TRANSFINITE EQUATIONS IN TRANSFINITE STRINGS [1]

CHRISTIAN CHOFFRUT
L.I.A.F.A., Université Paris 7,
Tour 55-56, 1 er étage,
2 Pl. Jussieu – 75 251 Paris Cedex – France
Christian.Choffrut@liafa.jussieu.fr

SÁNDOR HORVÁTH
Dept. of Computer Science, Eötvös Loránd University
Budapest, Rákóczi út 5., IV. em.
H–1088, Hungary
horvath@cs.elte.hu

1 Introduction

Transfinite strings have long been introduced. Logicians were the first to study them with the pioneering works of Büchi on the decidability of the monadic second order logic on the ordinal ω_1 [3], followed by other investigations beyond ω_1 [19] or more specifically on the "automata" aspect of the theory [5]. In the area of theoretical computer science such objects can be viewed as modelling the behaviour of sequential processes consisting of infinite actions whose times define a convergent sequence, also known in the literature as Zeno strings. The vivid area of timed automata which takes the duration of transitions of finite automata into account makes explicit use of this notion see, e.g., [8]. In another, more algebraic direction, the classical approach of "rational languages" and its connection to the theory of "varieties" of algebraic structures developed by Eilenberg for finite strings [6, Chapter VII], was recently extended to transfinite strings, see [2].

Combinatorics in strings is an old and wide area whose origin is usally traced back to Axel Thue, but that matured in the sixties with the works of Marcel-Paul Schützenberger and his followers. Numerous publications have already given account of state of the art (e.g. [15] or the chapter [4]) and they testify for the richness and profoundness of the topic. One of the main aspects of this theory is concerned with solving equations with strings as unknowns. Let us mention for example the effective possibility of determining whether or not an equation with constants has a solution which was established by Makanin or the possibility of expressing all solutions of an equation having at most three variables by means of "parameters" due to Hmelevskii, see [16] and [9]. The picture for transfinite strings is quite different since apart from isolated results (see, e.g., [10] and [11]) little is developed yet. We believe though that some elementary combinatorial properties on transfinite strings might help the study of subsets of transfinite strings as they have helped the study of subsets of finite strings.

Among the numerous issues on transfinite strings, we focus on that of solving equations where the two handsides are arbitrary finite or transfinite strings in the alphabet of unknowns and where the unknowns are assumed to take on values in finite or transfinite strings in the target aphabet. By solving an equation we mean giving some type of satisfactory description of the set of their solutions. The lengths of the strings that we

[1]This work was partially supported by the French-Hungarian research project "Balaton" (1997-1998)

consider are usually arbitrary ordinals apart from section 4 where we must assume they have length less than ω^ω.

Our main contributions are as follows. First we give a thorough description of the finite and transfinite solutions of finite and transfinite equations in two unknowns. Actually, except for very specific equations, the solutions are of the form $x = (u^\omega)^\alpha u^i$ and $y = (u^\omega)^\beta u^j$ for some ordinals α and β and some integers i and j, i.e., as for finite strings, the solutions are "parametrizable". Next we consider the particular case of what we call the "regular" equations, i.e., those whose two handsides are built from the individual variables by applying the two operations of concatenation and ω-product finitely many times. The peculiarity of this family is that the equation can be given effectively. We show that whatever the number of unknowns, each solution of a regular equation "factors" through a regular solution, i.e., a solution assigning regular strings to all variables.

Turning to finite equations, we solve the equations $x^m y^p = z^q$ for $m, p, q \geq 2$. Observe that together with the equations on two unknowns we have practically exhausted all cases that can be solved in finite strings. Solving these last equations requires studying the notion of periodicity. We thus investigate the concepts of periodicity, primitivity, conjugacy, ... In particular the famous result attributed to Fine and Wilf concerning the periods of a (finite) string can be extended to strings whose lengths are arbitrary ordinals with little damage but gives rise to different versions.

2 Preliminaries

We refer the reader to the numerous standard handbooks such as [18] or [17] for a comprehensive exposition of the theory on ordinals. For the sake of completeness, we recall here some well known results that are necessary for a good understanding of our paper. We denote by Ord the collection of all ordinals.

2.1 Ordinals

Every ordinal α admits a unique representation, known as Cantor's normal form, as a finite sum of non-increasing prime components. By grouping the equal prime components, an arbitrary element can thus be written as

$$\alpha = \omega^{\lambda_n} a_n + \omega^{\lambda_{n-1}} a_{n-1} + \ldots + \omega^{\lambda_1} a_1 + \omega^{\lambda_0} a_0 \tag{1}$$

where $n, a_n, \ldots, a_1, a_0 < \omega$, $a_n \neq 0$ and $\lambda_n > \lambda_{n-1} > \ldots > \lambda_0$. The ordinal λ_n is the *degree* of the ordinal α also denoted by ∂_α. Addition of two ordinals is associative and satisfies for all $a, b < \omega$, $\mu, \nu \in \mathrm{Ord}$

$$\omega^\mu a + \omega^\nu b = \begin{cases} \omega^\nu b & \text{if } \nu > \mu \\ \omega^\mu (a+b) & \text{if } \nu = \mu \end{cases}$$

Clearly, $\partial_{\alpha+\beta} = \max\{\partial_\alpha, \partial_\beta\}$ holds. The product of ordinals may also be defined. It is associative, distributes over the sum to the left ($\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$) and satisfies in particular

$$(\omega^{\lambda_n}a_n + \omega^{\lambda_{n-1}}a_{n-1} + \ldots + \omega^{\lambda_0}a_0)\alpha = \begin{cases} \omega^{\lambda_n+1} & \text{if } \alpha = \omega \\ \omega^{\lambda_n}a_n\alpha + \omega^{\lambda_{n-1}}a_{n-1} + \ldots + \omega^{\lambda_0}a_0 & \text{if } \alpha < \omega \end{cases}$$

Addition is not commutative but characterizing the pairs of ordinals that commute is easy.

PROPOSITION **1** (For example, [18], Theorem. 1., p. 346) *For two ordinals α, β the following conditions are equivalent*

i) $\alpha + \beta = \beta + \alpha$

ii) there exist an ordinal γ and two integers $n, p \geq 0$ such that $\alpha = \gamma n$ and $\beta = \gamma p$.

iii) either $\alpha\beta = 0$ or else for some $n, p, q < \omega$ and $\mu < \omega^n$ we have $\alpha = \omega^n p + \mu$ and $\beta = \omega^n q + \mu$

We shall not explicitely give any reference to the elementary properties of ordinals though they are used heavily at numerous points in the proofs. The reader should keep this in mind.

2.2 Strings of ordinal lengths

Given a finite alphabet Σ, a *string* is a mapping u of Ord into Σ. Equivalently, u is a sequence of Σ indexed by an ordinal α. We write u_i for all $i < \alpha$. The collection of all strings is denoted by Σ^{Ord}. The ordinal α is the *length* of u, denoted by $|u|$. The *empty string*, denoted by 1 is the string of length 0 and is the unit ofOrd as a monoid. The collection of all strings of length less than ω^ω is denoted by $\Sigma^{<\omega^\omega}$. By extension, the *degree* of a string x is the degree of its length and it is denoted by ∂_x. For $a \in \Sigma$, $|u|_a$ denotes the *length in the letter* a of the string u, i.e., the ordinal of the subsequence consisting of all the positions $i < \alpha$ for which $u_i = a$.

The *concatenate* of two strings u and v is the mapping from the ordinal $|u| + |v|$ to Σ defined by

$$(uv)_i = \begin{cases} u_i & \text{if } i < |u| \\ v_j & \text{if } i = |u| + j, j < |v| \end{cases}$$

Observe that $|uv| = |u| + |v|$ holds for all $u, v \in \Sigma^{\mathrm{Ord}}$ by definition, but the condition $uv = w$, $|u| > 0$ does not imply $|v| < |w|$ (consider $u = a, v = w = a^\omega$), which is a main departure from finite strings. Another difference is the fact that caoncatenation (also called product), like addition for ordinals, is left cancellative but not right cancellative ($a^\omega a a^\omega = a^\omega a^\omega$). Also, concatenation can be defined for a collection of strings indexed by an arbitrary ordinal. As a particularly important case, given a string u, we denote by u^ω the string infinitely concatenated with itself, i.e., the string of length $|u|\omega$ defined by

$$u^\omega_{\ell k+i} = u_i$$

where $\ell = |u|$, $k < \omega$ and $i < \ell$. Finally, u^* represents the set of all strings that are finite powers of u, empty string 1 included.

The notions of *prefix, suffix, substring* extend naturally from finite to arbitrary strings. In particular, an *occurrence* of a string $w \in \Sigma^{\mathrm{Ord}}$ in a string $u \in \Sigma^{\mathrm{Ord}}$ is a pair $(\alpha, w) \in \mathrm{Ord} \times \Sigma^{\mathrm{Ord}}$ such that $u = xwy$ for some strings $x, y \in \Sigma^{\mathrm{Ord}}$ with $|x| = \alpha$. For example, consider the string $x = yy$ where $y = aba^2ba^3b \ldots a^ib \ldots$ of length ω. The underlined occurrence $aba^2b\underline{a^3b} \ldots a^ib \ldots aba^2b \ldots a^ib \ldots$ is specified by the pair $(5, a^3b)$ while the underlined occurrence $aba^2ba^3b \ldots a^ib \ldots ab\underline{a^2b} \ldots a^ib \ldots$ is specified by the pair $(\omega{+}2, a^2b)$. The occurrences of a given string are linearly ordered by their first component, i.e., (α_1, w_1) is *before* (α_2, w_2) if $\alpha_1 < \alpha_2$.

