Suppose that for some $i < n$, $a_0 a_1 \cdots a_{2n} \notin U_i$ and $t_i = 0$, contradicting rule (3) of G'. This means that there is some U'_i such that

$$\tau'(a_0, a_1, \ldots, a_{2i+1}, U'_i) = (1, a_0 a_1 \cdots a_{2n}).$$

We can then change U_i for U'_i. This will change some further moves in G' as we follow the strategy τ', but none of the moves $(a_0, a_1, \ldots, a_{2n})$ by the definition of G'. Playing in this way, Player II will keep all his commitments because his strategy is a winning one and win G', thus winning G. □

One may note that the proof above is not effective in any sense, because of the uncountably many choices required by the simulation on the game G'. Consequently, the previous proof gives a general existence result on determinacy but it does not address the problem of the computational complexity of a solution.

4 Games on graphs.

We now consider games played on graphs, in which each player chooses in turn a vertex adjacent to the current vertex. The abstract games $G(X)$ considered until now can be considered as games on the Cayley graph of A^*. But actually, a game $G(X)$ such that X is an ω-rational set can also be considered as a game on a finite graph by playing the game on an automaton recognizing X. Thus the notion of a game played on a graph will give us more flexibility by allowing us to choose the more appropriate graph to play the game. As a counterpart, the properties of the game graph obtained by playing the game on an automaton will the depend on the automaton chosen to recognize the winning set.

Let us define formally a game on a graph. A graph $G = (V, E)$ on a countable set of vertices V is called an *arena* or *game graph* if

(1) it is bipartite, i.e. its vertex set V is partitionned into V_1 and V_2 and the edges connect vertices belonging to different subsets.

(2) there is at least one edge starting from every vertex (i.e. there are no dead ends).

If G is an arena, a game on G is given by a set X of winning paths. We shall always suppose that the set of winning paths does not depend on the starting vertex, i.e. that the set X is suffix-closed. All the particular winning sets considered later have this property.

We consider that Player I plays on vertices in V_1 and Player II on vertices in V_2. A play is thus an infinite path in G. Thus, if the first vertex is in V_1, then Player I plays first and otherwise, Player II plays first. Player I wins the play if it is a winning path, i.e. belongs to X. Otherwise, Player II wins the play.

Thus a game on a graph is essentially a particular case of the notion of game defined in Section 2, the alphabet being the set of vertices of the graph. The only difference is that the first player is not always Player I. This defines the notion of a strategy and of a winning strategy for each player as a function from the set of paths of even (or odd length) into the set V of vertices.

Example 4.1 Let G be the graph of Figure 4.1. We use simple circles for the positions of Player I and double ones for those of Player II. Thus $V_1 = \{1, 3\}$ and $V_2 = \{2\}$.

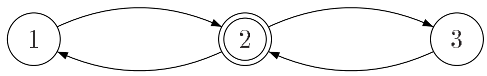

Figure 4.1. A game graph.

If X is the set of paths passing infinitely often by 1, Player II wins the game by always choosing vertex 3.

Let G be a game graph. A *memoryless strategy*, say for Player I, is a strategy which depends only on the last vertex of the path. When it is moreover a winning strategy, we will speak of a *memoryless winning strategy*. Actually, a memoryless strategy can be considered as a subgraph since it consists in selecting one edge for each vertex on which Player I makes a move.

Formally, we say that a pair (P, F) consisting of a set $P \subset V$ of vertices and a subset $F \subset E \cap (P \times P)$ is a *winning policy* for Player I if

(a) for each $q \in P \cap V_1$, there is exactly one edge in F starting at q.

(b) for each $q \in P \cap V_2$, all edges starting at q are in F.

(c) all paths in (P, F) are winning for Player I.

The corresponding notion for Player II is symmetrical. It is clear that each player has a memoryless winning strategy from vertex p if and only if he or she has a winning policy (P, F) such that $p \in P$ (we use here the hypothesis that the set of winning paths is suffix-closed).

We say that a player has a *winning policy on a set W* if he or she has a winning policy of the form (W, F).

Example 4.2 In the game of Example 4.1, Player II has a memoryless winning strategy from every vertex.

