability condition).

Also set $\mu = y'(0)$. Then $\frac{X_n - \mu n}{\sqrt{\sigma n}}$ converges in distribution to a normal random variable of mean 0 and variance 1, i.e. for every $x \in \mathbb{R}$

$$\lim_{n \longrightarrow +\infty} Pr\left\{\frac{X_n - \mu n}{\sqrt{\sigma n}} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt .$$

Proof. By the previous hypothesis, both μ and σ are well-defined and in a neighbourhood of $z = 0$ the function $y(z)$ admits an expansion of the form

$$y(z) = \left\{1 + \mu z + \frac{\sigma + \mu^2}{2} z^2 + \mathrm{O}(z^3)\right\} \tag{3}$$

Let us define $U_n = \frac{X_n - \mu n}{\sqrt{\sigma n}}$. Then the moment generating function of U_n is given by

$$\Psi_{U_n}(z) = \mathbb{E}\left\{ e^{z \frac{X_n - \mu n}{\sqrt{\sigma n}}} \right\} = e^{-z\mu\sqrt{n/\sigma}} \Psi_{X_n}\left(z/\sqrt{\sigma n} \right).$$

Now, replacing (3) in (2), from the previous equation we obtain

$$\Psi_{U_n}(z) = e^{-z\mu\sqrt{n/\sigma}} \cdot \left(1 + \mu \frac{z}{\sqrt{\sigma n}} + \frac{\sigma + \mu^2}{2} \frac{z^2}{\sigma n} + \mathrm{O}(n^{-3/2}) \right)^n \cdot \left(1 + \mathrm{O}(n^{-1/2}) \right)$$

The right hands side tends to $e^{z^2/2}$, which is the moment generating function of a normal random variable of mean 0 and variance 1. $\square$

The main advantage of this theorem, with respect to other classical statements of this kind, is that it does not require any condition of independency concerning the random variables X_n. For instance, the standard central limit theorems assume that each X_n is a partial sum of the form $X_n = \sum_{j \le n} U_j$, where the U_j's are independent random variables [8].

3 A general criterion for local convergence laws

Convergence in law of a sequence of r.v.'s $\{X_n\}$ does not yield an approximation of the probability that X_n takes a specific value. Theorems concerning approximations for expressions of the form $\Pr\{X_n = x\}$ are usually called local limit theorems and often give a property stronger than a traditional convergence in distribution [2]. A typical example is given by the so-called de Moivre-Laplace Local Limit Theorem [8], which intuitively states that, for certain sequences of binomial random variables X_n, up to a factor $\Theta(\sqrt{n})$ the probabity that X_n takes a value x approximates a Gaussian density at x.

In this section we present a general criterion that guarantees, for a sequence of discrete random variables, the existence of a local convergence property of a Gaussian type, in the same sense of de Moivre-Laplace's Theorem mentioned above. In the subsequent section, using such criterion, we show that the same local convergence property holds for certain pattern statistics.

Theorem 2 (Local Limit Criterion) *Let $\{X_n\}$ be a sequence of random variables such that each X_n takes values in $\{0, 1, \ldots, n\}$. Assume that conditions C1 and C2 of Theorem 1 hold true and let μ and σ be the positive constants defined in the same theorem. Moreover assume the following condition:*

C3 *For all* $0 < \theta_0 < \pi$ $\qquad \lim_{n \to +\infty} \left\{ \sqrt{n} \sup_{\theta_0 \le \theta \le 2\pi - \theta_0} |\Psi_{X_n}(i\theta)| \right\} = 0$

Then, as n grows to $+\infty$ the relation

$$\sqrt{\sigma n} \cdot Pr\{X_n = j\} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(j - \mu n)^2}{2\sigma n}} \cdot (1 + o(1)) \tag{4}$$

holds uniformly for every $j = 0, 1, \ldots, n$.

Observe that $\Psi_{X_n}(i\theta)$ is the so-called characteristic function of X_n, it is periodic of period 2π, it takes the value 1 at $\theta = 0$. The condition C3 states that, for every constant $0 < \theta_0 < \pi$, as n grows to $+\infty$ the value $\Psi_{X_n}(i\theta)$ is of the order $o(n^{-1/2})$ uniformly with respect to θ in the interval $[\theta_0, 2\pi - \theta_0]$; this also implies that its modulus assumes the maximum value only at the points $\theta = 2k\pi$ ($k \in \mathbb{Z}$).

When X_n is a sequence of binomial distributions of parameter n and p (i.e. the number of successes over n indipendent trials of probability p), equation (4) corresponds to de Moivre-Laplace Local Limit

[2]For this reason, theorems concerning convergence in distribution are sometimes called global or integral limit theorems.