We may also say that two strings $x, y \in \Sigma^{\mathrm{Ord}}$ are *comparable* if x is a prefix of y or y is a prefix of x. The most elementary property of finite strings is known as Levi's Lemma. It trivially holds for strings of arbitrary lengths. We recall it here.

Proposition 2 *Let $x, y, z, t \in \Sigma^{Ord}$ satisfy the equality $xy = zt$. Then there exists $u \in \Sigma^{Ord}$ such that either $x = zu$ and $t = uy$ or $z = xu$ and $y = ut$ holds.*

2.3 Equations

The traditional notion of (finite) equations in finite strings can be extended to equations where the legnths of both handsides are arbitrary ordinals and whose legnths fo the solutions are strings of arbitrary ordinals. Given a finite subset $\Xi = \{x, y, \ldots\}$ of *unknowns*, an *equation* is a pair of strings in Ξ^{Ord} written

$$L(x, y, \ldots) = R(x, y, \ldots) \tag{2}$$

An equation is *trivial* if the strings $L(x, y, \ldots)$ and $R(x, y, \ldots)$ are the same (as elements of Ξ^{Ord}). For example, $xy^\omega = xy^\omega$ is trivial but $xy = x$ and $xy^\omega = yx^\omega$ are not. The equation is *finite* if the lengths of the two handsides are finite, thus $xy = x$ is finite but $xy^\omega = yx^\omega$ is not. A *substitution* is an assignment $\phi : \Xi \to \Sigma^{\mathrm{Ord}}$ which can be extended to Ξ^{Ord} by setting $\phi(\prod_{\beta<\alpha} x_\beta) = \prod_{\beta<\alpha} \phi(x_\beta)$. A *solution* of equation (2) is a substitution $\phi : \Xi \to \Sigma^{\mathrm{Ord}}$ such that $\phi(L(x, y, \ldots)) = \phi(R(x, y, \ldots))$ holds. When convenient, we will indifferently speak of the solution as the morphism ϕ or the set $\phi(\Xi)$. Also we will adopt the traditional convention using the same symbol x for the unknown and its assignment in Σ^{Ord}, i.e., we will identify $x \in \Xi$ and $\phi(x) \in \Sigma^{\mathrm{Ord}}$. For example, the equation $xy = y$ admits the solution $x = a$ and $y = a^\omega a$.

We say that a substitution $\phi : \Xi \to \Sigma^{\mathrm{Ord}}$ is a *solution in length of the equation* $L(x, y, \ldots) = R(x, y, \ldots)$ if the equality (in Ord) $|L(\phi(x), \phi(y), \ldots)| = |R(\phi(x), \phi(y), \ldots)|$ holds. , $x = a$ and $y = b^\omega a$, $a \neq b \in \Sigma$ is a solution in length of the equation $xy = y$ but is not a solution of $xy = y$.

3 Equations in two unknowns

Let us recall shortly what holds in the case of finite equations with two finite unknowns. Let $L(x, y) = R(x, y)$ be a non trivial equation and consider a solution $\phi : \Xi^* \to \Sigma^*$, i.e., a morphism of the free monoid Ξ^* generated by Ξ into the free monoid Σ^* generated by Σ for which $\phi(L(x, y)) = \phi(R(x, y))$ holds. Then there exist a string $u \in \Sigma^*$ and two integers $n, m \geq 0$ such that $\phi(x) = u^n$ and $\phi(y) = u^m$. Such solutions are called *cyclic* in the literature.

This is clearly no longer valid in the case of equations (whether finite or infinite) with two unknowns taking on values that are strings of arbitrary lenghts. For example, the equation $xyy = yxy$ has the solution $\phi(x) = a^\omega ba$ and $\phi(y) = a^\omega b$. We shall prove that all equations on two unknowns have solutions of a specific type that generalizes that of cyclicity. Because our result is general, it cannot be effective. However, it shows that the set of solutions is parametrizable.

3.1 Generalized-cyclic strings

If two finite strings x, y are cyclic, i.e., if there exists a string z such that $x, y \in z^*$ then all elements of the monoid generated by x and y are comparable. This is this property that is captured in the next definition.

DEFINITION 1 *Two strings $x, y \in \Sigma^{Ord}$ are* generalized cyclic *if there exist two strings $u, v \in \Sigma^{Ord}$, two ordinals $\alpha, \beta \geq 0$ and two integers $0 \leq i, j < \omega$ such that*

$$x = (u^\omega v u^i)^\alpha \quad y = (u^\omega v u^j)^\beta \tag{3}$$

The following justifies the previous definition

PROPOSITION 3 *Let x and y be defined as in (3) and let $\lambda \in$ Ord . Every product $\prod_{i<\lambda} z_i$ where $z_i \in \{x, y\}$ for all $i < \lambda$ is of the form $(u^\omega v u^r)^\gamma$ for some $\gamma \in$ Ord and $r = 0, i, j$.*

Proof. Assume by contradiction there exists an element $\xi \in \{x, y\}^{\mathrm{Ord}}$ which is not of the given form and choose it of shortest length λ (in $\{x, y\}$).

Case 1: λ is a successor ordinal. Then the string is of the form $\xi = \eta x$ (or $\xi = \eta y$) with $|\eta| = \lambda - 1$. Then it is easy to conclude by induction.

Case 2: λ is a limit ordinal. For all $0 \leq i < \lambda$ let ξ_i be the prefix of ξ of length i: $\xi_i = (u^\omega v)^{\gamma_i} u^{r_i}$. The sequence $(\gamma_i)_{i<\lambda}$ is increasing and thus has a limit γ. We verify the equality $\xi = (u^\omega v)^\gamma$. Clearly, all ξ_i's are prefixes of $(u^\omega v)^\gamma$. Conversely, let z be a prefix of $(u^\omega v)^\gamma$ of length $(u^\omega v)^{\delta'}$ with $\delta' < \gamma$. Then there exists $j < \lambda$ such that $\delta' < \gamma_j < \gamma$ and thus z is a prefix of ξ_j. ■

3.2 Balanced equations

Some particular equations possess special solutions which have no counterpart for finite strings. This is intuitively due to the "absorption" property of concatenation of transfinite strings illustrated by the fact that $y = xy$ if, e.g., $x = a$ and $y = a^\omega$.

Given an equation $L(x, y) = R(x, y)$ we set $|L(x, y)|_x = \lambda$ and $|R(x, y)|_x = \rho$. Then $L(x, y)$ (respectively $R(x, y)$) has a unique decomposition

$$L(x, y) = \left(\prod_{i<\lambda} y^{\alpha_i} x\right) y^{\alpha_\lambda} \text{ (respectively } R(x, y) = \left(\prod_{i<\rho} y^{\beta_i} x\right) y^{\beta_\rho} \text{)} \tag{4}$$

where $\alpha_i \in$ Ord (respectively $\beta_i \in$ Ord).

DEFINITION **2** *The equation* $L(x,y)=R(x,y)$ *is* x-balanced *if the following two conditions hold*

i) $\lambda=\rho$

ii) $\alpha_\lambda=\beta_\rho$

Similarly, y-balanced *equations can be defined.*

The next definition makes sense essentially for balanced equations. The idea is to compare the numbers of occurrences of y's between two consecutive occurrences of x in each handside of the equation. Given two ordinals α,β, the expression $|\alpha-\beta|$ is shorthand for $\alpha-\beta$ if $\alpha\geq\beta$ and $\beta-\alpha$ otherwise.

DEFINITION **3** *The* x-gap *(or simply* gap *when no confusion may arise) of the equation* $L(x,y)=R(x,y)$ *is the smallest ordinal which is greater than the degree of all* $|\alpha_i-\beta_i|$*'s for* $i<\min\{\lambda,\rho\}$*. The* y-gap *is defined similarly.*

EXAMPLE **3.1** The equation $(yx)^\omega y^{\omega^2}xy=(y^\omega x)^\omega xy$ is x-balanced and its gap is 3. The equation $yxyxy=yxyx^2y$ is not x-balanced but is y-balanced and its gap is 1. □

3.3 Some simple equations

We first start with studying the equations where the lengths of the two handsides are less than or equal to ω.

PROPOSITION 4 *Let* $L(x,y)=R(x,y)$ *be a non trivial equation where the lengths of the two handsides are less than or equal to* ω*. Then a solution in length* $\phi:\Xi\rightarrow\Sigma^{Ord}$ *is a solution if and only if there exist* $u,v\in\Sigma^{Ord}$ *and four integers* k,i,l,j *such that* $\phi(x)=(u^\omega v)^k u^i$ *and* $\phi(y)=(u^\omega v)^l u^j$ *holds.*

Proof. The condition is clearly sufficient since the two handsides are prefixes of the same length of the string $(u^\omega v)^\omega$. We prove it is necessary. The case where one of the handside is a prefix of the other is trivial so we rule it out. We assume without loss of generality that the maximum common prefix of the two handsides is the empty string.

Assume first $\partial_{\phi(x)}\neq\partial_{\phi(y)}$, say $\partial_{\phi(x)}>\partial_{\phi(y)}$. Then x is an occurrence of both L and R. Indeed, if this were not the case, by considering the lengths, the equation would reduce to $x=y^\omega$. Thus by interchanging L and R if necessary, for some $0<k<\omega$ we have

$$L(x,y)=xL_1(x,y),\quad R(x,y)=y^kxR_1(x,y)$$

Arguing on the lengths, we see that the solution satisfies the equation $x=y^kx$. We now verify that x has a prefix of the form y^ω, which will prove the result. Assume by contradiction there exists a minimal ordinal $0\leq i<|y|\omega$ such that $x_i\neq x_{(|y|+i)}$ holds. Clearly, $i\geq|y|(k-1)$. Then $i-|y|(k-1)$ exists and by the minimality of i we have $x_{i-|y|(k-1)}=x_{i-|y|(k-2)}=\ldots=x_i$. Because $x=y^kx$ holds we have $x_{i-|y|(k-1)}=x_{|y|+i}$, i.e., $x_i=x_{(|y|+i)}$, a contradiction.