The following auxiliary result allows one to merge different memoryless winning strategies into one winning policy. It shows that there is a maximal set on which a player has a winning policy.

Proposition 4.1 *Each player has a winning policy on the set of all vertices from which he or she has a memoryless winning strategy.*

Proof. Let W be the set of all vertices from which Player I has a memoryless winning strategy. Thus, for each $p \in W$ we can choose a winning policy (P_p, F_p) for Player I such that $p \in P_p$ (note that this requires the axiom of choice). Since the set W, as a subset of V, is countable, we may index the set of these strategies by integers. For each vertex $p \in W \cap V_1$, we select the pair (P_x, F_x) such that $p \in P_x$ which has minimal index. This defines a unique edge going out of p. Let F be the set formed by all edges of this type and by those in $(W \cap V_2) \times W$. Then the pair (W, F) is a winning policy on W for Player I.

The same result is of course true for Player II. $\square$

For a vertex set U, we define the *attractor* of U for player I, denoted $A_1(G, U)$, or simply $A_1(U)$, as the set of vertices from which Player I can force a visit in U. The complement W of $A_1(U)$ is a set which is a *trap* for Player I: Player II can force Player I to remain inside W. This implies that for each vertex of W, there is at least one edge leading to a vertex in W. Thus the subgraph restricted to W is again a game graph, sometimes called a *subgame*.

The attractor of U for Player II, denoted by $A_2(U)$, is defined in the same way. In the same way, a trap for Player II is a set X of vertices such that Player I can force Player II to remain inside X. The complement of $A_2(U)$ is a trap for Player II.

4.1 Simple games.

It is interesting to come back with these new definitions to the simple games defined by open or Π_2-sets that we have treated before.

Let us first consider open games. Such a game can always be obtained as a game on a graph G in which the set of winning paths is formed by the paths which pass through a given set $F \subset V$ of vertices. Let us denote by (G, F) such a game.

Proposition 4.2 *In an open game (G, F), Player I has a memoryless winning strategy on the attractor $A_1(F)$ and Player II on the complement.*

Proof. The winning strategies of Players I and II can be computed using rank functions defined directly using definitions similar to those of Section 3.1. Indeed, the rank of a vertex q can be defined as the smallest integer i such that $q \in W_i$ where W_i is an increasing sequence of subsets of Q defined by $W_0 = F$ and inductively

$$
\begin{aligned}
W_{i+1} = W_i &\cup \{p \in Q_1 \mid q \in W_i \text{ for some } (p, q) \in E\} \\
&\cup \{p \in Q_2 \mid q \in W_i \text{ for every } (p, q) \in E\}.
\end{aligned}
$$

The attractor of F is $\cup_{k \geq 0} W_k$.

The strategy of Player I on this set consists in decreasing the rank. The strategy of Player II on the complement consists in keeping off the positions of $A_1(F)$. Both strategies are memoryless strategies. □

The case of Π_2-games corresponds to games on a graph G in which the winning paths are those which pass infinitely often through F. Let us consider here the case where the graph G is finite. For a set W of vertices, we denote by $R(W)$ the set of vertices from which Player I can force a visit to W after a path of length ≥ 1. The set $R(W)$ is close to the attractor $A_1(W)$ and can be computed in a similar way. We then consider the decreasing sequence of sets defined by $W_1 = F$ and

$$W_{i+1} = R(F \cap W_i)$$

Since the set Q of vertices is finite, the sequence W_i is stationnary. Let k be such that $W_k = W_{k+1}$. Then Player I has a memoryless winning strategy from the set $U = W_k$ consisting in reaching a vertex of $F \cap U$. Player II has a memoryless winning strategy on the other vertices. It consists in avoiding U.

Example 4.3 Let G be the graph represented in Figure 4.2 where $2, 4$ are positions of Player I and $1, 3, 5$ are positions of Player II. Player I wins if vertex 1 is visited infinitely often. Player I can force an infinity of visits of 1 from 1 and 2 but not from 3, 4 or 5. Thus Player I has a memoryless strategy on $\{1, 2\}$ and Player II on $\{3, 4, 5\}$.