Theorem. Actually, one can easily prove that condition C3 holds for such a sequence whenever $0 < p < 1$ (thus obtaning another proof of the same theorem).

We note that this criterion seems to be different from the hypothesis of the general theorem for local laws given in [7, Theorem 9.10] where, together with a condition on the unicity of maximum for the characteristic function, an expansion of the form (2) is required for the probability generating functions $p_n(z)$ of X_n, with z belonging to an annulus $1-\varepsilon \leq |z| \leq 1+\varepsilon$ $(\varepsilon > 0)$. In particular, this last hypothesis implies condition C1.

The proof of Theorem 2 can be split in two steps. First we show an approximation for the characteristic functions of $\{X_n\}$. Then we use the Discrete Fourier Transform to estimate the asymptotic behaviour of the values $\Pr\{X_n = j\}$ for $j = 0, 1, \ldots, n$.

3.1 Approximating the characteristic function

The first step is based on the evaluation of the modulus and the argument for the function y defined in condition C1.

Proposition 3 *Let $\{X_n\}$ be a sequence of random variables such that each X_n takes values in $\{0, 1, \ldots, n\}$. Assume that conditions C1 and C2 of Theorem 1 hold true and let $y(z)$ be the function defined by condition C1. Then, in a neighbourhood of $\theta = 0$, the function $y(i\theta)$ of real variable θ satisfies the equalities*

$$|y(i\theta)| = \left(1 - \frac{\sigma}{2}\theta^2 + O(\theta^3)\right) \tag{5}$$

$$arg(y(i\theta)) = \mu(\theta) + O(\theta^3)\ , \tag{6}$$

where μ and σ are defined as in Theorem 1.

Proof. >From (3) we have

$$\begin{aligned} |y(i\theta)| &= \left|1 + i\mu\theta - \frac{\sigma+\mu^2}{2}\theta^2\right| \cdot |1 + \mathrm{O}(\theta^3)| = \\ &= \sqrt{\left(1 - \frac{\sigma+\mu^2}{2}\theta^2\right)^2 + \mu^2\theta^2} \cdot |1 + \mathrm{O}(\theta^3)| = \\ &= \left(1 - \frac{\sigma}{2}\theta^2 + \mathrm{O}(\theta^4)\right) |1 + \mathrm{O}(\theta^3)| \end{aligned}$$

which proves the first relation. As far as the second one is concerned, observe that

$$\begin{aligned} \arg(y(i\theta)) &= \arg\left(1 + i\mu\theta - \frac{\sigma+\mu^2}{2}\theta^2\right) + \arg(1 + \mathrm{O}(\theta^3)) = \\ &= \operatorname{arctg}\left(\frac{\mu\theta}{1 - \frac{\sigma+\mu^2}{2}\theta^2}\right) + \mathrm{O}(\theta^3) = \mu(\theta) + \mathrm{O}(\theta^3)\ . \end{aligned}$$

□

Proposition 4 *Under the same hypothesis of Proposition 3, for every $1/3 < \varepsilon < 1/2$ and every θ such that $|\theta| \leq 2\pi(n+1)^{-\varepsilon}$, we have*

$$\left|y(i\theta)^n - e^{-(\sigma/2)\theta^2 n + i\mu\theta n}\right| = O\left(n^{-1/2}\right)$$

Proof. >From equations (5) and (6) one has

$$y(i\theta)^n = \left(1 - \frac{\sigma}{2}\theta^2 + \mathrm{O}(\theta^3)\right)^n \cdot e^{in(\mu\theta + \mathrm{O}(\theta^3))} = \mathrm{Exp}\left(-\frac{\sigma}{2}n\theta^2 + i\mu n\theta + n\mathrm{O}(\theta^3)\right)$$

Since $|\theta| = o(n^{-1/3})$ the last expression yields

$$y(i\theta)^n = \mathrm{Exp}\left(-\frac{\sigma}{2}n\theta^2 + i\mu n\theta\right) \cdot (1 + \mathrm{O}(n\theta^3))$$

and hence

$$\left|y(i\theta)^n - e^{-(\sigma/2)n\theta^2 + i\mu n\theta}\right| = \mathrm{O}\left(n|\theta^3|e^{-n\theta^2\sigma/2}\right) = \mathrm{O}\left(n^{-1/2}\right) ,$$

the last equality being obtained by deriving with respect to θ. □

Proposition 5 *Under the same hypothesis of Proposition 3, assume further that condition C3 holds true. Then there exists a constant* $0 < \theta_0 < \pi$ *such that, for every* $1/3 < \varepsilon < 1/2$ *and every* $-\pi < \theta \leq \pi$,