For the case $\partial_{\phi(x)}=\partial_{\phi(y)}$ we proceed by induction on $|\phi(x)|+|\phi(y)|$. Without loss of generality set $L(x,y)=xL_1(x,y)$ and $R(x,y)=yR_1(x,y)$. If $|\phi(x)|=|\phi(y)|$ we are done. If $|\phi(x)|>|\phi(y)|$, since $\phi(y)$ is a prefix of $\phi(x)$ we may consider the substitution

$$\phi(x)=\phi(y)\psi(x),\quad\psi(y)=\phi(y)$$

Then ψ is a solution of the equation

$$xL_1(yx, y) = R_1(yx, y)$$

If $\partial_{\psi(x)} < \partial_{\psi(y)}$ then we may conclude by the previous case. Otherwise we observe that the two handsides of the equation have lengths less than or equal to ω, that they are not prefix of one another and that $|\psi(x)| + |\psi(y)| < |\phi(x)| + |\phi(y)|$ holds. ∎

Because of its simplicity, the case where the solutions are of the same degree may be treated directly.

COROLLARY **5** *Let $L(x, y) = R(x, y)$ be an equation and let $\phi : \Xi \to \Sigma^{Ord}$ be a solution in length such that $\partial_{\phi(x)} = \partial_{\phi(y)}$. Then ϕ is a solution if and only if there exist $u, v \in \Sigma^{Ord}$ and four integers $0 \leq i, j, k, l$ such that $\phi(x) = (u^\omega v)^k u^i$ and $\phi(y) = (u^\omega v)^l u^j$.*

Proof. Clearly, $L(x, y)$ is infinite if and only if so is $R(x, y)$. If both are finite the result follows from Proposition 4. Assume thus that they are both infinite and set

$$L(x, y) = L_1(x, y)L_2(x, y) \text{ and } R(x, y) = R_1(x, y)R_2(x, y)$$

where $|L_1(x, y)| = |R_1(x, y)| = \omega$. If λ is the common degree of $\phi(x)$ and $\phi(y)$, then the common length of $L_1(\phi(x), \phi(y))$ and $R_1(\phi(x), \phi(y))$ is $\omega^{\lambda+1}$. Thus $L_1(\phi(x), \phi(y)) = R_1(\phi(x), \phi(y))$ holds and we may conclude again by Proposition 4.

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3.4 Solving equations in two unkowns

We are now in a position to describe the solutions of all equations in two unknowns. By Proposition 3 it is clear that if the two unknowns are generalized cyclic, then they satisfy the equation. The following theorem says that the converse is true unless the equation is x-balanced (or y-balanced). Indeed, in Example 3.1 the first equation possesses the solution $x = y^{\omega^3}$ and the second equation possesses the solution $y = x^{\omega^2}$ as can be readily verified.

THEOREM **1** *Let $L(x, y) = R(x, y)$ be an equation and let $\phi : \Xi \to \Sigma^{Ord}$ be a solution in length such that $|\phi(x)| \geq |\phi(y)|$. If the equation is not x-balanced then ϕ is a solution if and only if $\phi(x)$ and $\phi(y)$ are generalized cyclic.*

If the equation is x-balanced then ϕ is a solution if and only if $\phi(x)$ and $\phi(y)$ are generalized cyclic or if ϕ satisfies

$$\phi(x) = u^{\omega^g} v \quad \phi(y) = u$$

where $u, v \in \Sigma^{Ord}$ and g is the gap of the equation.

Proof. If $\partial_{\phi(x)} \geq \partial_{\phi(y)} + g$ where g is the gap of the equation and if the equation is x-balanced, then ϕ is a solution if and only if , with the notation of condition (4), it is a solution of

$$y^{\alpha_i} x = y^{\beta_i} x, \ i < \lambda (= \rho)$$

i., e., if and only if it is a solution of

$$x = y^{|\alpha_i - \beta_i|}x, \ i < \lambda$$

It is clear that the condition is sufficient. Conversely, consider such an equation. By Proposition 4 we have $\phi(x) = u^{\omega^{\mu_i+1}}v$ and $\phi(y) = u$ for some $u, v \in \Sigma^{\mathrm{Ord}}$ and $\mu_i \leq g$ where μ_i is the degree of the difference $|\alpha_i - \beta_i|$. If g is a successor ordinal then there exists an ordinal j for which $\mu_j = g - 1$ and we are done. Otherwise g is the limit of an infinite increasing sequence of ordinals that are the degrees of some $|\alpha_i - \beta_i|$'s, thus u^{ω^g} is a prefix of $\phi(x)$.

From now on we assume the equation is not x-balanced or it is x-balanced and $\partial_{\phi(x)} < \partial_{\phi(y)} + g$. We may assume by Corollary 5 that $\partial_{\phi(x)} > \partial_{\phi(y)}$ is satisfied. The condition is clearly sufficient by Proposition 3 so we prove it is necessary. We start with a technical result.

Lemma **3.1** *Let $xL(x,y) = y^\beta R(x,y)$, $\beta > 0$ be an equation and let $\phi : \Xi \to \Sigma^{Ord}$ be a solution such that*

$$|\phi(y^\beta)| < |\phi(x)| < |\phi(y^{\beta+1})| \tag{5}$$

holds. Then $\phi(x)$ and $\phi(y)$ are generalized cyclic.

Proof. Indeed, if $\beta < \omega$ then $\partial_{\phi(x)} = \partial_{\phi(y)}$ and we are done by Corollary 5. We thus assume $\beta \geq \omega$ and therefore $\partial_{\phi(x)} > \partial_{\phi(y)}$.

Assume first $L(x,y) = y^k$ for some $k < \omega$. Then the equation is actually $xy^k = y^{\beta'}$ for some $\beta' > \beta$. Arguing on the lengths we get $\phi(xy) = \phi(y^{\beta''})$ for some successor ordinal $\beta'' \leq \beta'$. There are two possibilities.

i) $\phi(y^{\beta''-1})$ is a prefix of $\phi(x)$. Then for some $u \in \Sigma^{\mathrm{Ord}}$ we have

$$\phi(x) = \phi(y^{\beta''-1})u, \phi(y) = u\phi(y)$$

thus $\phi(y) = u^\omega v$ for some $v \in \Sigma^{\mathrm{Ord}}$ and we are done.

ii) $\phi(x)$ is a prefix of $\phi(y^{\beta''-1})$. Then for some $u \in \Sigma^{\mathrm{Ord}}$ we have

$$\phi(x) = \phi(y^{\beta''-2})v, \phi(y) = vu = u\phi(y)$$

This implies $vu = uvu$, which via Proposition 4 achieves the verification in this case.

We are thus left with the case where $L(x,y)$ has a prefix of the form y^ω or of the form $y^k x$ with $k < \omega$. In either case $\phi(y^\omega)$ is a prefix of $\phi(L(x,y))$. The same holds for $R(x,y)$, i.e., $\phi(y^{\beta+\omega})$ is a prefix of $\phi(y^\beta R(x,y))$. Arguing again on the lengths we have

$$\phi(xy^\omega) = \phi(y^{\beta+\omega})$$

This implies $\phi(x) = \phi(y^\beta)u$, $\phi(y) = uv$, $\phi(y^\omega) = v\phi(y^\omega)$ for some $u, v \in \Sigma^{\mathrm{Ord}}$ and furthermore $(uv)^\omega = y^\omega = vy^\omega = v(uv)^\omega = (vu)^\omega$. Again we may conclude via Proposition 4. ■

We now return to the proof of the theorem.

Case 1: there exists $i < \min\{\lambda, \rho\}$ such that $\phi(y^{\alpha_i} x) \neq \phi(y^{\beta_i} x)$.

We assume i is the minimal ordinal with this property. By interchanging the two handsides if necessary, because of the hypothesis ϕ is a solution of an equation of the form

$$xL'(x, y) = y^\gamma x R'(x, y) \tag{6}$$

where $\phi(x)$ is a proper prefix of $\phi(y^\gamma x)$. Since $\phi(x)$ has a prefix equal to $\phi(y)$, either $\phi(y^{\gamma+1})$ is a prefix of $\phi(x)$ or $\phi(x)$ is a prefix of $\phi(y^{\gamma+1})$. If $\phi(x)$ and $\phi(y^{\gamma+1})$ are equal we are done, so we assume one is a proper prefix of the other.

We proceed by induction on $|\phi(x)|$. If $\phi(x)$ is a prefix of $\phi(y^{\gamma+1})$ the result is a consequence of Lemma 3.1. So we may assume that $\phi(y^{\gamma+1})$ is a prefix of $\phi(x)$. Since $\phi(y^\gamma)$ is a proper prefix of $\phi(x)$ we may define a substitution ψ by

$$\phi(y^\gamma)\psi(x) = \phi(x), \ \ \psi(y) = \phi(y)$$

It is a solution of the equation

$$xL'(y^\gamma x, y) = y^\gamma x R'(y^\gamma x, y) \tag{7}$$

Clearly if the initial equation (6) is not x-balanced then equation (7) cannot be x-balanced either. Also $\partial_{\phi(x)} \leq \partial_{\phi(y)} + g$ implies $\partial_{\psi(x)} \leq \partial_{\psi(y)} + g'$ where g' is the gap of equation (7) and $|\psi(x)| \geq |\psi(y)|$ holds because $\phi(y^{\gamma+1})$ is a prefix of $\phi(x)$. We conclude by induction.