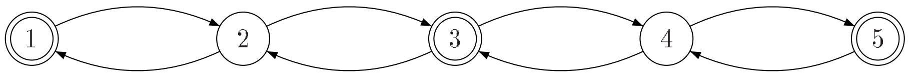

Figure 4.2. A game graph.

4.2 Winning conditions.

We shall consider games on graphs in which the winning set is defined through a finite set of colors in the following way. Let $G = (V, E)$ be a game graph and let $c : V \to Q$ be a map from the set of vertices into a finite set Q of colors. If x is an infinite path on G, we denote by $\mathrm{Inf}_c(x) = \mathrm{Inf}(c(x))$ the set of infinitely repeated colors in x.

We choose a particular collection $\mathcal{F}$ of subsets of Q and we define the set X of winning paths as those paths in G such that $\mathrm{Inf}_c(x)$ belongs to $\mathcal{F}$. In this section, we study particular classes $\mathcal{F}$ of sets of states used to define the winning paths. This game will be denoted by $(G, \mathcal{F})$.

We shall denote by $\mathcal{F}^c$ the complement of $\mathcal{F}$ in $\mathcal{P}(Q)$, that is, the set of subsets of Q which are not in $\mathcal{F}$. We define the *split tree* of $\mathcal{F}$ as follows. It is a tree T whose vertices are pairs $(1, X)$ for $X \in \mathcal{F}$ or $(2, X)$ for $X \notin \mathcal{F}$. The root of T is (σ, Q) with $\sigma = 1$ or 2 according to $Q \in \mathcal{F}$ or not. Inductively, if $x = (1, X)$ is a vertex of T, then

(1) if X contains subsets which do not belong to $\mathcal{F}$, then the children of x are all the $(2, Y)$'s where Y is a maximal subset of X which do not belong to $\mathcal{F}$.
(2) otherwise, x is a leaf of T.

A symmetrical condition holds if $x = (2, X)$

(1) if X contains subsets which belong to $\mathcal{F}$, then the children of x are all the $(1, Y)$ where Y is a maximal subset of X which is in $\mathcal{F}$.
(2) otherwise, x is a leaf of T.

Since the collection $\mathcal{F}$ is finite, the split tree of $\mathcal{F}$ has a finite heigth.

Example 4.4 Let G be the graph of Figure 4.1 with $\mathcal{F} = \{\{1, 2, 3\}\}$. The corresponding split tree is represented in Figure 4.3. If $\mathcal{F} = \{\{1, 2, 3\}, \{1\}\}$, the split tree is represented in Figure 4.4.

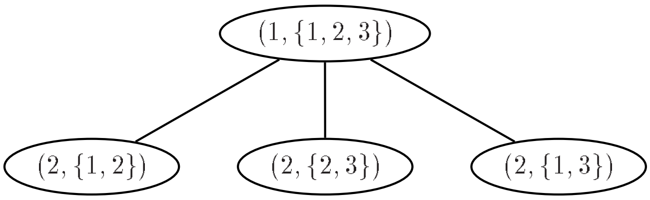

Figure 4.3. A split tree.

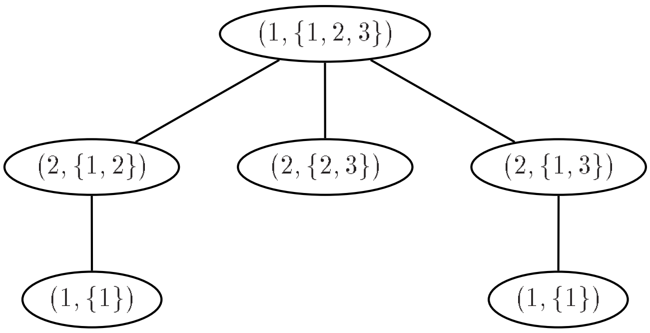

Figure 4.4. Another split tree.

The following proposition shows that the collection $\mathcal{F}$ can be computed from its split tree T.

Proposition 4.3 *A set X is in $\mathcal{F}$ if and only if there is a vertex $x = (1, Y)$ such that $X \subset Y$ and $X \not\subset Z$ for every child $(2, Z)$ of x.*

Proof. If X satisfies the condition, then X has to be in $\mathcal{F}$ since otherwise there would be a child $(2, Z)$ of x with $X \subset Z$. Thus $X \in \mathcal{F}$.