$$\left|\Psi_{X_n}(i\theta) - e^{-(\sigma/2)\theta^2 n + i\mu\theta n}\right| = \Delta_n(\theta)$$

where

$$\Delta_n(\theta) = \begin{cases} O(n^{-\varepsilon}) & \text{if } |\theta| \leq \frac{2\pi}{(n+1)^\varepsilon} \\ O\left(e^{-\sigma\pi^2 n^{1-2\varepsilon}}\right) & \text{if } \frac{2\pi}{(n+1)^\varepsilon} \leq |\theta| \leq \theta_0 \\ o(n^{-1/2}) & \text{if } \theta_0 \leq |\theta| \leq \pi \end{cases} \tag{7}$$

Proof. For the sake of brevity let Ψ_n stand for Ψ_{X_n}. We prove the proposition by considering separately the three intervals for the variable θ given in (7). Thus, first assume $|\theta| \leq \frac{2\pi}{(n+1)^\varepsilon}$. By condition C1 of Theorem 1 and Proposition 4 we have

$$\Psi_n(i\theta) = r(i\theta)y(i\theta)^n(1 + \mathrm{O}(n^{-1})) = \left(e^{-(\sigma/2)\theta^2 n + i\mu\theta n} + \mathrm{O}(n^{-1/2})\right)(1 + \mathrm{O}(n^{-\varepsilon}))$$

which proves the relation since $|e^{-(\sigma/2)\theta^2 n + i\mu\theta n}| \leq 1$ for every real θ.

Now, let θ_0 be a constant such that $0 < \theta_0 < c$, where c is defined as in condition C1 of Theorem 1. Consider the interval $\frac{2\pi}{(n+1)^\varepsilon} \leq |\theta| \leq \theta_0$. Then we have

$$\left|\Psi_n(i\theta) - e^{-(\sigma/2)\theta^2 n + i\mu\theta n}\right| \leq |\Psi_n(i\theta)| + e^{-(\sigma/2)\theta^2 n}.$$

Since $|\theta| \geq 2\pi(n+1)^{-\varepsilon}$ the second term of the right hand side is smaller or equal to $e^{-\sigma\pi^2 n^{1-2\varepsilon}}$. Hence we simply have to show an analogous bound for the first term. To this end, by equations (2) and (5) we have

$$|\Psi_n(i\theta)| \;=\; |r(i\theta)y(i\theta)^n(1 + \mathrm{O}(n^{-1}))| \;\leq\; r\left|1 - \frac{\sigma}{2}\theta^2 + \mathrm{O}(\theta^3)\right|^n |1 + \mathrm{O}(n^{-1})| \tag{8}$$

where $r = \sup_{|\theta| \leq \theta_0} |r(i\theta)|$. By the arbitrariness of θ_0, for some constant $d > 0$ and every $|\theta| \leq \theta_0$ we have

$$\left|1 - \frac{\sigma}{2}\theta^2 + \mathrm{O}(\theta^3)\right| \leq \left|1 - \frac{\sigma}{2}\theta^2\right| + d\theta^3 .$$

By the same reason we may assume $\theta_0 \leq \min\{\sqrt{2/\sigma},\ \sigma/(4d)\}$, which proves

$$\left|1 - \frac{\sigma}{2}\theta^2 + \mathrm{O}(\theta^3)\right| \leq 1 - \frac{\sigma}{2}\theta^2 + d\theta_0\theta^2 \leq 1 - \frac{\sigma}{4}\theta^2 ,$$

for every $|\theta| \leq \theta_0$. Replacing this value in (8) we get

$$|\Psi_n(i\theta)| = \mathrm{O}\left(\left|1-(\sigma/4)\theta^2\right|^n\right) = \mathrm{O}\left(e^{-(\sigma/4)\theta^2 n}\right) ,$$

which is bounded by $\mathrm{O}\left(e^{-\sigma\pi^2 n^{1-2\varepsilon}}\right)$ because of the range of θ . This concludes the analysis of the second interval.

Finally assume $\theta_0 \leq |\theta| \leq \pi$. Again, we have

$$\left|\Psi_n(i\theta) - e^{-(\sigma/2)\theta^2 n + i\mu\theta n}\right| \leq \sup_{\theta_0 \leq |\theta| \leq \pi} |\Psi_n(i\theta)| + e^{-(\sigma/2)\theta_0^2 n} .$$

The first term is bounded by $\mathrm{o}(n^{-1/2})$ by condition C3 while the second one is $\mathrm{O}(\tau^n)$ for some $0 < \tau < 1$. □

The last property suggests to define the following approximation of the characteristic function of X_n , where μ and θ are supposed to be positive constants. Let $\hat{F} : \mathbb{R} \to \mathbb{C}$ be the function such that:
- $\hat{F}(\theta) = e^{-(\sigma/2)\theta^2 n + i\mu\theta}$ for every $-\pi < \theta \leq \pi$;
- $\hat{F}(\theta) = \hat{F}(\theta + 2k\pi)$ for every $\theta \in \mathbb{R}$ and every $k \in \mathbb{Z}$.