Case 2: for all $i < \min\{\lambda, \rho\}$, $\phi(y^{\alpha_i} x) = \phi(y^{\beta_i} x)$ holds. Then the initial equation cannot be x-balanced. If $\lambda = \rho$ holds then $\alpha_\lambda \neq \beta_\lambda$ and $\phi(y^{\alpha_\lambda}) = \phi(y^{\beta_\lambda})$ implies $\phi(y) = 1$. We are left with $\lambda < \rho$ (or $\rho < \lambda$), i.e.,

$$\phi(y^{\alpha_\lambda}) = \phi\left((\prod_{\lambda \leq i < \rho} y^{\beta_i} x) \ y^{\beta_\rho} \right)$$

By cancelling out the prefix $\phi(y^{\beta_\lambda})$, either $\phi(x)$ is a power of $\phi(y)$ or by Lemma 3.1 $\phi(x)$ and $\phi(y)$ are generalized cyclic. This completes the proof. ∎

4 Systems of regular equations

In this section we make the assumption that the lengths of the strings are less than ω^ω. We are concerned not with single equations but more generally with arbitrary systems of equations where each equation is "regular" in a sense that will be made precise. The main result (Theorem 2) states that any solution is a "specialization" of a simpler one.

4.1 Regular equations

In the previous section we described the set of solutions of equations in two unknowns based on a generalization of the notion of cyclic solutions for equations on finite strings. For more than two unknowns the description is much more complex (even for finite strings, [15]). The first difficulty to face is related to the effectiveness of an equation. Indeed, since the cardinality of all possible finite and transfinite equations is non denumerable, there is no hope we can state effective results in general. Said differently, if we want to deal with effective objects then we must restrict attention to "effective" equations. This notion is made precise now.

A *regular string* over an alphabet A is a string that can be obtained from the single letters by a finite number of applications of concatenation and ω-closure. For example, with $A = \{a, b\}$, the string $(a^{\omega^2}b)^\omega a$ is regular but $aba^2ba^3b\dots a^nb\dots$ is not. Thus by construction, all regular strings have length less than or equal to ω^ω.

Denote by $\mathrm{Reg}(A)$ the set of regular strings over A. A *regular equation* is an equation whose handsides are regular strings, i., e., $L(x_1, \dots, x_n), R(x_1, \dots, x_n) \in \mathrm{Reg}(\Xi)$. For example, $x^\omega = y^\omega$ is a regular equation but $\prod_{0\leq i<\omega} x^i y = \prod_{0\leq i<\omega} y^i x$ is not. Equally, we will speak of a *regular solution* meaning a solution $\phi : \Xi \to \Sigma^{<\omega^\omega}$ such that $\phi(x)$ is a regular string for all $x \in \Xi$.

The following elementary properties can be found in [5]. They follow from the fact that a string is regular if and only if it can be recognized by some Choueka automaton.

PROPOSITION **6** *Let $\xi \in Reg(\Xi)$ be a regular string. Then all prefixes and suffixes of ξ are regular. Furthermore, the number of different suffixes is finite.*

It is quite straightforward that the set of regular strings is closed under regular substitutions.

PROPOSITION **7** *Let Ξ and Σ be two alphabets. Let $\xi \in Reg(\Xi)$ be regular and let $\sigma : \Xi \to \Sigma^{<\omega^\omega}$ be a substitution such that $\sigma(x) \in Reg(\Sigma)$ for all $x \in \Xi$. Then $\sigma(\xi) \in Reg(\Sigma)$.*

4.2 Factorizing solutions

Consider the equation $xy = zt$ and three arbitrary strings $w_1, w_2, w_3 \in \{a, b\}^{\mathrm{Ord}}$. Then the substitution $\phi(x) = w_1, \phi(z) = w_1w_2, \phi(y) = w_2w_3, \phi(t) = w_3$ is a solution. Though very complex it might be (the strings w_1, w_2, w_3 may not be effective), this solution is a specification of the very simple solution ψ over the alphabet $\Delta = \{u, v, w\}$ defined by $\psi(x) = u$, $\psi(z) = uv$, $\psi(y) = vw$, $\psi(t) = w$ (the images have length 1 and 2 !). This property holds more generally when the system consists of regular equations only. It is akin to that of "principal" solutions of the theory of Lentin on equations.

THEOREM **2** *Given a finite or infinite set I of indices, let $L_i = R_i$, $i \in I$, be a family of regular equations over Ξ and let ϕ be a solution. Then there exist a finite alphabet $\overline{\Xi}$, a regular solution $\bar\phi : \Xi \to Reg(\overline{\Xi})$ of the system and a substitution $\sigma : \overline{\Xi} \to \Sigma^{Ord}$ such that $\phi(x) = \sigma(\bar\phi(x))$ holds for all $x \in \Xi$.*

$$\begin{tikzcd}
\Xi^{<\omega^\omega} \arrow[r, "\overline{\phi}"] \arrow[dr, "\phi"'] & \overline{\Xi}^{\omega^\omega} \arrow[d, "\sigma"] \\
 & \Sigma^{Ord}
\end{tikzcd}$$

Consider the equation $z = W(x,y)$ on the alphabet $\{x,y,z\}$ where W is a non regular string. Then the solution $\phi(x) = a, \phi(y) = b, \phi(z) = W(a,b)$ on the alphabet $\Sigma = \{a,b\}$ does not satisfy the condition of the theorem which shows that the hypothesis on regularity is necessary.

Proof. To fix ideas assume the system of equations has n unknowns $x_1, x_2, \ldots, x_n$. At each induction step we make, if necessary, the following possible transformations which facilitate the verification of the statement without modifying the ordinal on which the induction is performed. We will not recall these operations in the course of the proof.

1) for all $x \in \Xi$, there exists $i \in I$ such that x occurs in the equation $L_i = R_i$.

2) for all $i \in I$, L_i and R_i are not prefix from one another. Indeed, if say L_i is a prefix of R_i, then the images of some unknowns are necessarily the empty string. It suffices to transform each equation by deleting all the variables that map onto the empty string. By Proposition 7 the result is still a regular system of equation.

3) for all $i \in I$ there is no proper prefix L'_i of L_i (respectively no proper prefix R'_i of R_i) such that $\phi(L'_i) = \phi(R'_i)$ holds. Indeed, this can be achieved by splitting the equation. If the initial equation is regular so are the results of the splitting operation.

4) Say $x \in \Xi$ is of *maximum degree* (relative to ϕ) if $\partial_{\phi(x)} = \max\{\partial_{\phi(y)} \mid y \in \Xi\}$ holds. Then for all $i \in I$, if the equation $L_i = R_i$ contains an occurrence of a symbol of maximum degree, then so does L_i and the first occurrence of maximum degree in L_i occurs before the first occurrence of maximum degree in R_i if it exists at all. This can be done by simply exchanging the two hansides if necessary.

After these preliminaries, we turn to the proof and we proceed by induction on the following sum where $k \to i_k$ is a permutation of the set $\{1, \ldots, n\}$

$$S(\phi) = |\phi(x_{i_1})| + |\phi(x_{i_2})| + \ldots + |\phi(x_{i_n})| \text{ with } |\phi(x_{i_1})| \geq |\phi(x_{i_2})| \geq \ldots \geq |\phi(x_{i_n})| \qquad (8)$$

We consider only the equations $L_i = R_i$ which have an occurrence of maximal degree. For any such equation we denote by $y_1^{(i)} \in \Xi$ the first occurrence in L_i of such a symbol and by $z_1^{(i)} \in \mathrm{Reg}(\Xi)$ the prefix of L_i just before $y_1^{(i)}$, i., e., $z_1^{(i)} y_1^{(i)}$ is a prefix of L_i. If R_i contains an occurrence of maximum degree we denote it by $y_2^{(i)} \in \Xi$ and by $z_2^{(i)} \in \mathrm{Reg}(\Xi)$ the prefix of R_i just before $y_2^{(i)}$, i., e., $z_2^{(i)} y_2^{(i)}$ is a prefix of R_i. If R_i does not contain an occurrence of maximum degree, $z_2^{(i)}$ denotes the string R_i itself. The proof consists in studying different ways in which the image $\phi(y_1^{(i)})$ is covered by factors in $\phi(R_i)$.

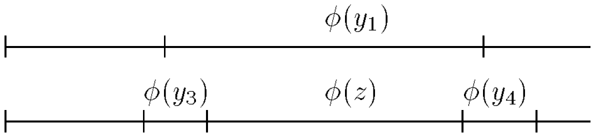
The general method is as follows. Assume for each variable x_i the image $\phi(x_i)$ can be expressed as $X_i(u_1, \ldots, u_p)$ where each u_j is a string of Σ^{Ord} and X_i is a *regular expression*, i.e., a regular string over these values as formal symbols. Consider the alphabet Δ containing p different symbols that are in one-to-one correspondence with the u_j's and denote by α_j the element of Δ in correspondence with u_j. Define the regular substitution $\theta : \Xi \to \mathrm{Reg}(\Delta)$ as $\theta(x_i) = X_i(\alpha_1, \ldots, \alpha_p)$ and the substitution $\psi : \Delta \to \Sigma^{\mathrm{Ord}}$ by $\psi(\alpha_j) = u_j$ for all $0 \leq j \leq p$. Then $\phi = \psi \circ \theta$ holds by construction and ψ is a solution of the equation $L(\theta(x_1), \ldots, \theta(x_n)) = R(\theta(x_1), \ldots, \theta(x_n))$. Assume $S(\psi) < S(\phi)$ holds. Then by the induction hypothesis there exist an alphabet $\overline{\Delta}$, a solution $\overline{\psi} : \Delta^{<\omega^\omega} \to \mathrm{Reg}(\overline{\Delta})$

of $L(\theta(x_1),\ldots,\theta(x_n)) = R(\theta(x_1),\ldots,\theta(x_n))$ and a substitution $\sigma : \overline{\Delta} \to \Sigma^{\mathrm{Ord}}$ such that $\psi(\alpha) = \sigma(\overline{\psi}(\alpha))$ holds for all $\alpha \in \Delta$. This yields $\phi(x) = \sigma(\bar{\phi}(x))$ for all $x \in \Xi$ where $\bar{\phi}$ is the regular substitution $\bar{\psi}\circ\theta$ which shows that ϕ is a solution of $L(x_1,\ldots,x_n) = R(x_1,\ldots,x_n)$.

$$\begin{tikzcd}
\Xi^{<\omega^\omega} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \Delta^{<\omega^\omega} \arrow[r, "\overline{\psi}"] \arrow[d, "\psi"] & \overline{\Delta}^{\omega^\omega} \arrow[dl, "\sigma"] \\
& \Sigma^{\mathrm{Ord}} &
\end{tikzcd}$$

For $S(\phi) = 0$ the result clearly holds so we assume $S(\phi) > 0$. We consider the following exclusive conditions where each condition assumes the previous ones are not satisfied.

