Relations over words and logic: a chronology

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The purpose of this short note is to give credit to the right people who produced original work on the connection between rational relations and logic. Indeed, my experience is that some authors seem to partially ignore the literature or at least neglect to cite it correctly. It is probably due to the fact that language theory and logic, though having largely filled the original gap which separated them, still have different backgrounds. I hope that recalling the chronology might be of some help. If I had some doubt about the necessity of this reminder, a recent experience proved it was justified. Indeed, I posted an early version of the present work on my web page. Kamal Lodaya from the University of Chennai happened to come across it, got interested and posed a few questions. Doing some bibliographical search he found that the relations which after Läuchli and Savioz I had called "special", had in fact been introduced three years earlier by D. Angluin and D. N. Hoover as "regular prefix relations".

Now we come to the point. Given n finite, nonempty alphabets Σ_i, $i = 1, \ldots, n$, I'm interested in the class of subsets, also called *relations*, of the direct product $\Sigma_1^* \times \cdots \times \Sigma_n^*$ which are *rational* (known as *regular* in the anglo-saxon literature). A simple example: the relation which is the graph of the operation of concatenation of two words and which consists of all triples of the form (u, v, uv) where $u, v \in \Sigma^*$, is defined by the rational expression $\Delta_1^*\Delta_2^*$ where $\Delta_1 = \bigcup_{a\in\Sigma}(a, 1, a)$ and $\Delta_2 = \bigcup_{a\in\Sigma}(1, a, a)$. These relations are also defined via an extension of the finite automata operating on tuples of words rather than on words, introduced by Rabin and Scott in the late fifties, [8]. They were studied by Elgot and Mezei who proved most of their general properties, [6]. The main decision issues were settled by Fischer and Rosenberg, [7]. It just happens that this class does not form a Boolean algebra unless $n = 1$ or all alphabets Σ_i's are reduced to a single symbol. Until the mid eighties, only two subclasses of the rational relations were known to be closed under the Boolean operations, to wit the recognizable and the synchronous relations which are therefore natural candidates for logical definability. A new class was discovered by D. Angluin and D. N. Hoover, [1], then rediscovered by Läuchli and C. Savioz.

The reader curious of learning more on rational relations is referred to the standard textbooks, such as [3, 4]. I only fix the notations by saying that the free monoid generated by an alphabet Σ is denoted by Σ^*. An n-ary relation is a subset of $(\Sigma^*)^n$. It is assumed that the reader has a minimum knowledge on Rabin's monadic second order logic of k successors. Finally, the comprehension of the remainder is facilitated if the reader bears in mind the following inclusions of classes of relations whose precise definitions are given in due time, representing respectively the class of recognizable, special, synchronous and rational relations.

$$\text{Rec} \subseteq \text{Spec} \subseteq \text{Sync} \subseteq \text{Rat}$$

The three inclusions are strict exactly under the same conditions that the class of rational relations is not a Boolean algebra.

1969: synchronous relations

Though discovered 36 years ago, the logical characterization of this class of relations due to Eilenberg, Elgot and Shepherdson is practically never cited. Intuitively, it can be described as follows. Given an n-tuple of words $(w_1, \dots, w_n) \in \Sigma^* \times \cdots \times \Sigma^*$, pad all components by as few occurrences of a new symbol, say # to fix ideas, as necessary in order to achieve equal length, i.e., transform $(abaa, bb, bba)$ into $(abaa, bb\#\#, bba\#)$. Such an n-tuple of words may be viewed as a word over the composite alphabet $\Delta = (\Sigma \cup \{\#\})^n - \{\#\}^n$, e.g., $(abaa, bb\#\#, bba\#)$ can be viewed as the length 4 word $[abb][bbb][a\#a][a\#\#]$