3.2 Discrete Fourier Transform

Now, let us go back to our initial goal, which is the proof of relation (4). We prove it by using the Discrete Fourier Transform (see for instance [5]). To this end, let $p^{(n)}$ be the array $(p_0^{(n)}, p_1^{(n)}, \ldots, p_n^{(n)})$ where

$$p_j^{(n)} = \Pr\{X_n = j\} \qquad (j = 0, 1, \ldots, n)$$

and let $f^{(n)} \in \mathbb{C}^{n+1}$ be its Discrete Fourier Transform, i.e. the array $(f_0^{(n)}, f_1^{(n)}, \ldots, f_n^{(n)})$ such that

$$f_s^{(n)} = \sum_{j=0}^{n} p_j^{(n)} e^{i\frac{2\pi s}{n+1}j} = \Psi_{X_n}\left(i\frac{2\pi s}{n+1}j\right)$$

for every $s = 0, 1, \ldots, n$. By antitransforming, each $p_j^{(n)}$ can be obtained by

$$p_j^{(n)} = \frac{1}{n+1}\sum_{s=0}^{n} f_s^{(n)} e^{-i\frac{2\pi s}{n+1}j} = \frac{1}{n+1}\sum_{s=0}^{n} \Psi_{X_n}\left(i\frac{2\pi s}{n+1}j\right) e^{-i\frac{2\pi s}{n+1}j}. \tag{9}$$

The idea is to approximate the values of Ψ_{X_n} in (9) by using Proposition 5, thus obtaining a tight approximation of the coefficients $p_j^{(n)}$. First we define $\hat{p}_j^{(n)}$ by

$$\hat{p}_j^{(n)} = \frac{1}{n+1}\sum_{s=0}^{n} \hat{F}_n\left(\frac{2\pi s}{n+1}j\right) e^{-i\frac{2\pi s}{n+1}j}. \tag{10}$$

Then, by using standard mathematical tools, one can prove that as n grows to $+\infty$ the relation

$$\hat{p}_j^{(n)} = \frac{1}{\sqrt{2\pi\sigma n}} e^{-\frac{(j-\mu n)^2}{2\sigma n}} + \mathrm{o}(n^{-1/2}) \tag{11}$$

holds uniformly for every $j = 0, 1, \ldots, n$ (a detailed proof can be found in [3, Section 5.3]).

Moreover, one can bound the error associated with the above approximation.

Proposition 6 *Under the same hypothesis of Proposition 5 the relation*

$$\left|p_j^{(n)} - \hat{p}_j^{(n)}\right| = o(n^{-1/2})$$

holds uniformly for every $j = 0, 1, \ldots, n$.

Proof. >From (9) and (10), recalling the periodicity of $\Psi_{X_n}(\theta)$ and $\hat{F}_n(\theta)$, we have

$$\left|p_j^{(n)} - \hat{p}_j^{(n)}\right| \leq \frac{2}{n+1} \sum_{s=0}^{\lfloor \frac{n+1}{2} \rfloor} \left| \Psi_{X_n}\left(i \frac{2\pi s}{n+1} j\right) - \hat{F}_n\left(\frac{2\pi s}{n+1} j\right) \right| \; ;$$

now, by Proposition 5, we get

$$\left|p_j^{(n)} - \hat{p}_j^{(n)}\right| \leq \frac{2}{n+1} \sum_{s=0}^{\lfloor \frac{n+1}{2} \rfloor} \Delta_n\left(\frac{2\pi s}{n+1} j\right)$$

This sum can be computed by splitting the range of s in three parts, corresponding to the intervals studied in Proposition 5. Thus, for any $1/3 < \varepsilon < 1/2$, the previous right hand side is bounded by

$$\frac{2}{n+1}\left\{(n+1)^{1-\varepsilon}\mathrm{O}(n^{-\varepsilon}) + \frac{\theta_0}{2\pi}(n+1)\mathrm{O}\left(e^{-\sigma\pi^2 n^{1-2\varepsilon}}\right) + \frac{n+1}{2}\mathrm{o}(n^{-1/2})\right\} \; = \; \mathrm{o}(n^{-1/2}) \; .$$

□

Proof of Theorem 2. The statement is a straightforward consequence of Propostion 6 and equation (11).

