A short introduction to automatic group theory

Christian Choffrut
L.I.A.F.A, Université Paris VII
email: cc@liafa.jussieu.fr

Abstract

This is a brief presentation of the rich topic introduced in the late eighties of discrete groups for which the product can be performed by finite automata. We focus on the simplest family of such groups, the so-called automatic groups and mention briefly the further extensions. We report at the end a few still open issues.

1 Introduction

This contribution is part of an introductory course on automatic groups. It claims no originality or novelty in the treatment of an already well-established theory. In fact it is widely inspired by the historic papers listed in the references, [2, 6]. A good introduction and tutorial concerning a more comprehensive approach of language theory as a tool for group theoreticians can be found in [8]. Our purpose is to leave a faithful image of the Thematic Term held in Coimbra in spring 2001 during which the lectures were given and to hopefully raise the interest of some theoretical computer scientists for the topic. It is indeed a priviledged field where mathematicians and computer scientists may collaborate fruitfully.

The basic idea behind automatic groups is to extend the notion of multiplication table from finite to discrete infinite groups. In order to actually manipulate elements of an infinite group, an encoding is needed. This is achieved by ways of normal forms which are specific strings over a suitable set of symbols, the "generators", and whose set is recognized by a finite state machine. Then a computing device is needed to effectively calculate the product of two elements. This second task is done by again using a finite state machine provided with two tapes, one for the input element and the second for the result of the multiplication of that element by some generator.

The main properties of automatic group are: 1) the notion is intrinsic and does not depend on a particular set of generators 2) the family of automatic groups is closed under several operations such as finite direct product, free products, subgroups of finite index 3) the word problem is solvable in quadratic time and 4) many natural groups are automatic and the notion seems to be relevant in the attempt of classifying 3-dimensional manifolds by their fundamental groups, see e.g., [3] for a background on the topic. However there exist such groups that are not automatic and not even asynchronously automatic (the non-deterministic extension). In [3, Theorem B] it is shown that by allowing the language of normal forms to be generated by index-grammars (much more powerful devices than finite automata), one is able to capture the subclass of fundamental groups of "compact geometrizable 3-manifolds".

This presentation focuses on synchronous automatic groups and is self-contained except for the mild prerequisite of a certain familiarity with finite automata theory. After all, finite automata theory is taught at an undergraduate level in the computer science curriculum all over the world and dealt with in widely used textbooks (e.g., [1, 12]). It has also become well-known to non-scientits with the use languages like Perl for example. The background on automata theory necessary for asynchronous automatic groups which is the natural continuation of automatic groups is a bit more advanced, so in spite of its great importance, I only mention this extension in the last section.

Several papers apply the notion of automatic structures to semigroups and monoids. We refer to the contribution of Hoffman and al., [10], in these proceedings for more detail and a wider account on the literature concerning this aspect. Interestingly, the so-called geometrical characterization of automatic structures in terms of Cayley graphs can be extended but it seems to be relevant essentially in the case of cancellative monoids, [16].

2 Finite automata and relations in free monoids

Given a finite set Σ, called an *alphabet*, whose elements are *letters*, the free monoid it generated is denoted by Σ^*. Its elements are *words* or *strings*, and its identity is the *empty word* denoted by ϵ. The product of two strings u and v is simply written uv where the operation symbol is understood. The *length* of a string $w \in \Sigma^*$ is denoted by $|w|$. A subset of Σ^* is also called a *language* and a subset of $\Sigma^* \times \Sigma^*$ a *relation.* We recall that for any two subsets $X, Y \subseteq \Sigma^*$, their *concatenation product* or simply *product* is the set $\{xy \mid x \in X, y \in Y\}$.

We recall that a finite automaton is a quadruple $\mathcal{A} = (Q, q_-, Q_+, E)$ where Q is the set of states, $q_- \in Q$ the unique *initial* state, $Q_+ \subseteq Q$ the subset of *final* states and $E \subseteq Q \times \Sigma \times Q$ the set of *transitions.* A sequence of the form

$$(q_0, a_1, q_1), (q_1, a_2, q_2), \ldots, (q_{n-1}, a_n, q_n) \in E$$

is called a *path* taking state q_0 to state q_n and its *label* is $a_1 a_2 \ldots a_n$. If, in the above sequence, we assume further that $q_0 = q_-$ and $q_n \in Q_+$, then we say that the path is *successful* and that the string $a_1 \dots a_n$ is *recognized* by the automaton $\mathcal{A}$. It will not matter, unless otherwise stated, whether the finite automata are deterministic or not. However, we will always assume that all states q are *accessible* from some initial state and *coaccessible* to some final state, in other words that they lie on a successful path. It is also well-known that it is no loss of generality to assume that the automaton is *normalized* in the sense that the subset Q_+ possesses exactly one state.

A language in Σ^* is *regular* (we also say *recognizable*) if it the subset of strings which are recognized by some finite automaton.

Theorem 1 *The family of recognizable languages is closed under the Boolean operations, the concatenation product, the image under direct and inverse morphism.*

The closure under product is a direct consequence of Kleene Theorem asserting recognizable languages are exactly the *rational* languages which are those that have a well-formed expression using the symbols of the alphabet and the operations of set union, product and Kleene star (which consists of taking the submonoid generated). Closure under direct and inverse morphisms can be easily established by structural induction of rational languages.

The shortest way to reduce a relation over the free monoid to a subset of this free monoid is to pad the shortest component of each pair of words with the minimal number of occurrences of a dummy symbol # so that both components have the same length, e.g., starting with $(abaa, bb)$ or with $(cab, bbac)$ one gets $(abaa, bb\#\#)$ and $(cab\#, bbac)$ respectively. Formally, consider the *extended alphabet* $\Sigma^\# = \Sigma \cup \{\#\}$ where # is a new symbol. With every pair $(u, v) \in \Sigma^* \times \Sigma^*$ associate the pair

$$(u,v)^\# = \begin{cases} (u\#^n, v) & \text{if } n = |v| - |u| \geq 0 \\ (u, v\#^n) & \text{if } n = |u| - |v| \geq 0 \end{cases} \tag{1}$$

The pair $(z, t) \in \Sigma^*$ is a *synchronous prefix* of the pair (u, v) if the pair (z, t) is componentwise a prefix of the pair (u, v) and if furthermore $|z| = |t| \leq \min\{|u|, |v|\}$ or if $\min\{|z|, |t|\} = \min\{|u|, |v|\}$. E.g., the synchronous prefixes of $(abaa, bb)$ are (ϵ, ϵ), (a, b), (ab, bb), (aba, bb) and $(abaa, bb)$. Those of $(cab, bbac)$ are (ϵ, ϵ), (c, b), (ca, bb), (cab, bba) and $(cab, bbac)$. Now we extend the notation (1) to subsets $R \subseteq \Sigma^* \times \Sigma^*$ by setting

$$R^\# = \{(u,v)^\# \mid (u,v) \in R\}$$

By converting R to the relation $R^\#$ we can view it as a subset of the free monoid $(\Sigma^\# \times \Sigma^\#)^*$ and therefore it becomes eligible for automaton recognition. We say the relation $R \subseteq \Sigma^* \times \Sigma^*$ is *synchronous* if the relation $R^\#$ can be recognized by a finite automaton over the alphabet $\Sigma^\# \times \Sigma^\#$. The class of synchronous relations, satisfies many closure properties. Here are those needed for a good understanding of our exposition.