Conversely, let $x = (1, Y)$ be a node of T as low as possible such that $X \subset Y$. Such a node exists since the root satisfies this condition. No child $z = (2, Z)$ of x can satisfy $X \subset Z$ since otherwise z would have a child $(1, W)$ with $X \subset W$, a contradiction with the choice of x. Thus the property holds for x. □

Example 4.5 Let $F \subset Q$ and let $\mathcal{F} = \{X \subset Q \mid X \cap F \neq \emptyset\}$. The split tree of $\mathcal{F}$ has two vertices: the root $(1, Q)$ with one child $(2, Q \setminus F)$.

Let $\mathcal{P} = (L_i, R_i)_{i \in I}$ be a family of pairs of subsets of a set Q. A subset X of a set Q is said to satisfy *Streett condition* $S(\mathcal{P})$ if for every $i \in I$, $L_i \cap X \neq \emptyset$ or $R_i \cap X = \emptyset$. Thus a Streett condition is just the negation of a Rabin condition (see Chapter I). A collection $\mathcal{F}$ of subsets of Q is said to be expressible by a Streett condition if there is a set $\mathcal{P}$ of pairs such that $X \in \mathcal{F}$ if and only if X satisfies $S(\mathcal{P})$.

Proposition 4.4 *A collection $\mathcal{F}$ of nonempty subsets of a finite set Q is expressible by a Streett condition if and only if it is closed under union.*

Proof. If X and Y satisfy the Streett condition $S(\mathcal{P})$, then so does $X \cup Y$. Indeed, if $\mathcal{P} = (L_i, R_i)_{i \in I}$, then for every i, either one of X, Y meets L_i and so does $X \cup Y$, or none of them meets U_i and neither does $X \cup Y$.

Conversely, let $\mathcal{F}$ be a collection of nonempty subsets of Q closed under union and let T be the split tree of $\mathcal{F}^c$. Let I be the set of all subsets U of Q such that $(1, U)$ is a node of T (thus none of the U's are in $\mathcal{F}$). Each node $(1, U)$ can have at most one child since $\mathcal{F}$ is closed under union. Let V_U be the label of this child if there is one and let V_U be the empty set otherwise.

Then, by Proposition 4.3, $X \in \mathcal{F}^c$ if and only if, for some U, $X \subset U$ and $X \not\subset V_U$. Let $L_U = U^c$ and $R_U = V_U^c$. Then $X \in \mathcal{F}^c$ if and only if for some U, one has $X \cap R_U \neq \emptyset$ and $X \cap L_U = \emptyset$. Thus $X \in \mathcal{F}$ if and only if for each $U \in I$, one has $X \cap L_U \neq \emptyset$ or $X \cap R_U = \emptyset$. It follows that X is defined by the Streett condition $S(\mathcal{P})$ with $\mathcal{P} = (L_U, R_U)_{U \in I}$. □

Let $\mathcal{C}$ be an increasing sequence of subsets of Q

$$\mathcal{C}: \ E_1 \subset F_1 \subset E_2 \subset F_2 \subset \ldots \subset E_n \subset F_n$$

A subset P of Q is said to satisfy the *Rabin chain condition* $\mathcal{C}$ if there is an index k such that $P \cap E_k = \emptyset$ and $P \cap F_k \neq \emptyset$.

There is an alternative formulation of Rabin chain condition using a *parity condition*. Given a function $\mu : Q \to \mathbb{N}$, we say that X *satisfies the parity condition* μ if and only if

$$\min \{ \mu(q) \mid q \in X \} \text{ is odd}$$

This formulation of the chain condition makes it extremely easy to use since it entails a very compact representation.

We say that a collection $\mathcal{F}$ of subsets of Q is *expressible* by a Rabin chain condition, (resp. by a parity condition μ) if there exists a chain $\mathcal{C}$ such that $X \in \mathcal{F}$ if and only if X satisfies $\mathcal{C}$ (resp. μ).