4 Pattern statistics over rational series

In this section we turn our attention to sequences of random variables defined by means of rational formal series in two noncommuting variables.

Let us consider the binary alphabet $\{a, b\}$ and, for $n \in \mathbb{N}$, let $\{a, b\}^n$ denote the set of all words of length n in $\{a, b\}^*$. Given a formal series $r \in \mathbb{R}_+\langle\langle a, b\rangle\rangle$, let n be a positive integer such that $(r, x) \neq 0$ for some $x \in \{a, b\}^n$. Consider the probability space of all words in $\{a, b\}^n$ equipped with the probability measure given by

$$\Pr\{w\} = \frac{(r, w)}{\sum_{x \in \{a,b\}^n}(r, x)} \qquad (w \in \{a, b\}^n).$$

In particular, if r is the characteristic series χ_L of a language $L \subseteq \{a, b\}^*$, then Pr is just the uniform distribution over the set of words on length n in L: $\Pr\{w\} = \sharp(L \cap \{a, b\}^n)^{-1}$ if $w \in L$, while $\Pr\{w\} = 0$ otherwise. We define the random variable $Y_n : \{a, b\}^n \to \{0, 1, \ldots, n\}$ such that $Y_n(w) = |w|_a$ for every $w \in \{a, b\}^n$. Then, for every $j = 0, 1, \ldots, n$, we have

$$\Pr\{Y_n = j\} = \frac{\sum_{|w|=n, |w|_a=j}(r, w)}{\sum_{x \in \{a,b\}^n}(r, x)} \; .$$

For sake of brevity, we say that Y_n *counts the occurrences of* a *in the stochastic model defined by* r. If $r = \chi_L$ for some $L \subseteq \{a, b\}^*$, then Y_n represents the number of occurrences of a in a word chosen at random in $L \cap \{a, b\}^n$ under uniform distribution.

When r is rational this approach defines a stochastic model for random generation (called *rational* stochastic model) which is of interest in the area of pattern statistics. A typical goal in that context is to estimate the limit distribution of the number of occurrences of patterns in a word of length n generated at random according a given probabilistic model (usually a Markovian process) [12]. The rational stocastic models extend the Markovian models in the following precise sense [3, Section 2.1]. For any finite alphabet Σ, any Markovian process Π generating random words in Σ^* and every regular set of patterns $R \subseteq \Sigma^*$, let $O_n(\Pi, R)$ be the number of (positions of) occurrences of strings of R in a word of length n generated by Π. It turns out that for every such Σ, Π and R there exists a formal series $r \in \mathbb{R}_+^{Rat}\langle\langle a, b\rangle\rangle$ such that the corresponding r.v. Y_n has the same distribution as $O_n(\Pi, R)$, i.e. for every $n \in \mathbb{N}$ and every $k = 0, 1, \ldots, n$,

$$\Pr\{Y_n = k\} = \Pr\{O_n(\Pi, R) = k\} .$$

The limit distribution of Y_n, both in the global and in the local sense, is studied in [3] when r is rational and admits a primitive representation. In this case it turns out that Y_n has a Gaussian limit distribution. A local limit property of the form (4) holds if we add a further hypothesis on the primitive linear representation of r. More precisely, we have the following general property (see Theorems 4 and 6 in [3]).

Theorem 7 *Given a formal series* $r \in \mathbb{R}_+^{Rat}\langle\langle a, b\rangle\rangle$ *, assume that r is primitive and nondegenerate and let* $\{Y_n\}$ *be the sequence of r. v. counting the occurrences of* a *in the stochastic model defined* r*. Then there exist two constants* α *and* β*,* $0 < \alpha$ *,* $0 < \beta < 1$*, such that* $\frac{Y_n - \beta n}{\sqrt{\alpha n}}$ *converges in distribution to a normal random variable of mean value* 0 *and variance* 1*.*

Further, let (ξ, μ, η) *be a primitive linear representation of* r *and let* λ *be the Perron–Frobenius eigenvalue of* $\mu(a) + \mu(b)$ *. Assume that, for every* $0 < \theta < 2\pi$ *, all the eigenvalues of the matrix* $\mu(a)e^{i\theta} + \mu(b)$ *are smaller in modulus than* λ *. Then, as* n *grows to* $+\infty$ *the relation*

$$\sqrt{\alpha n} \cdot Pr\{Y_n = j\} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(j-\beta n)^2}{2\alpha n}} \cdot (1 + o(1)) \tag{12}$$