Theorem 2 *The family of synchronous relations is closed under the Boolean operations and the composition of relations. The projections over either component are regular.*

If X is regular then the relation $\{(u,u) \mid u \in L\}$ is synchronous.

Furthermore, if X and Y are regular languages then $X \times Y$ is a synchronous relation.

Sketch of the proof. Let R and S be two synchronous relations. Then $(R \cup S)^{\#} = R^{\#} \cup S^{\#}$ which shows the closure under union by Kleene theorem. Concerning the complement $\overline{R}$, define

$$\Sigma_0 = \{(a,b) \mid a,b \in \Sigma\},\ \Sigma_1 = \{(a,\#) \mid a \in \Sigma\} \text{ and } \Sigma_2 = \{(\#,a) \mid a \in \Sigma\}$$

Then we obtain

$$\overline{R}^{\#} = \overline{R^{\#}} \bigcap (\Sigma_0^* {\Sigma_1}^* \cup \Sigma_0^* {\Sigma_2}^*)$$

which again in virtue of Kleene theorem shows the closure under complementation.

Given two automata recognizing X and Y it is an easy exercise left to the reader to construct an automaton which recognizes $X \times Y$.

Consider a finite automaton recognizing the synchronous relation R. Its projection over the first (resp. second) component is recognized by the same automaton were the second (resp. first) component of the labels of each tranisition is ignored.

The statement on the "diagonal" of a regular subset is trivial.

Finally, we prove the closure of synchronous relations under composition. We need to make an excursion into ternary relations, i.e., subsets $T \subseteq (\Sigma^*)^3$. Resort to the same trick as for binary relations and complete, if necessary, each triple of strings with occurrences of the symbol $\#$ so that the three components have the same length. Then T is synchronous if $T^{\#}$ is a regular language over the alphabet $(\Sigma^{\#})^3$. This is now applied to R and S in the following way. Consider the ternary relations $R \times \Sigma^*$ and $\Sigma^* \times S$. As for the last assertion of the theorem, it is easy to verify that these two relations are synchronous. Now observe that the composition $R \circ S$ is the image by the projection π_{13} of $(\Sigma^*)^3$ over the first and third components of the following relation

$$R \circ S = \pi_{13}((R \times \Sigma^*) \cap (\Sigma^* \times S))$$

The closures under intersection and projections easily extend to ternary synchronous relations and this allows us to complete the proof.

■

3 Automatic groups

3.1 Basic definitions

A *rational structure* for a group G, is a pair (Σ, L) consisting of a finite alphabet Σ and a regular subset $L \subseteq \Sigma^*$ such that there exists a morphism φ of Σ^* onto G with $\varphi(L) = G$. It is therefore possible to view Σ as a set of generators of the group as a monoid and the set L as a set of representatives of the equivalence classes defined by the congruence φ^{-1} though it is not a priori required that there exist only one representative per class. Furthermore, it is also assumed, unless otherwise stated, that for all symbols in $a \in \Sigma$ there exists a symbol $b \in \Sigma$ which maps onto the inverse of the image of a, in other words there exists a involution ι of Σ so that

$$\varphi(\iota(a)) = \varphi(a)^{-1} \tag{2}$$

holds for all $a \in \Sigma$. It is also convenient to represent $\varphi(u)$ for all $u \in \Sigma^*$ as $\overline{u}$ and by abuse of notation to write a^{-1} for $\iota(a)$ (but a^{-1} belongs to Σ).

Given a rational structure (Σ, L) for all $a \in \Sigma$ we set

$$R_a = \{(u, v) \in \Sigma^* \times \Sigma^* \mid \overline{u} = \overline{va}, u, v \in L\}$$

and furthermore

$$R_\epsilon = \{(u, v) \in \Sigma^* \times \Sigma^* \mid \overline{u} = \overline{v}, u, v \in L\}$$

The structure (Σ, L), is *automatic* if the relations R_a, for all $a \in \Sigma \cup \{\epsilon\}$ are synchronous. We call L the *language of representatives*, R_ϵ the *equality relation* and the different R_a with $a \in \Sigma$, the *multiplier relations*. More generally, we may define multiplier relations R_w for arbitrary strings $w = a_1 a_2 \ldots a_p \in \Sigma^*$ consisting of all the pairs of strings $(u, v) \in L \times L$ such that $\overline{u} = \overline{vw}$ holds. Such a relation is equal to the composition of relations $R_{a_1} \circ R_{a_2} \ldots \circ R_{a_p}$. In [6], instead of languages and relations composing an automatic structure it is spoken of automata defining them but this leads to unncessary technicalities.

Convention: the morphism φ is usually understood. Observe that whenever the language L is specified, the equality and the multiplier relations are completely determined. Therefore an automatic structure is unambiguously specified by the pair (Σ, L). As a consequence, each time we deal with automatic structures, we shall specify a pair (Σ, L) and verify that it is indeed automatic.

3.2 Examples

Many natural groups are automatic such as the Braid groups, [6, Theorem 9.3.1] and the Coxeter groups, [4, 9]. We content ourselves with the simple example of free groups. In order to keep notation simple, consider the two-generator free group $F(a, b)$. Put $\Sigma = \{a, b, \alpha, \beta\}$ where a and b map onto the generators of F and α and β onto their inverses. The language L of representatives is composed of all the reduced strings in Σ^*, i.e., having no factor of the form $a\alpha$, αa, $b\beta$ or βb and is therefore regular. Furthermore we have

$$R_\epsilon = \{(u, u) \in \Sigma^* \times \Sigma^* \mid u \in L\}$$

and

$$R_a = \{(ua, u) \in \Sigma^* \times \Sigma^* \mid ua \in L\} \cup \{(u, u\alpha) \in \Sigma^* \times \Sigma^* \mid u\alpha \in L\}$$

The relations R_b, R_α and R_β are defined similarly and are clearly synchronous as is R_a as well.

Observe that the language of representatives has two nice properties which will be studied more extensively in section 5. First there exists exactly one representative per class and second the set is prefix-closed because whenever a string is reduced, so are all of its prefixes.

Examples would not be satisfactory if they were not accompanied by counter-examples. In [6, p. 154], the first example of non-automatic group is referred to Thurston. It consists of the so-called Baumslag-Solitar groups whose presentation is

$$< \{a, b\}; ab^p a^{-1} b^{-q} >$$

with $p \neq q$ (for $p = q$, the groups are automatic). The technique for proving that this is indeed a counter-example uses the notion of isoperimetric inequality, see Theorem 7.