Case 1: for some $i \in I$, $y_2^{(i)}$ exists and $|\phi(z_1^{(i)}y_1^{(i)})| < |\phi(z_2^{(i)})|$ holds. For simplicity we will drop the superscripts, i.e., we write y_1 for $y_1^{(i)}$ for example. Observe also that because of assumption 1), the inequality is strict.

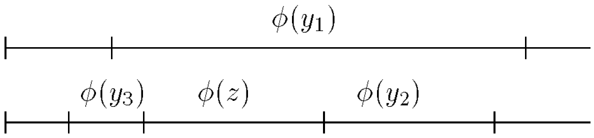


Denote by $y_3, y_4 \in \Xi$, the variables which have an occurrence such that $\phi(y_3)$ and $\phi(y_4)$ overlap the beginning and the end of $\phi(y_1)$ respectively. Then for some $z \in \Xi^{<\omega^\omega}$, $u, u', v, v' \in \Sigma^{\mathrm{Ord}}$ we have

$$\phi(y_1) = u'\phi(z)v,\, \phi(y_3) = uu',\, \phi(y_4) = vv'$$

Then we can write $\phi(y_1) = u'Ev$ where because of Proposition 6, E is a regular expression over the strings u, u', v, v' and the $\phi(y_i)$'s of degree less than that of $\phi(y_1)$. Because all degrees $\partial_u, \partial_{u'}, \partial_v, \partial_{v'}$ are less than $\partial_{\phi(y_1)}$ we may conclude.

Case 2: for some $i \in I$, $|\phi(z_2^{(i)}y_2^{(i)})| < |\phi(z_1^{(i)}y_1^{(i)})|$ holds. As in the previous case we drop the superscripts.

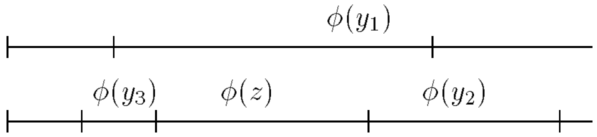


Denote by $y_3 \in \Xi$ the variable which has an occurrence such that $\phi(y_3)$ overlaps the beginning of $\phi(y_1)$. Then for some $z \in \Xi^{<\omega^\omega}$, $u, u', v \in \Sigma^{\mathrm{Ord}}$ we have

$$\phi(y_1) = u'\phi(zy_2)v,\, \phi(y_3) = uu'$$

Then we can write $\phi(y_1) = u'Ev$ where because of Proposition 6, E is an ω-regular expression over the strings u', v and the $\phi(y_i)$'s of degree less than that of $\phi(y_1)$ and $\phi(y_2)$. Because $\partial_{u'}, \partial_v < \partial_{\phi(y_1)}$ holds and $\phi(y_2)$ has the same degree as $\phi(y_1)$ but a smaller coefficient of highest degree we may conclude.

Case 3: for some $i \in I$, $|\phi(z_1^{(i)} y_1^{(i)})| < |\phi(z_2^{(i)} y_2^{(i)})|$ holds. As in the previous cases we drop the superscripts.



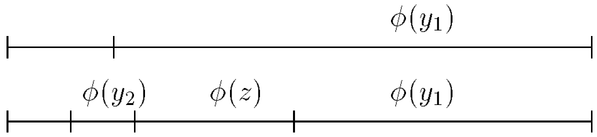
Denote by $y_3 \in \Xi$, the variable which has an occurrence such that $\phi(y_3)$ overlaps the beginning of $\phi(y_1)$. Then for some $z \in \Xi^{<\omega^\omega}$, $u, u', v, v' \in \Sigma^{\mathrm{Ord}}$ we have

$$\phi(y_1) = u'\phi(z)v, \ \phi(y_2) = vv', \ \phi(y_3) = uu'$$

Then we can write $\phi(y_1) = u'Ev$ where because of Proposition 6, E is an ω-regular expression over the strings u', v and the $\phi(y_i)$'s of degree less than that of $\phi(y_1)$. Because $\partial_{u'} < \partial_{\phi(y_1)}$ and either $\partial_v < \partial_{\phi(y_1)}$ or $\partial_v = \partial_{\phi(y_1)}$ holds but then $|v|$ has a smaller coefficient of highest degree than $|\phi(y_1)|$, we may conclude.

We are left with the following case.

Case 4: for each $y_1 \in \Xi$ of maximum degree, each equation $L_i = R_i$ where y_1 occurs has the form $z_1 y_1 = z_2 y_1$ where for all letters y in z_1 and z_2 the condition $\partial_{\phi(y)} < \partial_{\phi(y_1)}$ holds



Assuming without loss of generality $|\phi(z_1)| \leq |\phi(z_2)|$, we consider an equation $L_i = R_i$ with $|\phi(z_2)| - |\phi(z_1)|$ maximum. If this maximum is 0 we can easily conclude, so we assume it is not.

Denote by $y_2 \in \Xi$, the variable which has an occurrence whose image $\phi(y_2)$ overlaps the beginning of $\phi(y_1)$. Then for some $u, u' \in \Sigma^{\mathrm{Ord}}$ we have $\phi(y_2) = uu'$. By canceling out a common prefix, for some $z \in \Xi^{<\omega^\omega}$ we obtain by $\phi(y_1) = u'\phi(z)\phi(y_1)$. This can be considered as the solution of an equation in two unknowns whose images are $u'\phi(z)$ and $\phi(y_1)$, so by Proposition 4 we get $\phi(y_1) = (u'\phi(z))^\omega v$ for some $v \in \Sigma^{\mathrm{Ord}}$. Let us verify that the substitution ψ equal to ϕ except for y_1 where it is defined as $\psi(y_1) = (u'\phi(z))^\omega$ is a solution of the same system of equations. Indeed, consider an equation of the form $z_3 y_1 = z_4 y_1$ where $z_3, z_4 \in \mathrm{Reg}(\Xi - \{y_1\})$. Then for some $w \in \Sigma^{\mathrm{Ord}}$ we have $\phi(z_3)w = \phi(z_4)$ and $\phi(y_1) = w\phi(y_1)$. Thus $(u'\phi(z))^\omega v = w(u'\phi(z))^\omega v$ which by considerations on the lengths implies $(u'\phi(z))^\omega = w(u'\phi(z))^\omega$. Since $\partial_z < \partial_{\phi(y_1)}$ we may conclude by induction on $S(\psi)$.

■

5 Elementary combinatoric properties

Here we investigate the basic combinatoric elementary properties of strings whose lengths are arbitrary ordinals. In other words, we study conjugacy, primitivity, periodicity and the like.

5.1 Periodicity

This section deals with the notion of periodicity and can be viewed as establishing preliminary results for solving the equations $x^m y^p = z^q$ with $2 \leq m, p, q < \omega$. Indeed, recall that in the case of finite strings, if m and q are "large" then the fact that the powers x^m and z^q have a "large" common prefix is the key argument, known as Fine and Wilf's result, for solving the equation. This is precisely what we want to reproduce here but it requires some care and extends in different ways.

5.1.1 Fine and Wilf revisited

As a result of Proposition 4 and of easy considerations on the lengths, two strings commute if and only if they are powers of a third string. To draw further the analogy with the case of finite strings, we will show that under the hypothesis $|xy| = |yx|$, a direct analog of Fine and Wilf theorem holds. Observe that because of Proposition 1, $|x|$ and $|y|$ have a maximal common left divisor $|x| \wedge |y|$ such that $|x| = (|x| \wedge |y|)p$ and $|y| = (|x| \wedge |y|)q$ holds for some integers $0 < p, q < \omega$ and $p \wedge q = 1$. The next statement applies when interpreted over finite strings and is known as Fine and Wilf's theorem, [7].

Proposition 8 *Let $x, y \in \Sigma^{Ord}$ be two non empty strings satisfying $|xy| = |yx|$. Then the following conditions are equivalent*

i) $xy = yx$

ii) there exist $0 < r, s < \omega$ and z such that $x = z^r$ and $y = z^s$

iii) there exist $0 < r, s < \omega$ such that $x^r = y^s$

iv) x^ω and y^ω have a common prefix of length $|x| + |y| - (|x| \wedge |y|)$.

Observe that the condition is sharp for any degree. For example, with $\Sigma = \{a, b, c\}$, $x = a^\omega aba^\omega ac$ and $y = a^\omega aba^\omega aca^\omega ab$ the strings x^ω and y^ω have a common prefix of length equal to the predecesor of $|x| + |y| - (|x| \wedge |y|)$.