holds uniformly for every $j = 0, 1, \ldots, n$ *.*

Some comments are now necessary on the additional hypothesis for local convergence in the previous theorem. We note that, given a matrix $T \in \mathbb{C}^{m\times m}$ and a primitive matrix $M \in \mathbb{R}_+^{m\times m}$, if $|T_{ij}| \leq M_{ij}$ for all indices i, j then all eigenvalues of T are in modulus smaller or equal to the Perron–Frobenius eigenvalue of M [17]. Thus, if (ξ, μ, η) is a primitive linear representation and λ is the Perron–Frobenius eigenvalue of $\mu(a) + \mu(b)$, then $|\gamma| \leq \lambda$ for every eigenvalue γ of $\mu(a)e^{i\theta} + \mu(b)$ and every $\theta \in \mathbb{R}$. However, it may occur that $|\gamma| = \lambda$ for some γ and some θ . The previous theorem essentially states that if this happens only when $\theta = 2k\pi$ $(k \in \mathbb{Z})$, then a local limit property holds for Y_n .

Our goal in the following section is to extend the previous theorem to stochastic models defined by powers of primitive formal series. Note that in this case the model is not primitive anymore. Our analysis extends some results presented in [6] dealing with the case when the model is defined by the product of two primitive formal series.

5 Main results

This section contains our main results, that is the asymptotic evaluation of the maximum coefficients of rational series in commuting variables. This approximation is a consequence of Theorems 9 and 10 presented in this section, in which we study of the pattern statistics Y_n, counting the number of symbol occurrences in a stochastic model defined by the power of a primitive rational series.

We start with a technical lemma concerning the generating functions of a power of formal series. Given a formal series $r : \{a, b\}^* \to \mathbb{R}_+$, for every $n, j \in \mathbb{N}$, let $r_{n,j}$ be the coefficient $(\varphi(r), a^j b^{n-j})$, i.e.

$$r_{n,j} = \sum_{|x|=n, |x|_a=j} (r, x) \ .$$

We define the function $r_n(z)$ and the corresponding generating function $\mathbf{r}(z,w)$ by

$$r_n(z) = \sum_{j=0}^{n} r_{n,j}\ e^{jz} \qquad \text{and} \qquad \mathbf{r}(z,w) = \sum_{n=0}^{+\infty} r_n(z)\ w^n = \sum_{n=0}^{+\infty}\sum_{j=0}^{n} r_{n,j}\ e^{jz}\ w^n$$

where $z, w \in \mathbb{C}$.

Now let us consider the series $s = r^k$, where k is a positive integer. Its coefficients are given by

$$(s,w) = \sum_{w=x_1x_2\cdots x_k} (r,x_1)(r,x_2)\cdots(r,x_k) \qquad (w \in \{a,b\}^*) \tag{13}$$

and the corresponding terms $s_{n,j}$, $s_n(z)$ and $\mathbf{s}(z,w)$ are well-defined.

Lemma 8 *If* s *and* r *are defined as above then the relation* $\mathbf{s}(z,w) = \mathbf{r}(z,w)^k$ *holds.*

Proof. By the definition of s (see 13), we have

$$\begin{aligned}
s_n(z) &= \sum_{j=0}^{n} s_{n,j}\ e^{jz} = \sum_{j=0}^{n}\left\{ \sum_{\substack{|x_1|+\cdots+|x_k|=n \\ |x_1|_a+\cdots+|x_k|_a=j}} (r,x_1)(r,x_2)\cdots(r,x_k)\right\} e^{jz} \\
&= \sum_{j=0}^{n}\ \sum_{i_1+\cdots+i_k=n}\ \sum_{j_1+\cdots+j_k=j}\ \sum_{\substack{|x_1|=i_1 \quad\ |x_k|=i_k \\ |x_1|_a=j_1\cdots|x_k|_a=j_k}} (r,x_1)(r,x_2)\cdots(r,x_k)\ e^{jz} \\
&= \sum_{i_1+\cdots+i_k=n}\left(\sum_{j=0}^{n}\ \sum_{j_1+\cdots+j_k=j} r_{i_1,j_1}\cdots r_{i_k,j_k} e^{jz}\right) \\
&= \sum_{i_1+\cdots+i_k=n}\left(\sum_{j=0}^{i_1} r_{i_1,j}e^{jz}\right)\cdot\left(\sum_{j=0}^{i_2} r_{i_2,j}e^{jz}\right)\cdots\left(\sum_{j=0}^{i_k} r_{i_k,j}e^{jz}\right) \\
&= \sum_{i_1+\cdots+i_k=n} r_{i_1}(z)r_{i_2}(z)\cdots r_{i_k}(z)
\end{aligned}$$

Hence, for every $z \in \mathbb{C}$, the function $s_n(z)$ is the k-th convolution of $r_n(z)$ with itself and this proves the statement. □

Theorem 9 *For any positive integer* k *and any primitive nondegenerate* $r \in \mathbb{R}_+^{Rat}\langle\langle a,b\rangle\rangle$ *, let* s *be defined by* $s = r^k$ *and let* Y_n *be the random variables counting the occurrences of* a *in the model defined by* s *. Then there exist two constants* α *and* β *, satisfying* $0 < \alpha$ *and* $0 < \beta < 1$*, such that* $\frac{Y_n-\beta n}{\sqrt{\alpha n}}$ *converges in distribution to a normal random variable of mean value* 0 *and variance* 1.