3 tapes

a	b	a	a
b	b		
b	b	a	

converted into 1 tape

a b b	b b b	a # a	a # #

Given a relation $R \subseteq (\Sigma^*)^n$ transform all its n-tuples by padding them as just explained and denote by $R^\# \subseteq \Delta^*$ the resulting subset. Then R is synchronous if there exists a finite automaton on the alphabet Δ which recognizes $R^\#$. E., g., the relation $\{(a^{n+1}, a^n) \mid n \geq 0\} \cup \{(a^{2n}, a^{2n+1}) \mid n \geq 0\}$ is synchronous but the relation $\{(a^{2n}, a^n) \mid n \geq 0\}$ is not. The authors introduce the first order theory

$$\text{Th} = (\Sigma^*, \text{Eq}, \leq_{\text{pref}}, (L_a)_{a\in\Sigma}) \tag{1}$$

where the binary predicate $\text{Eq}(u, v)$ is true if and only if the two words u and v have the same length, the binary predicate $u \leq_{\text{pref}} v$ is true if and only if u is a prefix of v and for each $a \in \Sigma$, the unary predicate $L_a(u)$ is true if and only if the word u ends with the letter a. Observe in passing that this signature is denoted by $\mathbf{S}_{\text{len}}$ in [2] and that this logic, in the context of infinite words, is called chain logic + E in [10]. The logical characterization is as follows, [5].

Theorem 1. *A subset $R \subseteq (\Sigma^*)^n$ is synchronous if and only if it is definable in the theory (1).*

The sufficiency of the condition is a simple consequence of the closure properties of the synchronous relations under the Boolean operations, composition of relations and projections. The authors prove the converse when the relation is given through a rational expression. Mimicking the automaton yields a much more intuitive proof and can be reconstructed by a good PhD student.

1984: special relations

I discovered this family in a paper of Wolfgang Thomas, [10] where he cited a publication of Läuchli and Savioz. However the correct reference as far as I know is [1]. The merit of this family is that it is a new Boolean class of relations with a neat logical characterization. The starting point is a restriction of the theory of the k successors SkS on the structure $(\Sigma^*, (s_a)_{a\in\Sigma})$ where for each $u \in \Sigma^*$, s_a is interpreted as the function defined by $s_a(u) = ua$ for each $u \in \Sigma^*$. Indeed, a formula $\varphi(x_1, \dots, x_n)$ with n free individual variables and no free set variable defines an n-ary relation $R \subseteq \Sigma^*$ by setting $R \models \varphi(x_1, \dots, x_n)$. Furthermore, Läuchli and Savioz introduce the first order theory

$$\mathrm{Th} = (\Sigma^*, 1, \mathrm{LCP}, (P_L)_{L\in\mathrm{Rat}\Sigma^*}) \tag{2}$$

where 1 is the empty word, LCP is the function that assigns the largest common prefix of two words and for all rational subsets $L \in \mathrm{Rat}\Sigma^*$, the predicate $P_L(x, y)$ is true whenever $y \in xL$. This structure is studied under the terminology $\mathbf{S}^+_{\mathrm{reg}}$ in [2]. The theory yields a new subclass of rational relations which is strictly intermediate between the recognizable and the synchronous relations. Indeed, consider a denumerable set of symbols $X = \{x_i\}_{i\in\mathbb{N}}$ and define the least collection C of sequences of strings on X which contains the sequences reduced to one symbol x_i and which satisfies the two conditions

(i) if $(u_1, \dots, u_p) \in (X^*)^p$ belongs to C and if σ is a permutation on the set $\{1, \dots, p\}$, then $(u_{\sigma_1}, \dots, u_{\sigma_p}) \in (X^*)^p$ belongs to C.

(ii) if $(u_1, \dots, u_p) \in (X^*)^p$ belongs to C and if x_i and x_j are two distinct symbols occurring in none of the u_i's, then $(u_1, \dots, u_{p-1}, u_p x_i, u_p x_j) \in (X^*)^{p+1}$ belongs to C.