4 Geometric properties

4.1 Cayley Graphs

Let G be a group generated, as a group, by Σ. Its Cayley graph $\Gamma(G)$ is the graph whose vertices are the elements of G and whose labelled edges are the triples (x, a, xa) where $x \in G$ and $a \in \Sigma \cup \Sigma^{-1}$. Given $x, y \in G$, their distance, denoted by $d(x, y)$, is the minimum length, in the number of edges, of a path from x to y in $\Gamma(G)$ (this clearly defines a distance). Equivalently, it is the minimum length of a path from the identity $1 \in G$ to the element xy^{-1} (or to the element yx^{-1}). The following properties are more or less trivial (where $|x|$ denotes the length of the group element as the length of the minimum number of generators in Σ for representing it)

$$d(x, y) = |y^{-1}x| = |x^{-1}y| \tag{3}$$

$$d(zx, zy) = d(x, y) = d(1, y^{-1}x) = d(1, x^{-1}y) \tag{4}$$

In case of possible confusion, we shall write $d_G(x, y)$ to emphasize that the distance is relative to the group G (it will always be clear how the group is generated).

Cayley graphs allow us to define automatic groups in geometric terms via the crucial notion of "fellow traveller property". This can be explained as follows. Consider two cyclists riding in the Cayley graph of a group at the same speed and starting at the origin which is the vertex representing the identity of the group. Both will eventually stop. The property captures the idea that at any time of the ride, the distance which separate the two cyclists measured by the minimum number of edges is bounded by some integer. Formally, we consider a pair $(u, v) \in \Sigma^* \times \Sigma^*$. For each integer $0 < i \leq |u|$ we denote by u_i the prefix of u of length i and we set $u_i = u$ whenever $i \geq |u|$ and likewise for v. Now we say that (u, v) has the k-*fellow traveller property* if $d(\overline{u_i}, \overline{v_i}) \leq k$ for all integers $i > 0$. This property is the crux for characterizing the automatic structures among the rational structures. More generally, a subset $R \subseteq \Sigma^* \times \Sigma^*$ has the k-fellow traveller property if all its elements (u, v) have it.

Theorem 3 (Characterization of automatic structures) *Given a rational structure (Σ, L) for G, it is automatic if and only if there exists an integer $k > 0$ such that for all $\Sigma \cup \{\epsilon\}$, all pairs $(u, v) \in R_a$ have the k-fellow traveller property.*

Proof. We first verify that the condition is necessary. Let K_a be the cardinality of an automaton recognizing R_a, which we assume normalized, and consider a path of $(u, v) \in R_a$ taking the initial state q_- to a final state q_+. Let (u_1, v_1) be a synchronous prefix of (u, v) and assume it takes q_- to some state, say q. There exists a path of length less than or equal to $K_a - 1$, labelled by (u_2, v_2) taking q to q_+ which implies $\overline{u_1 u_2} = \overline{v_1 v_2 a}$. Because of properties (3) and (4) we have

$$d(\overline{u_1}, \overline{v_1}) = d(1, \overline{v_1}^{-1}\overline{u_1}) = d(1, \overline{v_2 a}\ \overline{u_2}^{-1}) \leq |v_2 a u_2^{-1}| \leq 2K_a - 1$$

It suffices to set $k = \max\{2K_a \mid a \in \Sigma \cup \epsilon\}$.

Now we prove that the condition is sufficient. Let B be the sphere of $\Gamma(G)$ centered at the origin 1 and of radius k and consider an automaton $\mathcal{A} = (Q, q_-, q_+, E)$ recognizing L. We assume without loss of generality that there is no outgoing transition from the state q_+. The idea is that the B controls the distance between the two components. For all $a \in \Sigma \cup \{\epsilon\}$ we define the automaton $(Q \times Q \times B, (q_-, q_-, 1), (q_+, q_+, \overline{a}), E_a)$ where the set of transitions E_a is obtained as follows: given two transitions $(q, b, q'), (p, c, p') \in E$ and an element $x \in B$, the quadruple $((q, p, x), b, c, (q', p', \overline{b}^{-1} x\overline{c}))$ belongs to E_a whenever $x, \overline{b}^{-1} x\overline{c} \in B$. We also add the transitions $((q, p, x), b, \#, (q', p, \overline{b}^{-1} x))$ and $((q, p, x), \#, c, (q, p', x\overline{c}))$ for $b, c \in \Sigma$. The conclusion is met by verifying by induction on the length of $\max\{|u|, |v|\}$ that the pair (u, v) labels a path starting in $(q_-, q_-, 1)$ and ending in (q, p, x) for some states $q, p \in Q$ if and only if $\overline{u}^{-1}\overline{v} = x$.

∎

Corollary 1 *Given an automatic structure for a group and a string $w \in \Sigma^*$ the multiplier relation R_w has the fellow traveller property for some integer K.*

Proof. Indeed, let k be the maximum over the relations R_a where $a \in \Sigma \cup \{\epsilon\}$ of the constants as guaranteed by Theorem 3. If the string w equals $a_1 a_2 \dots a_p$ for $a_i \in \Sigma$, $i = 1, \dots, p$, then the multiplier relation R_w is the composition of the multiplier relations $R_{a_1} \circ R_{a_2} \dots \circ R_{a_p}$ and has K-fellow traveller property where $K = kp$.

∎

4.2 Automaticity is intrinsic

This paragraph shows that the notion of automaticity is a property of the group itself, not of a specific generating system for the group.

Theorem 4 *Let G be a group and let Σ and Δ be two sets of (semigroup) generators. Then G is automatic relative to Σ if and only if it is automatic relative to Δ.*

Proof. We prove the result under the assumption that there exists an involution as in (2) for both generating sets. It makes the proof somewhat easier to read while not modifying it substantially. We assume G has an automatic structure of the form (Σ, L) and we will prove that it is automatic relative to the generating set Δ. We denote by R_a^Σ the multipliers associated with (Σ, L).

Consider first the case where Δ contains an element e which maps onto the identity of the group. Then we can directly define an automatic strucure (Δ, L'). Indeed, with every $a \in \Sigma$ associate a string $u_a \in \Delta$ which maps to the same element of G. Since Δ contains a generator which maps to the identity we may assume the strings u_a for $a \in \Sigma$ have the same length, say ℓ, after possible padding with sufficiently many occurrences of e. Consider the morphism $\theta : \Sigma^* \to \Delta^*$ defined by $\theta(a) = u_a$. Observe that the length of all strings in $\theta(L) = L'$ is a multiple of ℓ. In particular if $d_\Sigma(x, y)$ is the distance of $x, y \in G$ in the Cayley graph defined by the set Σ of generators, then $d_\Delta(x, y) \leq \ell d_\Sigma(x, y)$ where $d_\Delta(x, y)$ is the distance relative to the set of generators Δ. Now we claim that the pair (Δ, L') is an automatic structure for G.