Proof. Let (ξ,μ,η) be a primitive linear representation of r . Define $A=\mu(a)$, $B=\mu(b)$ and let λ be the Perron–Frobenius eigenvalue of the matrix $M = A + B$. By the results presented in [3, Section 4], we know that the equation

$$\det(uI - Ae^z - B) = 0 \tag{14}$$

implicitely defines a function $u = u(z)$, non-null and analytic in a neighbourhood of $z = 0$, where it admits an expansion of the form

$$u(z) = \lambda\left(1 + \beta z + \frac{\alpha+\beta^2}{2}z^2 + \mathrm{O}(z^3)\right), \tag{15}$$

α and β being two constants such that $0 < \alpha$ and $0 < \beta < 1$. We recall that both α and β can be expressed as function of M and in particular of its eigenvectors.

By the definition of our stochastic model, $\Psi_{Y_n}(z)$ is given by

$$\Psi_{Y_n}(z) = \sum_{j=0}^{n} \Pr\{Y_n = j\} e^{jz} = \sum_{j=0}^{n} \left(\frac{\sum_{|w|=n, |w|_a = j} (s, w)}{\sum_{|w|=n} (s, w)} e^{jz} \right) = \frac{s_n(z)}{s_n(0)}$$

hence the result follows by showing that $\Psi_{Y_n}(z)$ satisfies conditions C1 and C2 of Theorem 1.

By the analysis of the primitive case, we know [3, Equation (28)] that near the point $(0, \lambda^{-1})$ the function $\mathbf{r}(z, w)$ admits a Laurent expansion of the form

$$\mathbf{r}(z, w) = \frac{R(z)}{1 - u(z)w} + \mathrm{O}(1)$$

where $R(z)$ is a complex function, non-null and analytic at $z = 0$, and $u(z)$ is defined in (14) and satisfies (15). By the previous Lemma, in the same neighbourhood, we have

$$\mathbf{s}(z, w) = \left(\frac{R(z)}{1 - u(z)w} \right)^k + \mathrm{O} \left(\frac{1}{1 - u(z)w} \right)^{k-1}$$

and hence the associated sequence is of the form

$$s_n(z) = R(z)^k \binom{n + k - 1}{k - 1} u(z)^n + \mathrm{O}\left(n^{k-2} u(z)^n\right). \tag{16}$$

This proves that $\Psi_{Y_n}(z)$ in a neighbourhood of 0 has an expansion of the form

$$\Psi_{Y_n}(z) = \frac{s_n(z)}{s_n(0)} = \left(\frac{R(z)}{R(0)} \right)^k \cdot \left(\frac{u(z)}{\lambda} \right)^n \cdot \left(1 + \mathrm{O}(n^{-1})\right)$$

and hence both conditions of Theorem 1 hold with $\mu = \beta$ and $\sigma = \alpha$. $\square$

Theorem 10 *Under the hypotheses of Theorem 9, let (ξ, μ, η) be a primitive linear representation of r and let λ be the Perron–Frobenius eigenvalue of the matrix $\mu(a) + \mu(b)$. Assume further that for every $0 < \theta < 2\pi$ all eigenvalues of the matrix $\mu(a)e^{i\theta} + \mu(b)$ are in modulus smaller than λ. Then, as n grows to $+\infty$ the relation*

$$\sqrt{\alpha n} \cdot Pr\{Y_n = j\} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(j - \beta n)^2}{2\alpha n}} \cdot (1 + o(1))$$

holds uniformly for every $j = 0, 1, \ldots, n$, where α and β are defined as in Theorem 9.