For example, $(x_1x_3, x_1x_2x_4, x_1x_2x_5)$ is of this form, (x_1x_2, x_2x_1) and (x_1x_2, x_3x_4) are not. Finally, a relation R is *special* if there exist a sequence $(u_1, \dots, u_p)$ in C and a rational subset L_i for each x_i such that R is the set of p-tuples obtained by substituting an arbitrary word of L_i for each occurrence of x_i. As particular cases of special relations, we have the generalized diagonal $\{\overbrace{(u, \dots, u)}^{n\text{times}} \mid u \in \Sigma^*\}$, all recognizable relations (see below), the relation $\{(uv, uw) \in \Sigma^* \times \Sigma^* \mid u \in \Sigma^*, |v| \equiv 0$

mod 2, $|w| \equiv 1 \mod 3\}$, etc .... I chose the formulation given by Läuchli and Savioz, rather than that of Angluin and Hoover. Indeed, the latter authors do not mention the theory (2). Also instead of the notion of special relations, they give a more complicated, though substantially equivalent, notion of prefix automata.

Theorem 2. *Given a subset $R \subseteq (\Sigma^*)^n$, the following conditions are equivalent*

(i) *R is definable in SkS by a formula having n free variables and no free subset variables*

(ii) *R is definable in the first order theory (2)*

(iii) *R is a finite union of special subsets*

1990: recognizable relations

The reason for this last characterization to remain hidden it that it was published in the proceedings of an ASMICS meeting. Furthermore, it was stated more generally in terms of trace monoids: these are quotients of free monoids by a congruence generated by a reflexive and symmetric relation called *relation of partial commutations* $I \subseteq \Sigma\times\Sigma$. Direct products of free monoids $\Sigma_1^*\times\cdots\times\Sigma_n^*$ are a special case where $I = \bigcup_{i\neq j}\Sigma_i\times\Sigma_j$.

We recall that a subset R of $\Sigma_1^*\times\ldots\times\Sigma_n^*$ is *recognizable* if there exists a finite monoid M and a morphism $f : \Sigma_1^*\times\ldots\times\Sigma_n^* \to M$ such that $R = f^{-1}f(R)$ holds. It is a classical exercise to prove that the class $\mathrm{Rec}(\Sigma_1^*\times\ldots\times\Sigma_n^*)$ of recognizable relations is a Boolean algebra. It can be shown that R is recognizable if and only if it is a finite union of direct products of the form $X_1\times\ldots\times X_n$ where X_i is a recognizable subset of Σ_i^*, a result which is attributed to Elgot and Mezei by Eilenberg. From a logical viewpoint, the idea is to consider an n-tuple of strings $(w_1,\ldots,w_n)$ as a disjoint union of n linear orders together with two constants for each component representing the first and the last positions and a predicate asserting that a certain position is labeled with a given letter. More precisely, the model theoretic structure is of the form

$$(\bigcup_{1\leq i\leq n} I_i, <, ((Q_a)_{a\in\Sigma_i})_{1\leq i\leq n}) \tag{3}$$

where $I_i = \{(0,i),\ldots,(|w_i|-1,i)\}$ for $i = 1,\ldots,n$. The predicate $(j_1,i_1) < (j_2,i_2)$ is true if and only if $i_1 = i_2$ and $j_1 < j_2$ holds and the predicate $Q_a((j,i))$ is true if and only if the j-th occurrence of w_i is equal to a. For example if $n = 2$, $\Sigma_1 = \{a,b\}$, $\Sigma_2 = \{a,c\}$ and $(w_1,w_2) = (babb, aca)$, then the corresponding structure is

$$\begin{array}{ccccccc} (0,1) & \text{———} & (1,1) & \text{———} & (2,1) & \text{———} & (3,1) \\ (0,2) & \text{———} & (1,2) & \text{———} & (2,2) & & \end{array}$$

We have, e.g., $Q_a(1,1) = \texttt{true}$, $Q_c(2,2) = \texttt{false}$, $(0,1) < (3,1) = \texttt{true}$ and $(2,0) < (1,2) = \texttt{false}$.

We insist that in the previous two cases, the monoid was the model, while here it is each element of the monoid which is a model in itself. Then the following holds, [9].

Theorem 3. *A subset R of is recognizable if and only if it is the set of models of some formula in the theory defined in (3).*

One direction of the proof is standard. The crux for showing that a recognizable relation is expressible through the above logic is the fact that each of the n components of the relation $R \subseteq \Sigma_1^* \times \cdots \times \Sigma_n^*$ can be independently controlled by a finite automaton.

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