Proposition 4.5 *Let $\mathcal{F}$ be a collection of finite nonempty subsets of a set Q. The following conditions are equivalent:*

(1) *$\mathcal{F}$ and $\mathcal{F}^c$ are closed under union.*

(2) *$\mathcal{F}$ can be defined by a Rabin chain condition.*

(3) *$\mathcal{F}$ can be defined by a parity condition.*

Proof. (1) implies (2). Let T be the split tree of the collection $\mathcal{F}$. Since $\mathcal{F}$ and $\mathcal{F}^c$ are closed under union, each vertex of T has at most one child. It follows that T has exactly one leaf. We may suppose that $Q \in \mathcal{F}^c$, so that the root of T is $(2, Q)$. Let $((2, V_0), (1, U_1), (2, V_1), \ldots)$ be the unique path from the root to the leaf. For $i \geq 0$, let $F_i = U_i^c$ and $E_i = V_i^c$. Then the sequence $\mathcal{C}$ formed by $E_1 \subset F_1 \subset \cdots$ is increasing and, by Proposition 4.3, one has $X \in \mathcal{F}$ if and only if X satisfies $\mathcal{C}$. Thus $\mathcal{F}$ can be defined by a Rabin chain condition.

(2) implies (3). Let

$$E_1 \subset F_1 \subset E_2 \subset F_2 \subset \ldots \subset E_n \subset F_n$$

be an increasing sequence of subsets of Q defining a Rabin chain condition. Let us define a function $\mu : Q \to \mathbb{N}$ by setting, for $1 \le k \le n$,

$$\mu(q) = \begin{cases} 0 & \text{if } q \in E_1 \\ 2k-2 & \text{if } q \in E_k \setminus F_{k-1} \text{ and } k > 1 \\ 2k-1 & \text{if } q \in F_k \setminus E_k \\ 2n & \text{if } q \notin F_n \end{cases}$$

Then a set X satisfies $\mathcal{C}$ if and only if $\mu(X)$ is odd. Thus X is defined by a parity condition.

(3) implies (1). If $\mathcal{F}$ is defined by a parity condition, then $\mathcal{F}$ and $\mathcal{F}^c$ are clearly closed under union. □

4.3 Parity games.

We now consider games on graphs, called *parity games*, in which the winning set is defined by a parity or Rabin chain condition defined by a chain $\mathcal{C}$

$$E_1 \subset F_1 \subset \ldots \subset E_n \subset F_n.$$

More precisely, let $G = (V, E)$ be a game graph and let $c : V \to Q$ be a coloring. The *parity game* defined by $\mathcal{C}$ is the game $(G, \mathcal{F})$, where $\mathcal{F}$ is the collection of subsets of Q defined by the Rabin chain condition $\mathcal{C}$. Therefore, the set X of winning paths consists of the paths x such that $\mathrm{Inf}_c(x) \cap E_k = \emptyset$ and $\mathrm{Inf}_c(x) \cap F_k \neq \emptyset$ for some $k \in \{1, \ldots, n\}$.

Observe that any play visiting infinitely often E_1-colored vertices is winning for Player II. Indeed, let x be a path in G and let $P = \mathrm{Inf}_c(x)$. If $P \cap E_1 \neq \emptyset$, any of the sets E_k is met infinitely often and thus there can be no index k satisfying the condition $P \cap E_k = \emptyset$.

Theorem 4.6 *In a parity game, one of the players has a memoryless winning strategy from each vertex.*

Proof. We shall prove by induction on the length of the chain $\mathcal{C}$ that there is a partition $Q = W_1 \cup W_2$ on the set of vertices such that Player I has a memoryless winning strategy on W_1 and Player II has one on W_2. We make the assumption that $E_1 \neq \emptyset$. Otherwise, we would exchange the roles of Players I and II in the forthcoming discussion. Thus, whenever we find a game with a chain of length n and $E_1 = \emptyset$, we can invoke the induction hypothesis.

If $n = 0$, then Player II wins anyway.

Let W be the set of vertices from which Player I has a memoryless winning strategy. By Proposition 4.1, Player I has a winning policy on the set W. We want to prove that Player II has a memoryless winning strategy from every vertex in $L = W^c$.