Proof.

$i) \Rightarrow ii)$ follows from Proposition 4.

$ii) \Rightarrow iii)$: trivial

$iii) \Rightarrow iv)$: trivial since powers of x and y have a common prefix of arbitrary length.

$iv) \Rightarrow i)$: Set $|x| = \lambda p$, $|y| = \lambda q$ where $p, q < \omega$ and $\lambda \in \mathrm{Ord}$ and $p \wedge q = 1$. Set

$$x = x_1 x_2 \dots x_p, \ y = y_1 y_2 \dots y_q$$

where $|x_i| = |y_j| = \lambda$, $1 \leq i \leq p$, $1 \leq j \leq q$. Hypothesis iv) means that the two infinite periodic sequences $x_1, x_2, \dots x_p, x_1, x_2, \dots$, and $y_1, y_2, \dots y_q, y_1, y_2, \dots$, (considered as sequences over the finite alphabet consisting of the p factors x_i and the q factors y_j)

have a common prefix of length $p+q-1$. The conclusion follows from the result on finite strings.

■

A similar result holds when considering suffixes instead of prefixes. However, it needs a special treatment as the "mirror image" of a transfinite string is not a transfinite string.

Proposition **9** *Let $x, y \in \Sigma^{Ord}$. If for some $k, l < \omega$, x^k and y^l have a common suffix of length $|xy|$ then $xy = yx$.*

Observe that the conditions are sharp (consider $x = ba^\omega b^2 a^\omega b^2$ and $y = a^\omega b^3 a^\omega b^2 a^\omega b^2$).

Proof. Simple considerations on the lengths show that $|xy| = |yx|$. Let λ be the greatest common left divisor of $|x|$ and $|y|$: $|x| = \lambda p$ and $|y| = \lambda q$ holds for some $p, q < \omega$ and $p \wedge q = 1$. Let z be the common suffix as assumed in the assertion and write $z = z_1 z_2$ where $|z_1| = \lambda$. Factorize x and y into factors of length λ: $x = x_1 x_2 \dots x_p$ and $y = y_1 y_2 \dots y_q$ ($|x_i| = |y_j| = \lambda$ for $1 \leq i \leq p$, $1 \leq j \leq q$).

For some $\bar{x}$ and $\bar{y}$ we have $x^k = \bar{x}z$ and $y^l = \bar{y}z$. It is not difficult to verify that $|\bar{x}z_1|$ is a multiple of λ thus that z_2 is the product of the last $r = p+q-1$ factors $x_{i_1} x_{i_2} \dots x_{i_r}$ where $i_{s+1} = (i_s + 1) \bmod p$ and $i_r = p$. Similarly, z_2 is a product of the last r factors $y_{j_1} y_{j_2} \dots y_{j_r}$ where $j_{s+1} = (j_s + 1) \bmod q$ and $j_r = q$. The conclusion follows from the result on finite strings. ■

5.1.2 Roots – Primitivity

The above properties yield, as in the case of finite strings, a few interesting consequences. In particular the notions of roots and primitive strings make sense in this more general setting. Indeed, by Proposition 4 we have

Proposition **10** *Given $1 \neq x \in \Sigma^{Ord}$ there exists a unique (length-) minimal element $z \in \Sigma^{Ord}$ and a unique integer finite n such that $x = z^n$*

We say z is the *root* of x and n its *exponent*. By convention, the root of the empty string is the empty string itself and its exponent is 0. A string x is *primitive* if it cannot be written as $x = z^n$ where $1 < n < \omega$. For example, $a^\omega a^3$ is primitive but $(a^\omega)^2 a^3$ is not since it is equal to $(a^\omega a^3)^2$.

5.2 Conjugacy

We concentrate here on the notion of conjugacy which is a central combinatorial concept in the theory of monoids. We recall that two elements x, y of a monoid are *conjugate* if there exist two elements z, t such that $x = zt$ and $y = tz$. The proof of the next property connects the notions of primitivity and conjugacy. It follows the same line as in the case of finite strings and it is left to the reader.

Proposition **11** *Two conjugate strings have conjugate roots and equal exponents.*

5.2.1 The conjugacy equation

As in the case of free monoids there is an alternative definition to the notion of conjugacy. The following result is a direct extension of [14], see also [15, p. 8].

PROPOSITION **12** *The strings* $x, y, z \in \Sigma^{Ord}$, $y \neq 1$ *satisfy the condition* $xz = zy$ *if and only if there exist* $u, v \in \Sigma^{Ord}$ *and* $n < \omega$ *such that* $x = uv$, $y = vu$ *and* $z = (uv)^n u$

Observe that the equality $xx^\omega = x^\omega 1$ holds. Therefore the condition $y \neq 1$ is necessary.

Proof. We observe that $\partial_z \leq \partial_x$. We argue by induction on the length of z. If $|z| = 0$ we are done. If $|z| \leq |x|$ then there exists x' such that $x = zx'$. By cancelling out the left factor z in the equation we get $x'z = y$. If $|z| > |x|$ then $\partial_x = \partial_z$ and there exists z' such that $z = xz'$ and $|z'| < |z|$. By cancelling out the left factor x in the equation we get $xz' = z'y$ and we may conclude by induction.

■

A finite string has a number of different conjugates equal to the length of its root. Said differently, there exists a natural bijection between the set of its prefixes of the root and the set of different conjugates. Expressed this way, the result can be extended to transfinite strings. The proof for finite strings carries over here, so we do not reproduce it.

PROPOSITION **13** *Consider the string* $x \in \Sigma^{Ord}$ *and let* n *be its exponent. Let* f *be the function that associates with every* $\alpha < |x|$ *the conjugate* x_2x_1 *where* $x = x_1x_2$ *and* $|x_1| = \alpha$. *Then the inverse image of each conjugate of* x *contains exactly* n *ordinals which are equal modulo the length of the root of* x.

Contrarily to finite strings, conjugates do not necessarily have the same length, e.g., a^ω, $a^\omega a$, $\ldots$, $a^\omega a^n$, ...are all conjugates. However, if we fix the length, then there are finitely many different conjugates of this length.

PROPOSITION **14** *The number of different conjugates of* $x \in \Sigma^{Ord}$ *of a given length is equal to the coefficient of highest degree of the length of its root.*

Proof. By the previous proposition we may assume x is primitive. Set $|x| = \omega^\lambda c + \nu$ where $0 < c < \omega$ and $\partial_\nu < \lambda$. The length of an arbitrary conjugate is of the form $\omega^\lambda c + \mu$ with $\partial_\mu < \lambda$. Assuming without loss of generality $\nu \leq \mu$, it suffices to verify that the system of the two equations $\alpha + \beta = \omega^\lambda c + \nu$ and $\beta + \alpha = \omega^\lambda c + \mu$ has exactly the c different solutions $\alpha = \omega^\lambda c' + \mu$ and $\beta = \omega^\lambda (c - c') + \nu$ where $0 \leq c' < c$.

■

5.2.2 Strong conjugacy

The following is a refinement of the notion of conjugacy.

DEFINITION **4** *Two strings* $x, y \in \Sigma^{Ord}$ *are* strongly conjugate *if and only if there exist* $z, t \in \Sigma^{Ord}$ *and two integers* m, n *such that*

$$x = z^\omega t z^n \text{ and } y = z^\omega t z^m \tag{9}$$

Strong conjugacy has nice properties which we proceed to state.

PROPOSITION **15** *Let $x, y \in \Sigma^{Ord}$. The following conditions are equivalent*

i) x, y are strongly conjugate

ii) $xy = yy$ and $yx = xx$

iii) x, y are conjugate and x is a prefix of y or y is a prefix of x

iv) there exist z, t such that $x = zt, y = tz$ and either $zt = t$ or $tz = z$

Furthermore, the relation "being strongly conjugate" is an equivalence relation

Proof. $ii) \Rightarrow i)$ (resp. $ii') \Rightarrow i)$): Clear from Proposition 4 arguing on the lengths of x and y.

The implications $i) \Rightarrow iv)$ and $iv) \Rightarrow iii)$ are trivial.

$iii) \Rightarrow ii)$ and $ii')$: $x = zt$ and $y = tz$ for some $z, t \in \Sigma^{\mathrm{Ord}}$. Assume x is a prefix of y. If $\partial_z < \partial_t$ then we have $|z| + |t| = |t|$. This implies $zt = t$ and therefore $xy = zttz = ttz = tztz = yy$ and $yx = tzzt = tzt = ztzt = xx$.

We now assume $\partial_z = \partial_t$. If $|t| \leq |z|$ then $z = tz_2$ for some $z_2 \in \Sigma^{\mathrm{Ord}}$ implying z_2t is a prefix of tz_2. If $|z| \leq |t|$ then $t = zt_2$ implying zt_2 is a prefix of t_2z. In both cases we may conclude by induction on the length that ii) and ii') holds for z and t_2. Thus z and t_2 are strongly conjugate and a simple computation shows that ii) and ii') hold.

Let us turn to the last assertion. The relation is clearly symmetric and reflexive. Let us verify that it is also transitive. Assume x, y and y, z are strongly conjugate. By condition iii), it suffices to consider the case where y is a prefix of x and z. Then $xy = yy, zy = yy$, i.e., x and z are comparable. ■

As a consequence we have

COROLLARY **16** *If two strings x, y are strongly conjugate then so are their roots.*

Proof. Because of Proposition 11 and Proposition 12, we have $x = (x_1x_2)^k$ and $y = (x_2x_1)^k$ for some $x_1, x_2 \in \Sigma^{\mathrm{Ord}}$ where k is the common exponent of x and y and where $x_1, x_2 \in \Sigma^{\mathrm{Ord}}$. Assume without loss of generality that x is a prefix of y. Then x_1x_2 is a prefix of x_2x_1 and we may conclude by assertion iii) of the previous proposition. ■

5.3 Quasi-cyclicity

The following notion captures the property enjoyed by the solutions of equations in two unknowns whose both handsides are finite (cf. Proposition 4). It will also prove crucial when studying the equations $x^m y^p = z^q$.