By Theorem 1 the language L' is regular. There exists an integer λ such that for all $b \in \Delta$ there exists a string of Σ^* of length less than or equal to

λ which maps to the same element of G. Denote by K an upper bound on the constants associated with the fellow traveller property for all multiplier relations R^Σ_w with $|w| \leq \lambda$. For each $b \in \Delta$ denote by R^Δ_b its multiplier in the rational structure (Δ, L') and let $(\theta(u), \theta(v)) \in R^\Delta_b$. By definition of the morphism θ, for each synchronous prefix (z, t) of $(\theta(u), \theta(v))$ there exist two strings u' and v' in Σ^* of the same length and two strings $z_1, t_1 \in \Delta^*$ of length less than ℓ such that $\theta(u')z_1 = z$ and $\theta(v')t_1 = t$ holds. Now $(u, v) \in R^\Sigma_w$ where w is a string of Σ^* which has the same image as b in G and is of length less than or equal to λ. This yields

$$d_\Delta(\overline{z}, \overline{t}) = d_\Delta(\overline{\theta(u')z_1}, \overline{\theta(v')t_1}) \leq d_\Sigma(\overline{u'}, \overline{v'}) + |z_1| + |t_1| \leq K + 2\ell$$

It thus suffices to deal with the special case where Σ contains a (unique) element e which maps to the identity $1 \in G$ and where $\Delta = \Sigma - \{e\}$. It is not possible to simply suppress all occurrences of the letter e in the strings of L since this could violate the fellow traveller property (imagine there is no upper bound on the difference of occurrences of e in u and v). Let K be an upper bound of the constants associated with the multipliers R^Σ_a, $a \in \Sigma$ of the automatic structures (Σ, L) guaranteed by the fellow traveller property. Because of our simplification hypothesis there exist two letters $a, b \in \Sigma$ such that $\overline{ab} = 1$. Consider the mapping $\theta : \Sigma^* \to \Delta^*$ which suppresses every second occurrence of e say at odd positions and substitute one occurrence of ab for each occurrence of e at even positions and observe that inequality $0 \leq |v| - |\theta(v)| \leq 1$ holds for all prefixes of $u \in \Sigma^*$. It is easy to construct a synchronous automaton realizing the function. Setting $T = \{(u, \theta(u)) \in \Sigma^* \times \Delta^* \mid u \in \Sigma^*\}$, we see that the subset $L' = \pi_2(T \cap (L \times \Delta^*)$ is rational because of Theorem 2 and maps onto the group. We check that (Δ, L') is automatic. Indeed, consider a pair $(u', v') \in R'_a$ for some $a \in \Delta$ and one of its synchronous prefixes $(u'_1, v'_1) \in L' \times L'$. Let $(u, v) \in R_a$ satisfy $\theta(u) = u'$ and $\theta(v) = v'$. Then there exist a pair $u_1, v_1 \in \Sigma^*$ which are prefixes of u and v respectively and such that $\theta(u_1) = u'_1$ and $\theta(v_1) = v'_1$ holds. This yields

$$d(\overline{u'_1}, \overline{v'_1}) \leq d(\overline{u'_1}, \overline{u_1}) + d(\overline{u_1}, \overline{v_1}) + d(\overline{v_1}, \overline{v'_1}) = d(\overline{u_1}, \overline{v_1}) + 2$$

and shows that R'_a satisfies the fellow traveller property.

■

5 Simplifying automatic structures

The definition of automatic structures is purely existential. When working with a group that one knows to be automatic one needs to effectively find an automatic structure and hopefully an efficient one for that group. This aspect is probably the most important combinatorial problem that one may have to face and it is intimately connected with the rewrite theory in term algebras as introduced by Knuth and Bendix, [11]. However it is more complex in that it solves at the same time two different problems: defining rewrite rules that reduce words to a finite regular set of representatives and guaranteeing the fellow traveller property for representatives which are at distance one apart in the Cayley graph. We refer the reader to [6, Chapter 6] for techniques and heuristic resolving this problem. Here with deal with existential results only and show in which ways one may simplify automatic structures.

The notion of automatic structure as defined in section 3 does not require the subset L to be a cross-section. The main result of this paragraph is to show that it is always possible to assume that L is indeed a cross-section. The idea is to single out an element in each congruence class. This element is the short-lexicographically minimum element in the class. The second simplification consists of showing that it is always possible to assume the set of representatives is closed under taking prefixes. Whether both conditions can be achieved simultanously seems to be still unsettled.

We recall that given a linear ordering $<$ on the alphabet Σ, it is possible to extend it to a total well ordering on Σ^* by setting $(u, v) \in$ ShortLex if $|u| < |v|$ or if $|u| = |v|$ and there exist $z, u', v' \in \Sigma^*$ $a, b \in \Sigma$ and $a < b$ such that $u = zau'$ and $v = zbv'$. The relation ShortLex is synchronous since it is recognized by the following finite automaton pictured in Figure 1 where x and y stand for arbitrary symbols of the alphabet Σ.

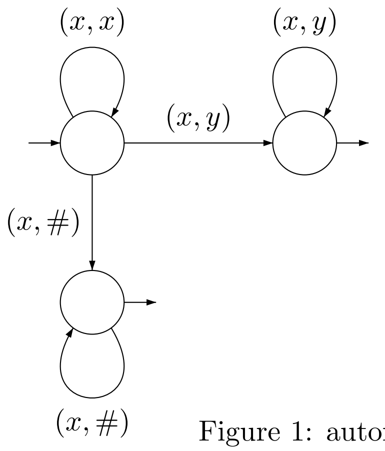


Figure 1: automaton for ShortLex

Theorem 5 (Unique representative propety) *If (Σ, L) is an automatic structure for a group G, then there exists a cross-section $L' \subseteq L$ such that (Σ, L') is also an automatic structure for G.*

Proof. The subset

$$L' = \Sigma - \pi_2(R_\epsilon \cap \text{ ShortLex })$$

is clearly a cross-section of the morphism $\varphi : \Sigma^* \longrightarrow G$ since it eliminates all elements in a congruence class which are not minimal. It is also synchronous because of Theorem 2. Let us now verify that the pair (Σ, L') defines an automatic structure for G. Is is clear that the relation $R'_\epsilon = \{(u, u) \in \Sigma^* \times \Sigma^* \mid u \in L'\}$ is synchronous. Concerning the relations R'_a with $a \in \Sigma$, we observe that

$$R'_a = R_a \cap (L' \times L') \tag{5}$$

holds which proves that they also are synchronous since synchronous relations are closed under the Boolean operations.

■

A second natural simplification consists of showing that there always exists a language of representatives which is prefix-closed for all automatic groups.

Theorem 6 (Prefix-closed representatives) *If* (Σ, L) *is an automatic structure for a group* G *then there exists an automatic structure* (Σ, L') *for* G *where the language* L' *is prefix closed.*

Proof. It suffices to define L' as the prefix closure of the language L and the relations R'_a as the intersections $R_a \cap L' \times L'$. Indeed, L' is clearly regular: take an automaton recognizing L and make all states final. Furthermore, consider now a pair $(u, v) \in R'_a$. If n is the number of states of the above automaton, then there exist z and t of length less than or equal to n such that $uz, vt \in L$. Let K be an upper bound of the constants of the fellow traveller property associated with the different relations R_w with $|w| \leq 2n + 1$ of the automatic structure (Σ, L). We have $(uz, vt) \in R_{z^{-1}aw}$. Then a synchronous prefix (u', v') of (u, v) is a synchronous prefix of (uz, vt) and therefore one has $d_G(\overline{u'}, \overline{v'}) \leq K$.