Proof. To prove the statement, we simply have to show that condition C3 holds. Observe that, by (16), $s_n(0) = \Theta(n^{k-1}\lambda^n)$. Moreover, the generating function of $r_n(i\theta)$ is given by

$$\begin{aligned}
\mathbf{r}(i\theta, w) &= \sum_{n=0}^{+\infty} r_n(i\theta) w^n = \sum_{n=0}^{+\infty} \xi^T (Ae^{i\theta} + B)^n \eta \, w^n \\
&= \xi^T [I - w(Ae^{i\theta} + B)]^{-1} \eta = \frac{\xi^T \mathrm{Adj}(I - w(Ae^{i\theta} + B))\eta}{\det(I - w(Ae^{i\theta} + B))}
\end{aligned}$$

By Lemma 8, this implies

$$\mathbf{s}(i\theta, w) = \left(\frac{\xi^T \mathrm{Adj}(I - w(Ae^{i\theta} + B))\eta}{\det(I - w(Ae^{i\theta} + B))} \right)^k$$

and hence all singularities of $\mathbf{s}(i\theta, w)$ are inverses of eigenvalues of $Ae^{i\theta} + B$. By the hypotheses, such eigenvalues are smaller than λ in modulus whenever $0 < \theta < 2\pi$. This means that, for every $0 < \theta_0 < \pi$, we can choose $0 < \tau < \lambda$ such that $|\mu| \le \tau$ for every eigenvalue μ of $Ae^{i\theta} + B$ and every $\theta_0 \le \theta \le 2\pi - \theta_0$. Then, for any $\theta_0 \le \theta \le 2\pi - \theta_0$, $\mathbf{s}(i\theta, w)$ has no singularities for $|w| \le \tau^{-1}$ and hence, choosing the contour $\mathcal{C} = \{w \in \mathbb{C} \mid |w| = \tau^{-1}\}$, by Cauchy's integral formula we have

$$|s_n(i\theta)| = \left| \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mathbf{s}(i\theta, w)}{w^{n+1}} dw \right| \le \frac{M}{2\pi} \int_0^{2\pi} \tau^{n+1} dt = \mathrm{O}(\tau^n)$$

where $M = \sup\{|\mathbf{s}(i\theta, w)| \mid |w| \le \tau^{-1},\ \theta_0 \le \theta \le 2\pi - \theta_0\}$. This implies

$$|\Psi_{Y_n}(i\theta)| = \frac{|s_n(i\theta)|}{|s_n(0)|} = \frac{\mathrm{O}(\tau^n)}{\Theta(n^{k-1}\lambda^n)} = \mathrm{O}(\rho^n)$$

for some $0 < \rho < 1$, which proves condition C3. □

The previous result provides us an asymptotic evaluation for the maximum coefficients of formal series in commuting variables that are commutative image of powers of primitive rational formal series.

Corollary 11 *Let* $r \in \mathbb{R}_+^{Rat}\langle\langle a, b\rangle\rangle$ *satisfy the hypothesis of Theorem 10 and let* k *be a positive integer. Then the growth function of* $\varphi(r^k)$ *satisfies the relation* $g_{\varphi(r^k)}(n) = \Theta(n^{k-(3/2)}\lambda^n)$ *for some* $\lambda > 0$.

Proof. Let r be defined as in Theorems 9 and 10 and let Y_n count the occurrences of a in the model defined by r^k. By (16) we have $r_n^k(0) = \Theta(n^{k-1}\lambda^n)$. Moreover, by the previous theorem, the set of probabilities $\{\Pr(Y_n = j) \mid j = 0, 1, \dots, n\}$ has the maximum at $j = \lfloor \beta/n \rfloor$, where it takes a value of the order $\Theta(n^{-1/2})$. Hence, since $\Pr(Y_n = j) = r_{n,j}^k / r_n^k(0)$, we have

$$g_{\varphi(r^k)}(n) \ = \ \max\{r_{n,j}^k \mid j = 0, 1, \dots, n\} \ = \ \Theta(n^{k-(3/2)}\lambda^n) \ .$$

□

Recalling that the sequence associated with the generating function $(1 - \lambda z)^{-k-1}$ is of the order $\Theta(n^k\lambda^n)$, we remark another typical behaviour of growth functions of series in $\mathbb{R}_+^{Rat}[[a, b]]$.

Corollary 12 *For every integer* $k \ge 1$ *, if* $s = \varphi(r^k)$ *for some primitive degenerate* $r \in \mathbb{R}_+^{Rat}\langle\langle a, b\rangle\rangle$*, then* $g_s(n) = \Theta(n^{k-1}\lambda^n)$ *for some* $\lambda > 0$.

6 Open problems

We conclude with a pair of, as far as we know, open problems on the tropical semiring $\mathcal{T}$.

Problem 1. Prove or disprove that every rational formal series over free monoids with coefficients in the semiring $\mathcal{T}$ has a growth function of the order $\Theta(n^{1/k})$ for some integer $k > 0$.

Problem 2. Does there exist an algorithm for computing the previous exponent?

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