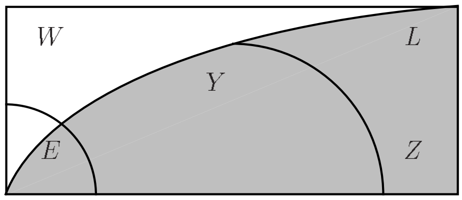

Figure 4.5. The memoryless strategies.

We first notice that, for Player I, W is its own attractor. Thus L is a trap for Player I. This implies that the graph induced by G on L is a game graph G'.

Let Y be the attractor for Player II of the set E_1 inside the game G'.

$$Y = A_2(G', L \cap E_1)$$

Let finally Z be the complement of Y in L. Since $Z \cap E_1 = \emptyset$, we may apply the induction hypothesis to the game restricted to Z. There can be no positions in Z on which Player I has a winning strategy because Z is disjoint from W. Thus Player II has a memoryless winning strategy on Z (provided the game remains within Z). Let us consider the strategy for Player II on L consisting in following the winning strategy on Z and to reach E_1 on the vertices of Y. This is clearly a memoryless strategy. It is actually winning because either the play passes infinitely often through E_1 or it stays out of Y from some moment on and then it stays within Z and is thus winning for Player II. □

4.4 Parity automata.

An *m-parity automaton* is an automaton $\mathcal{A} = (Q, i, \mu)$ where μ is a function from Q into $\{0, 1, \ldots, m\}$. For a path c in $\mathcal{A}$, we define

$$\mu(c) = \max\{\mu(q) \mid q \text{ occurs infinitely often in } c\}.$$

By definition, a path c in $\mathcal{A}$ is successful if it starts at i and the integer $\mu(c)$ is odd. As for parity games, an equivalent definition is obtained by considering an increasing sequence $\mathcal{C} = E_1 \subset F_1 \subset \ldots \subset E_n \subset F_n$. A *Rabin chain automaton* is an automaton $\mathcal{A} = (Q, i, \mathcal{C})$, with $\mathcal{C}$ as above. A path in $\mathcal{A}$ is final if the set of infinitely repeated states satisfies the Rabin chain condition $\mathcal{C}$. As a consequence of Proposition 4.5, any parity automaton can be viewed as a Rabin automaton and vice versa.

We shall use here a construction that allows one to build a parity automaton from a Muller automaton. We shall meet this construction later in Chapter V. It is based on the notion of a *memory extension* of a finite automaton.

Let $\mathcal{A} = (Q, i, \mathcal{F})$ be a Muller automaton. We build a deterministic automaton $\mathcal{B}$ as follows. Let $\mathrm{Arr}(Q)$ denote the set of sequences of elements of Q, each appearing at most once (sometimes called arrangements). The set of states of $\mathcal{B}$ is $S = \{(u, v) \mid uv \in \mathrm{Arr}(Q)\}$. An element of S can be called *last appearance record* since the transitions are defined in such a way that the arrangement uv

gives the order of last occurrence of each state. The division of uv into a pair (u,v) marks the previous position of the last state. The initial state of $\mathcal{B}$ is the pair (ε, i) where i is the initial state of $\mathcal{A}$ and where ε denotes the empty sequence. The transitions are defined as follows. Let $(u,v) \in S$ and $a \in A$. Let p be the last element of uv and let $q = p\cdot a$. Then

$$(u,v)\cdot a = \begin{cases} (x, yq) & \text{if } uv = xqy \\ (uv, q) & \text{if } q \notin uv \end{cases}$$

The automaton $\mathcal{B}$ is called the *memory extension* of $\mathcal{A}$.

Example 4.6 The memory extension of the automaton $\mathcal{A}_1$ of Figure 4.6 is pictured in Figure 4.7.

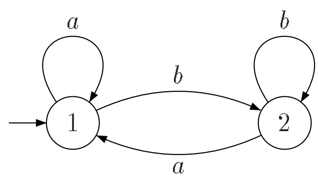

Figure 4.6. A Muller automaton.