■

6 The word problem

A *group presentation* is a pair written $< \Sigma; R >$ composed of a finite alphabet Σ (the set of *generators*) and a subset R of the free group $F(\Sigma)$ generated by Σ (the set of *relators*). The group thus presented is the quotient of the free group $F(\Sigma)$ by the normal subgroup generated by R. The presentation is finite if Σ and R are finite.

Given a presentation $< \Sigma; R >$ the word problem consists of the following

Instance: $w \in F(\Sigma)$

Question: do there exist an integer $m \geq 0$, relators $r_1, r_2, \ldots, r_m \in R$ and elements $y_1, y_2, \ldots, y_m \in F(\Sigma)$ such that

$$w = (y_1^{-1} r_1 y_1)(y_2^{-1} r_2 y_2) \ldots (y_m^{-1} r_m y_m) \tag{6}$$

holds.

Novikov in 1953 and Boone in 1956 proved that the problem is undecidable in general, cf. for example [14, Chapter 12]. With every element $w \in F(\Sigma)$ mapping to the identity in the group presented, associate its *area* which is the minimum integer m as defined in (6). The *isoperimetric function* of a presentation is the function $\varphi : \mathbb{N} \to \mathbb{N}$ which assigns to every integer n the maximum area of the reduced words of length less than or equal to n that map onto the identity of the group presented. Actually, it can be shown that this function, up to an affine change of variable [7], depends on the group, not on a specific presentation. As a consequence it makes sense to say that the function is, say polynomial of degree three or that it is exponential.

We first start with a technical Lemma which asserts that given an arbitrary string $u \in \Sigma^*$, one can find an string in L which is congruent to it in quadratic time relative to the length of u. This is done by incrementally associating a representative to each prefix of a given string.

Lemma 1 *Given an automatic structure* (Σ, L) *there exists an integer* N *with the following property:*

1) Let $u \in L$ *and* $a \in \Sigma$*. Then there exists* $v \in L$ *satisfying* $\overline{ua} = \overline{v}$ *and* $|v| < |u| + N$*.*

2) Such a representative v *can be computed in linear time relative to* $|u|$*.*

Proof. Consider the relation R_a recognized by an automaton $\mathcal{A}$ and view it as a (not necessarily deterministic) automaton $\mathcal{B}$ by ignoring the second component. Let N be the maximum size of the automata for all R_b's where $b \in \Sigma$. A path labelled by u can be computed in time $O(|u|)$. Indeed, pose $u = a_1 a_2 \ldots a_n$. We paraphrase the famous subset construction for determinizing finite automata by defining the sequence of subsets of Q: $Q_0 = \{q_-\}$, $Q_1, \ldots, Q_n$ where for each $i = 1, \ldots, n$, Q_i is the set of states reached from some state $q \in Q_{i-1}$ by a transition labelled by the letter a_i in the automaton $\mathcal{B}$ (by a transition labelled by (a_i, b) for some $b \in \Sigma^{\#}$ in the automaton $\mathcal{A}$). Choose an arbitrary transition from a state of Q_{n-1} to a final state in Q_n and let (a_n, b) be its label in the automaton $\mathcal{A}$. If $b = \#$ then this means that there exists a path labelled by some $(u, v) \in \Sigma^* \times \Sigma^*$ with thus $|v| < |u|$ in the automaton $\mathcal{A}$. Otherwise, the path is labelled by a pair of the form (u, v') with $|u| = |v'|$. This path can be extended by a path of length less than the cardinality N so that the concatenated paths eventually reaches a final state. This completes the proof.

■

Theorem 7 *Let* G *be an automatic group. The the following holds.*

1) it has a finite presentation

2) it has quadratic isoperimetric function

3) it has word problem solvable in quadratic time

Proof.1) We start with an automatic structure $(\Sigma \cup \Sigma^{-1}, L)$ with unique representative for G and prove that the group has a finite presentation as a monoid which will imply that it has a finite presentation as a group. For all strings $u \in (\Sigma \cup \Sigma^{-1})^*$ we write $u \sim v$ whenever two strings have the same image in the group.

Consider a word $w = a_1a_2 \dots a_n$ which maps onto the identity of the group. We devise a double factorization of w. Denoting by $u_i \in L$ the representative of the class of the prefix w_i of length i of w for $i = 0, \dots, n$ the first factorization is

$$w \sim (u_0a_1u_1^{-1})(u_1a_2u_2^{-1}) \dots (u_{n-1}a_nu_n^{-1}) \tag{7}$$

Now we shall verify that each element $u_ia_{i+1}u_{i+1}^{-1}$ in turn can be factored into strings of bounded length.

Assume without loss of generality that the length p of u_i is less than or equal to the length m of u_{i+1}. Set $u_i = b_1b_2 \dots b_m$ with $b_t \in \Sigma \cup \Sigma^{-1}$ if $0 \leq t \leq p$ and $b_t = \epsilon$ if $p < t \leq m$ and $u_{i+1} = c_1c_2 \dots c_m$, where $c_s \in \Sigma \cup \Sigma^{-1}$ for all $s = 1, \dots, m$. For all $0 \leq t \leq p$ set $u_i(t) = b_1b_2 \dots b_t$ and denote by $u_{i+1}(t)$ the prefix of u_{i+1} of length t. Because of $(u_i, u_{i+1}) \in R_{a_{i+1}}$, for all $0 \leq t \leq m$ we have $d(u_i(t), u_{i+1}(t)) \leq k$ where k is the constant of the fellow traveller property. We thus have the equality $u_{i+1}(t) \sim u_i(t)v(t)$ for some string $v(t)$ of length bounded by k. This yields the following (which is actually an equality in the free group)

$$u_ia_{i+1}u_{i+1}^{-1} \sim u_i(1)v(1)u_{i+1}(1)^{-1} \prod_{1 \leq t \leq m} u_{i+1}(t)v(t)^{-1}b_{t+1}v(t+1)u_{i+1}(t+1)^{-1} \tag{8}$$

Because of $u_{i+1}(t+1) = u_{i+1}(t)c_{t+1}$ the t-th factor in the product is a conjugate by $u_{i+1}(t)$ of the element

$$v(t)^{-1}b_{t+1}v(t+1)c_{t+1}^{-1} \tag{9}$$

whose length is bounded by $2k + 2$. The set of relators of the presentation can thus be taken as a subset of strings of length less than or equal to $2k+2$.

2) The assertion concerning the isoperimetric inequality follows from the previous considerations. Indeed, by Lemma 1 the length of each u_i is linearly bounded by the length of w. Now each element in (7) consists of a linearly bounded number of relators as in (9). This proves the assertion.