DEFINITION **5** *A set X of strings is* quasi-cyclic *if there exist two strings $u, v \in \Sigma^{Ord}$ such that $X \subseteq (u^\omega v)^* u^*$ holds. The two strings u and $u^\omega v$ may be assumed primitive.*

By extension two strings x, y are quasi-cylic if the set $\{x, y\}$ is quasi-cyclic.

Restricted to two strings, the definition is close to but not equivalent to that of generalized cyclicity. Indeed, $a^\omega b$ and a are quasi-cyclic, but not generalized cyclic. Conversely, $(a^\omega b)^{\omega+2}a$ and $(a^\omega b)^{\omega+1}a^2$ are generalized cyclic but not quasi-cyclic.

It is clear that u may be supposed primitive. Furthermore, let z be the root of $u^\omega v$, i.e., $z^n = u^\omega v$ for some $n \geq 1$. Arguing on the lengths we get $|z| \geq |u^\omega|$, so $z = u^\omega w$ for some $w \in \Sigma^{\mathrm{Ord}}$. Then $u^\omega v \in (u^\omega w)^*$.

With the help of this definition we may state a weaker version than that of Proposition 8 in the case where $|xy| \neq |yx|$.

PROPOSITION **17** *Let $x, y \in \Sigma^{Ord}$. Then the following conditions are equivalent*
i) $x^\omega = y^\omega$
ii) x *and* y *are quasi-cyclic and* $\partial_x = \partial_y$
iii) x^ω *and* y^ω *have a common prefix of length* $\min\{|xy|, |yx|\}$

By considering $x = a^n b^\omega a^{n+1} b^\omega$ and $y = a^n b^\omega a^{n+1} b^\omega a^n b^\omega a$ we see that we may not weaken the condition $iii)$.

Proof. The only non trivial implication is $iii) \Rightarrow ii)$. Clearly we may assume without loss of generality that $|xy| \geq |yx|$ is satisfied and that $\partial_x = \partial_y > 0$ holds. This implies that there exist some $\lambda, \mu \in \mathrm{Ord}, \partial_\lambda > \partial_\mu$ and some $p, q < \omega$, $p \wedge q = 1$ for which we have $|x| = \lambda p$ and $|y| = \lambda q + \mu$. Set

$$x = x_1 x_2 \ldots x_p,\ y = y_1' y_2 \ldots y_q z, y_1 = z y_1'$$

where $|x_i| = |y_j| = \lambda$, $1 \leq i \leq p$, $1 \leq j \leq q$, $|y_1'| = \lambda$, $|z| = \mu$. Then the sequences $x_2, \ldots, x_p, x_1$ and $y_2, \ldots, y_q, y_1$ (as sequences over the finite alphabet consisting of the p factors x_i and the q factors y_j) have a common prefix of length $p + q - 1$. Thus for all $1 \leq i \leq p$, $1 \leq j \leq q$, we have $x_i = y_j$. Now $x_1 = y_1 = y_1'$ and $y_1 = z y_1'$ implies $y_1 = y_1' = z^\omega t$ for some $t \in \Sigma^{\mathrm{Ord}}$ which completes the proof.

∎

The following fact is straightforward.

PROPOSITION **18** *Two strings $x, y \in \Sigma^{Ord}$ are quasi-cyclic if and only if so are x and xy.*

Quasi-cyclicity is preserved by taking root and raising to power.

PROPOSITION **19** *Two strings $x, y \in \Sigma^{Ord}$ are quasi-cyclic if and only if so are their roots.*

Furthermore, the relation "x and y are quasi-cyclic" restricted to the strings of equal degree is an equivalence relation.

Observe that restricting the relation to the strings having the same degree is necessary. Indeed, with $x = a^\omega a$, $y = a$, $z = (a^\omega b)^\omega a = a^\omega (b a^\omega)^\omega a$ we have that x, y are quasi-cyclic and so are y, z but x, z are not.

Proof. Clearly, if the roots of x and y are quasi-cyclic, so are x and y. Conversely, assume x and y are quasi-cyclic and let $\bar{x}$ and $\bar{y}$ be their respective roots. If $\partial_x < \partial_y$ holds and assuming u and $u^\omega v$ are primitive, we get $x = u^i$ and $y = (u^\omega v)^l u^j$ for some integers i, j, l.

Since $u^\omega v$ is primitive so is $u^\omega v u^j$ and thus $\bar{x} = u$ and $\bar{y} = u^\omega v u^j$. If $\partial_x = \partial_y$ holds then $x = (u^\omega v)^k u^i$ and thus $\bar{x} = u^\omega v u^i$ which proves the first assertion.

Assume that $\partial_x = \partial_y = \partial_z$ holds, that x and y are quasi-cyclic and that so are y and z. By the previous claim we have $x = \bar{x}^n$, $y = \bar{y}^m$ and $z = \bar{z}^p$ where the roots $\bar{x}$ and $\bar{y}$ are two primitive and strongly conjugate strings and so are the roots $\bar{y}$ and $\bar{z}$. By Proposition 15 this implies $\bar{x}$ and $\bar{z}$ are strongly conjugate.

■

We may reformulate the notion of quasi-cyclicity by refining it.

Proposition **20** *Let $x, y \in \Sigma^{Ord}$. The following conditions are equivalent*

i) x and y are quasi-cyclic

ii) xy and yx are comparable.

iii) if $\partial_y \geq \partial_x$ then $xyy = yxy$ else $xyx = yxx$

Proof.

ii) implies *i*). If say, yx is a prefix of xy then x is a prefix of yx. Assume first $\partial_x \neq \partial_y$. Then $\partial_x < \partial_y$, i.e., $xy = y$ holds and we may conclude by Proposition 4. Now assume $\partial_x = \partial_y$. If $|x| \leq |y|$, i.e., $y = xy_2$ for some y_2 where $|y_2| < |y|$ then $y_2 x$ is a prefix of $y = xy_2$ and we conclude by induction. If $|y| \leq |x|$, i.e., $x = yx_2$ for some x_2 where $|x_2| < |x|$ then $x = yx_2$ is a prefix of $x_2 y$ and again we conclude by induction. Checking the implications $i) \Rightarrow iii)$ and $iii) \Rightarrow ii)$ is trivial.

■

6 The equation $x^m y^p = z^q$

Before tackling the general case we treat the easier case where two of the variables are quasi-cyclic. We show that the three variables are then quasi-cyclic.

Proposition **21** *Let x, y, z be three strings satisfying the equality $x^m y^p = z^q$ where $2 \leq m, p, q < \omega$. If two of the three strings are quasi-cyclic, then so are the three strings.*

Proof. Let us first verify that if $\partial_x \neq \partial_y$ then x, y and z are quasi-cyclic. Indeed, assume $\partial_x > \partial_y$ and thus $\partial_x = \partial_z$. If $m = q = 2$, then $|z| + |x| = |x|2$ else $|x|(m-1) \geq |z|$, i., e., $|x|m \geq |z| + |x| = |zx|$. Thus in all cases by Proposition 17 we have $x = (u^\omega v)^k u^i$ and $z = (u^\omega v)^l u^j$ for some $i, j < \omega$, $0 \leq k, l < \omega$ and $u, v \in \Sigma^{\mathrm{Ord}}$ with u and u^ω primitive. By canceling out the prefix of length $|x^m|$ we get $y^p = u^{j-i}$ and thus $y \in u^*$.

Now if $\partial_x < \partial_y$ then $\partial_y = \partial_z$ and thus $|y^p| = |z^q| \geq |y| + |z| = |yz|$. By Proposition 9 we have $yz = zy$. Let w be their common root. Then equation $x^m y^p = z^q$ can be viewed as an equation in the two unknowns x and w. Thus there exist $u, v \in \Sigma^{\mathrm{Ord}}$ and $\ell > 0$ such that $x \in u^*$ and $w \in (u^\omega v)^\ell u^*$.

We return to the proof of the assertion. We may assume that the two unknowns that are quasi-cyclic belong to a set of the form $(u^\omega v)^* u^*$ where $u^\omega v$ and u are primitive and by the previous observation that the lengths of the three unknowns have equal degree. It is also clear that we can rule out the case where the two unknowns commute because then we are reduced to considering an equation with two unknowns and we may apply Proposition 4.

1) $x = (u^\omega v)^k u^i = (u^\omega v u^i)^k$ and $y = (u^\omega v)^\ell u^j = (u^\omega v u^j)^\ell$ for some i, j and $0 < k, \ell < \omega$. We have $z^q = x^m y^p = (u^\omega v)^h u^j = (u^\omega v u^j)^h$ for some $h \geq 0$ and therefore z is some power of the string $u^\omega v u^j$ by Proposition 10.

2) $y = (u^\omega v)^k u^i = (u^\omega v u^i)^k$ and $z = (u^\omega v)^\ell u^j = (u^\omega v u^j)^\ell$ for some $i, j \geq 0$ and $0 < k, \ell < \omega$. Equality $x^m (u^\omega v)^r u^i = (u^\omega v)^s u^j$ holds for some integers $r, s > 0$. By considerating the lengths, we have $i = j$ and $x^m (u^\omega v u^i)^r = (u^\omega v u^i)^s$. This is an equation in the two unknowns x and $u^\omega v u^i$ which implies that x and $u^\omega v u^i$ are quasi-cyclic. Thus by Proposition 19 the 3 unknowns are quasi-cyclic.