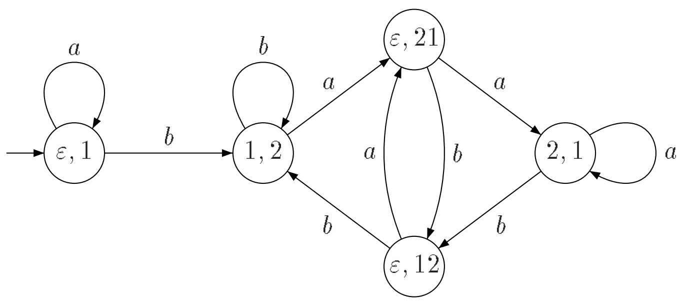

Figure 4.7. The memory extension of the automaton $\mathcal{A}_1$.

The fundamental property of the memory extension is the following one. For a path c in an automaton, we denote by $\mathrm{Inf}(c)$ the set of states occurring infinitely often in c. In the following proposition, we use the notation $\underline{v}$ to denote the set of elements appearing in a sequence v.

Proposition 4.7 *Let $\mathcal{A}$ be an automaton and let $\mathcal{B}$ be its memory extension. Let c be an initial path in $\mathcal{A}$ and let c' be the corresponding path in $\mathcal{B}$. Then $T = \mathrm{Inf}(c)$ if and only if all states $(u,v) \in \mathrm{Inf}(c')$ satisfy $\underline{v} \subset T$ and at least one satisfies $\underline{v} = T$.*

Proof. Let $(q_0, q_1, \ldots)$ be the sequence of states appearing along c. We first observe that all states of c' are ultimately of the form (uv'_n, v''_n) with $\underline{u} = S$ and $\underline{v_n} = T$, where $v_n = v'_n v''_n$ and where S is the set of states appearing finitely often along c. Next, for each state of this form with $v'_n \neq \varepsilon$, there is later on the path a state of the form (u, v_m). Let indeed $v'_n = qw'$ with $q \in Q$. Since $q \in T$, there is an occurrence of q on c later on. For the first index $m > n$ such that $q_m = q$, we have $v'_m = \varepsilon$.

This shows that the condition is necessary and sufficient. □

We define a chain $E_0 \subset F_0 \subset \cdots \subset E_n \subset F_n \subset \cdots$ as follows. For $i \geq 0$, let E_i be the set of states (u, v) of $\mathcal{B}$ such that either $|u| < i$ or $|u| = i$ and $\underline{v} \notin \mathcal{F}$. And let F_i be the union of E_i and the set of states (u, v) such that $|u| = i$ and $\underline{v} \in \mathcal{F}$. This defines a chain automaton which is clearly equivalent with $\mathcal{A}$.

We have thus proved the following result.

Theorem 4.8 *For any Muller automaton, there exists an equivalent parity automaton.*

It would not change anything to use as set of states the pairs (u, v) where uv is a permutation of Q. In this case, the initial state can be chosen as any of the states of the form (u, vi), where i is the initial state. This can be used to reduce the number of states of the resulting automaton, as in the following example.

Example 4.7 Let $\mathcal{A} = (Q, i, \mathcal{F})$ be the Muller automaton represented in Figure 4.8 with $i = 2$ and $\mathcal{F} = \{1, 2, 3\}$.

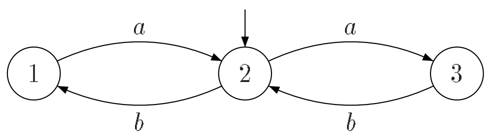

Figure 4.8. A Muller automaton.

It recognizes the set of infinite words in $(ab + ba)^\omega$ with both an infinite number of occurrences of ab and ba.

The memory extension $\mathcal{B}$ of $\mathcal{A}$ is represented in Figure 4.9. Actually, we have represented only the states which are permutations of Q. Both states $(3, 12)$ or $(1, 32)$ can be used as initial state.