3) Given $w = a_1a_2 \dots a_n$, by Lemma 1 there exists $u_1u_2 \dots u_n \in L$ such that

$$\overline{u_0} = 1, \overline{u_1} = \overline{a_1}, \dots, \overline{u_i} = \overline{u_{i-1}a_i}, \dots, \overline{u_n} = \overline{u_{n-1}a_n} = \overline{w}$$

where $|u_i| \leq |u_{i-1}| + N$ for $i = 1, \ldots, n$. If u_0 is known and its length is ℓ, the total cost is bounded by

$$\sum_{i=1}^{n} \ell + iN = O(n^2) + O(n\ell)$$

It thus suffices to prove that ℓ is bounded by a constant. It is possible to identify an element of L, say $v = b_1 b_2 \ldots b_r$, in constant time by considering an arbitrary succesful path in the automaton recognizing L. Define the sequence $v_i \in L$ for $1 \leq i \leq r$ by setting $v_0 = v$ and more generally

$$\overline{v_1} = \overline{v_0}\overline{b_1}^{-1}, \overline{v_2} = \overline{v_1}\overline{b_2}^{-1}, \ldots, \overline{v_r} = \overline{v_{r-1}}\overline{b_r}^{-1}$$

Then we have $\overline{v_r} = 1$ as desired and w maps to the identity if and only if $(v_r, u_n) \in R_\epsilon$ holds.

■

7 Closure properties

The numerous closure properties enjoyed by the class of automatic groups prove the robustness of the concept. We deal here with two main properties, to wit the closure under direct product and under taking a subgroup of finite index. The closure under free product also holds but uses the same type of techniques (see [10] where is it proven for monoids).

Theorem 8 *If G_1 and G_2 automatic groups, then so is their direct product $G_1 \times G_2$.*

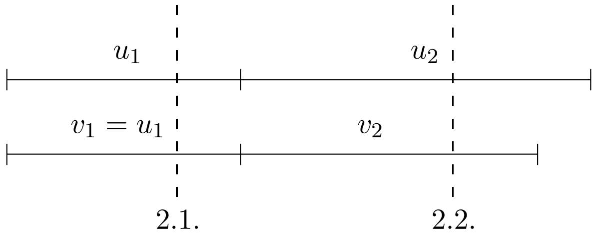
Proof. Let (Σ_i, L_i) be automatic structures for G_i with $i = 1, 2$. Without loss of generality we may assume that Σ_1 and Σ_2 are disjoint and by Theorem 5 that L_1 and L_2 have the unique representative property. Put $\Sigma = \Sigma_1 \cup \Sigma_2$ and $L = L_1 L_2$ and consider the morphism from Σ^* to $G_1 \times G_2$ which extends the natural morphisms from Σ_1^* and Σ_2^* to G_1 and G_2 respectively. In particular

$$u_1 u_2 \in L_1 L_2 \longrightarrow (\overline{u_1}, \overline{u_2}) \in G_1 \times G_2$$

Then the pair (Σ, L) is a rational structure because the family of rational subsets of a free monoids is closed under product. It now suffices to verify that it satisfies the Theorem 3. To that purpose, let k_1 and k_2 be two constants for the two structures (Σ_1, L_1) and (Σ_2, L_2) as claimed by Characterization Theorem. Because the two structures have the unique representative property, Lemma 1 asserts that there exists an integer N such that for all $a \in \Sigma_1 \cup \Sigma_2$ and for all pairs (u, v) belonging to the multiplier relation R_a of either structure, inequality $||u| - |v|| \leq N$ holds. Set $k = \max\{k_1, k_2\} + N$.

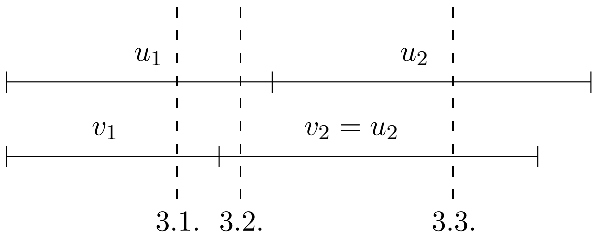
Case 1: $a = \epsilon$, then $R_\epsilon = \{(u, u) \in \Sigma^* \times \Sigma^* \mid u \in L\}$ is clearly synchronous.

Case 2: $a \in \Sigma_2$.



For all $(u_1u_2, v_1v_2) \in R_a$ we have $\overline{u_1u_2} = \overline{v_1v_2a}$. Since the group G is a direct product, this implies $\overline{u_1} = \overline{v_1}$ and $\overline{u_2} = \overline{v_2a}$ and because of the unique representative property for (Σ_1, L_1) we get $u_1 = v_1$. If (u_3, v_3) is a synchronized prefix of (u_1, v_1), we have $u_3 = v_3$ and therefore $d_{G_1 \times G_2}(\overline{u_3}, \overline{v_3}) = 0$. Now let (u_1u_3, v_1v_3) be a synchronized prefix of (u_1u_2, v_1v_2). We have $d_{G_1 \times G_2}((\overline{u_1}, \overline{u_3}), (\overline{v_1}, \overline{v_3})) = d_{G_2}(\overline{u_3}, \overline{v_3}) \leq k_2 \leq k$.

Case 3: $a \in \Sigma_1$. This is in turn divided into 3 subcases. We assume without loss of generality that $|u_1| \geq |v_1|$.



Case 3.1.: (u_3, v_3) is a synchronized prefix of (u_1, v_1). Then we obtain

$$d_{G_1 \times G_2}((\overline{u_3}, 1), (\overline{v_3}, 1)) = d_{G_1}(\overline{u_3}, \overline{v_3}) \leq k_1 \leq k$$

Case 3.2.: (u_3, v_1v_3) is a synchronized prefix of (u_1, v_1v_2). Then equality $\overline{v_1v_3} = (\overline{v_1}, \overline{v_3})$ implies

$$d_{G_1 \times G_2}(\overline{u_3}, \overline{v_1v_3}) = d_{G_1}(\overline{u_3}, \overline{v_1}) + d_{G_2}(1, \overline{v_3}) = d_{G_1}(\overline{u_3}, \overline{v_1}) + |v_3| \leq k_1 + N \leq k$$

since inequality $|v_3| \leq N$ is guaranteed by Lemma 1 and the unique representative property.

Case 3.3.: (u_1u_3, v_1v_3) is a synchronized prefix of (u_1u_2, v_1v_2). The string u_3 is a prefix of v_3 and $|v_3| - |u_3| = |u_1| - |v_1| \leq N$. Because of $(\overline{u_1u_3}, \overline{v_1v_3}) = (\overline{u_1}, \overline{u_3})(\overline{v_1}, \overline{v_3})$ we obtain

$$d_{G_1 \times G_2}(\overline{u_1u_3}, \overline{v_1v_3}) = d_{G_1}(\overline{u_1}, \overline{v_1}) + d_{G_2}(\overline{u_3}, \overline{v_3}) \leq k_1 + N \leq k$$

■

As an immediate corollary we have

Theorem 9 *Any finitely generated commutative group is automatic.*

We now turn to the second closure property.