3) $x = (u^\omega v)^k u^i = (u^\omega v u^i)^k$ and $z = (u^\omega v)^\ell u^j = (u^\omega v u^j)^\ell$ for some $i, j \geq 0$ and $0 < k, \ell < \omega$. This entails $(u^\omega v)^r u^i y^p = (u^\omega v)^s u^j$ for some integers $r, s > 0$. By cancelling out $(u^\omega v)^r u^i$ to the left we obtain: $y^p = (u^\omega v)^{s-r} u^j = (u^\omega v u^j)^{s-r}$ which completes the proof via Proposition 10. ■

Theorem **3** *All solutions of the equation $x^m y^p = z^q$ with $2 \leq m, p, q < \omega$ are quasi-cyclic.*

Proof. We shall prove the result by contradiction. Indeed, we shall assume that an equation of the form $x^m y^p = z^q$ admits a non-quasi-cyclic solution and we shall consider such a solution for which z has minimal length. We shall also assume that x, y and z are primitive strings. Because of Propositions 8 and 9 and of the previous Proposition we have

$$|x|(m-1) < |z| \quad \text{and } |y|(p-1) < |z|$$

For all $m, p \geq 2$ we have

$$|x^m y^p| \leq |x|(2(m-1)) + |y|(2(p-1)) < |z|4$$

which rules out the case $q \geq 4$.

Now if $m, p \geq 3$ we obtain

$$|x^m y^p|2 \geq |x|(3(m-1)) + |y|(3(p-1)) < |z|6$$

which rules out the case $m, p \geq 3, q = 3$.

Now assume $q = 3$ and let us first verify the inequality $|z| < |x^m| < |z^2|$. Indeed, $|z| \geq |x^m|$ implies $|y^p| \geq |z^2|$ thus $|y^{p-1}| \geq |z|$ and $|y^p| \geq |y| + |z|$. Equally, $|x^m| \geq |z^2|$ implies $|x^{m-1}| \geq |z|$ and $|x^m| \geq |x| + |z|$. In all cases, the solution is quasi-cyclic. This far we necessarily have

$$x^m = z_1 z_2 z_1, z = z_1 z_2 \text{ and } y^p = z_2 z_1 z_2 \text{ for some } z_1, z_2 \in \Sigma^{\mathrm{Ord}} \tag{10}$$

If $p = 2$ then we may assume that $\partial_{z_2} = \partial_{z_1}$. Indeed, $\partial_{z_2} > \partial_{z_1}$ implies $|z_2| = |z_1 z_2|$ and thus $y = z_2 = z_1 z_2 = z$ and the solution is quasi-cyclic. If $\partial_{z_2} < \partial_{z_1}$ then $|z_1| = |z_2 z_1|$ and thus $|x| \leq |z_1| = |z_2 z_1|$ thus $|x^m| \geq |z| + |x|$ and the solution is again quasi-cyclic. Similarly, $m = 2$ implies $\partial_{z_2} = \partial_{z_1}$. Indeed, if $\partial_{z_2} > \partial_{z_1}$ holds then $|y| \leq |z_2| = |z_1 z_2|$ thus $|y^p| \geq |y| + |z|$ and if $\partial_{z_2} < \partial_{z_1}$ then $|z_1| = |z_2 z_1|$ and thus $x = z_1 = z_2 z_1 = z$.

If $m = 2$ there exists a unique factorization $z_2 = z_2' z_2''$ such that $x = z_1 z_2' = z_2'' z_1$. This last equation implies $z_1 = (uv)^r u$, $z_2' = vu$, and $z_2'' = uv$, for some $u, v \in \Sigma^{\mathrm{Ord}}$, $r \geq 0$ hence

$$x = (uv)^{r+1} u, y^p = vu(uv)^{r+2} uuv \text{ and } z = (uv)^{r+1} uuv \tag{11}$$

Case 1: $m = 2, p \geq 3, q = 3$

Let us handle the case $r = 0$, i.e., $x = uvu, y^p = vuuvuvuuv = (vuuvu)vuuv$. A simple computation shows that $|y^p| > |vuuvu| + |y|$, thus y and $vuuvu$ are quasi-cyclic but also y^p and $vuuvu$ therefore so are $vuuv$ and $vuuvu$ by Proposition 18. Then we get $vuuvu\ vuuv\ vuuvu = vuuv\ vuuvu\ vuuvu$. This yields $uvuuv\ vuuvu = vuuvu\ vuuvu$ showing via Proposition 20 that u and v are quasi-cyclic, completing this particular case.

Now $r > 0$. There exists a conjugate y' of y for which $y'^p = (uv)^{r+2}uuvvu$ holds. By considering the coefficients of the highest degree in the expressions of the lengths of u, v, y, a tedious computation shows that $|y'| + |uv| \leq |(uv)^{r+2}u|$ and thus y' and uv are quasi-cyclic. Now y and vu are powers of two conjugate and by (11) comparable strings so they are also quasi-cyclic. Furthermore, by Proposition 18, vu and in $(uv)^{r+2}uuv$ are quasi-cyclic, thus uv and vu are comparable, hence u and v are themselves quasi-cyclic and we are done.

Case 2: $p = m = 2, q = 3$

Subcase 2.1) $r + 1 = 2r' + 1$ for some $r' \geq 0$. Observe first that $\partial_u \geq \partial_v$ holds else we have $vu(uv)^{r'} = u(vu)^{r'}uv$ meaning that u and v are quasi-cyclic. By standard considerations on the length, equation (11) implies

$$y = vu(uv)^{r'}u_1 = u_2(vu)^{r'}uv \tag{12}$$

with $u_1u_2 = u$. If $\partial_u > \partial_v$ then by considering the lengths of both handsides we have $|u_1| = |u_2v|$. Furthermore, the two handsides have the same prefix of length $|u_1| = |vu_1| = |u_2v|$ thus $vu_1 = u_2v$ implying $u_1 = vu_1$.

We are left with $\partial_u = \partial_v$. By considering the lengths of both handsides we have $|vu_1| = |u_2v|$ and then by considering both prefixes we get $vu_1 = u_2v$. By cancelling out this common prefix, we observe that u_1u_2 and u_2u_1 are comparable, i.e., for some k, i, j, we get $u_1 = (a^\omega b)^r a^i$ and $u_2 = (a^\omega b)^r a^j$ by Proposition 20. Then equality $vu_1 = u_2v$ yields $v = (a^\omega b)^s a^i$ for some $s \geq 0$ and thus $uv = (a^\omega b)^{r+s}a^i = (a^\omega ba^i)^{r+s}$. Finally, $y^2, z \in (a^\omega ba^i)^*$ which closes this subcase.

Subcase 2.2) $r + 1 = 2r'$ for some $r' > 0$. Observe first that $\partial_u \leq \partial_v$ holds else we have $vu(uv)^{r'+1} = (uv)^{r'}uuv$ meaning that u and v are quasi-cyclic.

Using considerations on the lengths we have

$$y = vuu(vu)^{r'}v_1 = v_2u(vu)^{r'}uv \tag{13}$$

By considering the lengths of both handsides, we have $|v_1| = |v_2|$. Since v_1 and v_2 are prefixes of y we get $v_1 = v_2$. Thus equation (13) is an equation in the two unknowns, and the verification follows from Proposition 4.

Case 3: $m \geq 3, p = 2, q = 3$

There exists a unique factorization $z_1 = z_1'z_1''$ such that $y = z_2z_1' = z_1''z_2$. This last equation implies $z_2 = (uv)^r u$, $z_1' = vu$, and $z_1'' = uv$, for some $u, v \in \Sigma^{\mathrm{Ord}}$, $r \geq 0$ hence

$$x^m = vu(uv)^{r+2}uuv, y = (uv)^{r+1}u \text{ and } z = vu(uv)^{r+1}u \tag{14}$$

The treatment of this case is a paraphrase as that of case 1.

There remains the case where $q = 2$.

Case 4: $p, m \geq 2, q = 2$

Because of Proposition 8 and Proposition 9 we either have $z = x^{m-1}x_1 = x_2y^p$ with $x = x_1x_2$ or $z = x^m y_1 = y_2y^{p-1}$ with $y = y_1y_2$. In the first case we have by concatenating both handsides by x_2 to the left: $(x_2x_1)^m = x_2^2y^p$. Because of the minimality of z, x_2 and x_2x_1 are quasi-cyclic, thus so are x_2 and x_1 by Proposition 18 and we may conclude by Proposition 21.

Similarly, the second case leads to the equation $x^m y_1^2 = (y_2y_1)^p$, i.e., y_2 and y_2y_1 are quasi-cyclic by induction, thus so are y_2 and y_1 by Proposition 18 and we may conclude by Proposition 21.

■

7 Conclusion

Investigating transfinite strings led us to new notions that are interesting modifications of standard ones. For example, the notion of generalized cyclicity of two transfinite strings corresponds to that of cyclicity of two finite strings. With these modifications, quite a few results on equations in finite strings extend to transfinite strings. In fact most of the equations that can be "solved" in finite strings can also be solved in transfinite strings.

There remains of lot to be undertaken: solving equations with constants, giving a more satisfying notion of periodicity, examining the notion(s) of rank of a set of transfinite strings etc.... Also the equations $x^m y^p = z^q$ can be considered for arbitrary ordinals $m, p, q \geq 2$ as suggested by one of the referee.

A last example among others. Consider a finite equation (the two handsides are finite) and assume we know that all of its solutions in finite strings are cyclic. Is it true that all the solutions in transfinite strings of equal degree are quasi-cyclic? If this were true the proof of Theorem 3 would reduce to the first few lines of Proposition 21!. Observe that equations in two unknowns enjoy this property.

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