Theorem 10 *Let H be a subgroup of finite index of a group G. Then H is automatic if and only if so is G.*

Proof. Set $G = \bigcup_{0 \le i \le n} Hx_i$ where the x_i's are representatives of the cosets of H in G and where $X = \{x_0, \ldots, x_n\}$ and x_0 is the identity 1 of the group.

Assume first G has an automatic structure (Σ, L) wth unique representative property. Associate with each pair $(x, a) \in X \times \Sigma$ the unique pair $(w, x') \in \Sigma^* \times X$ such that $\overline{xa} = \overline{wx'}$ holds, defining thus an action

$$X \times \Sigma \longrightarrow \Sigma^* \times X \tag{10}$$

We claim that the w's generate the subgroup H. Indeed, for all $z = a_0a_1 \ldots a_m$ we have $\overline{a_0a_1 \ldots a_m} = \overline{x_0a_0a_1 \ldots a_m}$ and more generally

$$\begin{array}{l}\overline{x_0a_0a_1 \ldots a_m} = \overline{w_0x_{i_1}a_1 \ldots a_m} = \overline{w_0w_1x_{i_2} \ldots a_m} = \\ \ldots = \overline{w_0w_1w_2 \ldots w_mx_{i_{m+1}}}\end{array} \tag{11}$$

Thus z belongs to H if and only if $x_{i_{m+1}} = x_0 = 1$ as claimed. Now we construct an automatic structure (Δ, L') for H where Δ is in one-to-one correspondence with the w's as found in (10). This yields an action

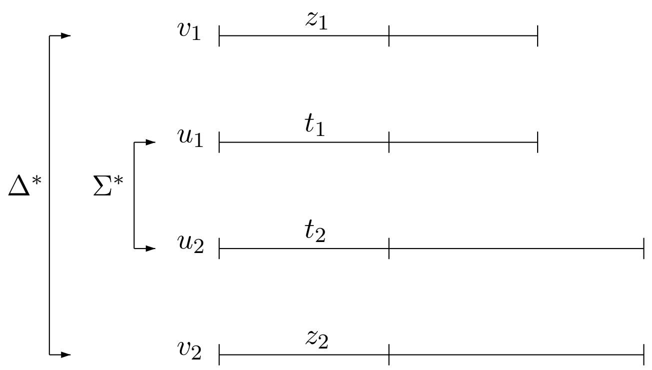
$$X \times \Sigma \longrightarrow \Delta \times X \tag{12}$$

which extends to $X \times \Sigma^*$ as in (11) and associates with every pair $(x, u) \in X \times \Sigma^*$ a pair $(v, x') \in \Delta^* \times X$ where u and v have the same length. We view (12) as defining the set of transitions of an automaton whose set of states is X and whose alphabet is $\Sigma \times \Delta$. Take x_0 as initial state and unique final state and define L' as the image of the relation recognized by the automaton by the projection onto its second component. This shows that L' is regular and because of the procedure (11) that $\overline{L'} = H$. Let us now verify that the pair (Δ, L') defines an automatic structure. Let $b \in \Delta$ and consider a pair $(v_1, v_2) \in \Delta^*$ in the multiplier relation R'_b of the structure (Δ, L'). The reader should follow the proof on the next Figure. By construction there exits a pair $(u_1, u_2) \in (\Sigma^*)^2$ such that u_1 and u_2 map onto v_1 and v_2 by (12). Consider a synchronous prefix (z_1, z_2) of (v_1, v_2). By the same action there exists a synchronous prefix (t_1, t_2) of (u_1, u_2) satisfying for some $y_1, y_2 \in X$ the equalities $\overline{x_0t_1} = \overline{z_1y_1}$ and $\overline{x_0t_2} = \overline{z_2y_2}$. This yields

$$d_H(\overline{z_1}, \overline{z_2}) = d_H(\overline{x_0t_1}\,\overline{y_1}^{-1}, \overline{x_0t_2}\,\overline{y_2}^{-1}) \le \lambda d_G(\overline{t_1}, \overline{t_2}) + |y_1| + |y_2|$$

where λ is the maximum of the lengths of the w's obtained in (10). Now (u_1, u_2) belongs to the multiplier relation R_w of the structure (Σ, L) where

$w \in \Sigma^*$ is the element assicoated with to the symbol $b \in \Delta$ in the above one-to-one correspondence. Let k be the constant of the fellow traveller property for R_w and let K be an upper bound of the lengths of the elements of X. Then we obtain $d_H(\overline{z_1}, \overline{z_2}) \leq \lambda k + 2K$.



Conversely, assume (Δ, L') is an automatic structure for H. Let Σ be in one-to-one correspondence with X: $a_i \in \Sigma \longleftrightarrow x_i \in X$ and assume $\Sigma \cap \Delta = \emptyset$. Then $(\Sigma \cup \Delta, L'\Sigma)$ is an automatic structure for G. Indeed, $L'\Sigma$ is regular and the mapping $L'\Sigma \longrightarrow G$ is clearly onto. Let now $(ub, vc) \in R_a$ where $u, v \in \Delta^*$, $b, c \in \Sigma$ and $a \in \Sigma \cup \{\epsilon\}$. Then we have $\overline{ub} = \overline{vca}$, i.e., $(u, v) \in R_{ba^{-1}c^{-1}}$. There are only finitely many different values for $ba^{-1}c^{-1}$, implying thus by Corollary 1, that the pair ub and vc have the fellow traveller property. ■

8 Asynchronous automatic groups and beyond

A wider family of groups, known as asynchronous automatic, is defined in terms of finite 2-tape automata as introduced by Rabin and Scott, [13] and further developed by Elgot and Mezi [5]. They can be viewed as two-tape Turing machines with read-only heads. The program of the machine is a set of instructions consisting of moving one of the heads to the right and changing state depending on the current state and the contents of the cells scanned by the heads. There is a deterministic version where the computation is unique and a non-deterministic one where the machine may choose among a finite set of instructions at each step. The properties enjoyed by asynchronously automatic groups are similar to those for automatic groups: they are closed under finite direct and free products and under taking subgroups of finite index, they have a solvable word problem (whose time complexity has an exponential upper bound though), they have the unique representative property and the property of being asynchronously automatic does not depend on the generating set for the group. These results can be found in [6] where ad hoc methods based on automata constructions are used. I would strongly recommend the version in [2] where the proofs are much better worked out and polished and the tedious constructions on automata avoided as much as possible.

Formally, a two-tape automaton is a construct (Q, q_-, Q_+, E) where Q is a finite set of *states*, q_- is the *initial state*, $Q_+ \subseteq Q$ is the set of *final states* and

$$E \subseteq (Q \times \Sigma \times \{\epsilon\} \times Q) \cup (Q \times \{\epsilon\} \times \Sigma \times Q)$$

is the set of *transitions*. A *path* is a sequence of transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots, (q_{n-1}, u_n, v_n, q_n)$$

whose *label* is the pair $(u_1 u_2 \ldots u_n, v_1 v_2 \ldots v_n)$. It is *successful* if $q_0 = q_-$ and $q_n \in Q_+$. The relation *recognized* by the two-tape automaton is the set of labels of the successful computations. The automaton is *deterministic* whenever there exists a partition $Q = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = \emptyset$ such that the set of transitions may be written as

$$E \subseteq (Q_1 \times \Sigma \times \{\epsilon\} \times Q) \cup (Q_2 \times \{\epsilon\} \times \Sigma \times Q)$$

with the extra condition that $(q_1, a, \epsilon, q), (q_1, a, \epsilon, p) \in E$ implies $q = p$ (resp. $(q_2, \epsilon, a, q), (q_2, \epsilon, a, p) \in E$ implies $q = p$).

The relationship between asynchronously automatic and non-deterministic asynchronously automatic is settled in [15, Theorem 1, p. 301] where it is shown that the two concepts lead to the same class of groups. More precisely, if (Σ, L) is non-deterministic asynchronous automatic then here exists a subset $L' \subseteq L$ such that (Σ, L') is asynchronous automatic. The author argues that non-deterministism is more natural. Indeed, characterizing non-deterministic asynchronously automatic structures can be done very much in the same way as automatic structures. More precisely, consider a computation of a pair $(u, v) \in \Sigma^*$ of strings on a two-tape automata, i.e., a successful path labelled by (u, v). Each prefix of this path is labelled by some pair (u', v') where u' and v' are prefixes of u and v respectively. Similarly to synchronous automata, it is clear that the distance $d(\overline{u'}, \overline{v'})$ is bounded by some integer k (there is always a path of length less than the number of states which leads to a final state) but unfortunately there is no a priori relationship between the lengths of u' and v'. In other words, given the label (u, v) of a successful computation, one is unable to infer the pairs of strings labelling the prefixes of its computation because the two tapes move separately at different speeds. In the literature this is considered as a parametrization issue. For non-deterministic asynchronous automatic structures, this does not matter because it suffices to consider all possible pairs (u', v') and control their distance in the Cayley graph by verifying that $\overline{u'}^{-1}\overline{v'}$ belongs to the ball centered at the origine and with radius k. The deterministic case is more involved since it is not enough to prevent $\overline{u'}$ and $\overline{v'}$ to be too far apart. One must keep the computation deterministic. This is achieved in two steps. First, it is shown that given a (deterministic) asynchronous automatic structure (Σ, L) for the group G, in is possible to identitfy a subset $L' \subseteq L$ such that (Σ, L') is again an asynchronous automatic structure for G such that every equivalence class has finitely many elements only. It is shown that this guarantees an upper bound of the number of letters that can be read from one tape in a row. Secondly, a deterministic two-tape automaton is devised by using a ball of appropriate radius to keep the images of the prefixes close, but determinism is ensured by switching from one tape to the other "at the last moment" when one tape has exhausted the allowed time of computation.

9 Open problems

The following problems were already presented as open in [6]. We suggest the reader to visit the following site

http://zebra.sci.ccny.cuny.edu/web/problems/probhyp.html

for more references on hyperbolic and automatic groups.

The first question is posed in [6] where it is conjectured to be wrong but I don't have any further reference to the problem.

Representatives. Is it always possible to find an automatic structure for an automatic group such that the language L has the uniquen representative property and is prefix closed at the same time?

A biautomatic structure for a group G is an automatic structure (Σ, L) such that (Σ, L^{-1}) is also an automatic structure for G where

$$L^{-1} = \{a_1^{-1}a_2^{-1}\dots a_n^{-1} \mid a_n a_{n-1}\dots a_1 \in L\}$$

with the convention $\epsilon^{-1} = \epsilon$ for the empty word $(n = 0)$.

Biautomatic. Are automatic groups also biautomatic?

The previous question is interesting for its own sake as a possible closure property but if the answer were positive, then the following question could also be answered positively.

Conjugacy. Is is possible to decide, given an automatic structure (Σ, L) and two strings $u, v \in \Sigma^*$ whether or not there exists a string $w \in \Sigma^*$ such that $\overline{u} = \overline{w}^{-1}\overline{v}\overline{w}$ holds?

Direct product. If the direct product $G \times H$ is automatic, is it true that G is automatic?

Retract. Let G be automatic and assume there exists a subgroup $H < G$ and a morphism $\varphi : G \to H$ such that φ is the identity on H. Is it true that H is automatic?

The word problem for asynchronous groups is solvable in exponential time, however it is not ruled out that it could be of lower complexity.

Word problem. Can the word problem for asynchronous groups be solvable in polynomial time or is exponential time a lower bound?

References

[1] A. Aho and J. Ullman. *Foundations of Computer Science.* W. H. Freeman Company, 1992.

[2] G. Baumslag, S. M. Gersten, M. Shapiro, and H. Short. Automatic groups and amalgams. *J. Pure Appl. Algebra*, 76:229–316, 1991.

[3] M. R. Bridson and R. H. Gilman. Formal languages theory and the geometry of 3-manifolds. *Commentarii Math. Helv.*, 71:525–555, 1996.

[4] B. Brink and R. Howlett. A finiteness property and an automatic structures for Coxeter groups. *Math. Ann.*, 296:179–190, 1993.

[5] C. C. Elgot and J. E. Mezei. On Relations Defined by Finite Automata. *IBM Journal*, 10:47–68, 1965.

[6] D. Epstein, J. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word processing in groups.* Jones and Bartlett, 1992.

[7] S. M. Gersten. Isoperimetric and isodiametric functions. volume 182 of *LMS Lecture Notes*, pages 79–96. Camb. University Press, 1993.

[8] R. Gilman. Formal languages and infinite groups. *Geometric and Computer Perspectives on Infinite Groups*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci.:27–51, 1996.

[9] Susan M. Hermiller. Rewriting systems for Coxeter groups. *J. Pure Appl. Algebra*, 92:137–148, 1994.

[10] M. Hoffman, D. Kuske, and R. M. Thomas. Some relatives of automatic and hyperbolic groups. these proceedings.

[11] Daniel D. Knuth and P. B. Bendix. Simple word problem in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebras*, pages 263–297. Pergamon Press, 1970.

[12] H. R. Lewis and C. H. Papadimitriou. *Elements of the Theory of Computation.* Allyn and Bacon, 1978.

[13] M. O. Rabin and D. Scott. Finite automata and their decision problems. *I.B.M. J. Res. and Develop.*, 3:114–125, 1959.

[14] J. J. Rotman. *The Theory of Groups.* Prentice-Hall, 1981.

[15] M. Shapiro. Deterministic and non-deterministic asynchronous automatic structures. *J. Algebra Comput.*, 2:297–305, 1992.

[16] P. V. Silva and B. Steinberg. A geometric characterization of automatic monoids, 2000. preprint.