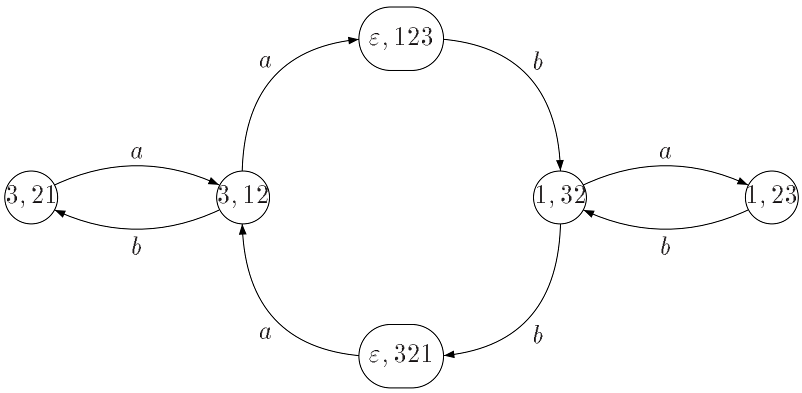

Figure 4.9. The memory extension.

The chain reduces to $F_0 = \{(\varepsilon, 123), (\varepsilon, 321)\}$ since $E_0 = \emptyset$ and $F_1 = F_0$. Thus $\mathcal{B}$ is actually a Büchi deterministic automaton.

4.5 Rational winning strategies.

Let G be a game graph in which the winning condition is given in Muller form, i.e. by a collection $\mathcal{F}$ of subsets of Q such that Player I wins the play if the set of infinitely repeated vertices belongs to $\mathcal{F}$. The following example shows that, in general, there is no memoryless winning strategy.

Example 4.8 Let G be the game graph of Figure 4.10 with $\mathcal{F} = \{\{1, 2, 3\}\}$. Player I has a winning strategy from each vertex consisting in choosing alternately 1 and 3 from vertex 2. However, there is no memoryless strategy since it would force Player I to always choose either 1 or 3 after 2, resulting in a loop either on $\{1, 2\}$ or on $\{2, 3\}$.

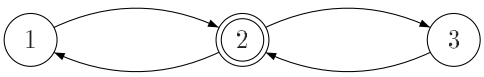

Figure 4.10. Player I has no memoryless strategy.

We now come back to abstract games given by the winning set $X \subset A^\omega$. A *rational* or *finite memory* strategy for Player I is given by a finite deterministic automaton $\mathcal{S} = (M, i, \delta)$ and a function

$$f : M \to A.$$

We say that Player I follows the strategy $(\mathcal{S}, f)$ in the play $a_0 a_1 \cdots$ if for every $n \geq 0$, $a_{2n} = f(m)$ where $m = \delta(i, a_0 \cdots a_{2n-1})$.

We prove the following result, known as the Büchi-Landweber theorem.

Theorem 4.9 *In a rational game, one of the players has a rational winning strategy.*

Proof. By Theorem 4.8, there is a parity automaton $\mathcal{A}$ recognizing X. We may suppose, by duplicating the states that the set of states Q is partitionned into $Q = Q_1 \cup Q_2$ in such a way that the initial state is in Q_1 and that the graph of $\mathcal{A}$ is bipartite. The game $G(X)$ defines a parity game on the graph of $\mathcal{A}$ and this game is equivalent to the original one. By Theorem 4.6, one of the players, say Player I, has a memoryless winning strategy in this game. This player has a rational winning strategy in $G(X)$. It uses the automaton $\mathcal{A}$ and the following function $f : Q_1 \to A$ (the value of f on Q_2 is irrelevant). For $p \in Q_1$, there is a state $q \in Q_2$ given by the memoryless strategy of Player I. Let a be a symbol such that (p, a, q) is a transition of $\mathcal{A}$. Then we define $f(p) = a$. This is clearly a rational winning strategy for Player I. □

Example 4.9 Let X be the set recognized by the Muller automaton of Example 4.7. The graph of the automaton coincides with the graph of Example 4.8 and the winning condition is the same. Accordingly, Player I wins $G(X)$ by choosing alternately the states 1 and 3, i.e. by playing alternately a and b (or any other strategy ensuring to play infinitely often a and b).

A parity automaton recognizing X is represented in Figure 4.9. This time, we have a memoryless strategy on the graph of the automaton. It consists in playing b in $(1, 32)$ and a in $(3, 12)$. It happens to be the same strategy as above, resulting in one of the two possible plays $(abba)^\omega$ or $(baab)^\omega$.

5 Wadge games.

Let $X \subset A^\omega$ and $Y \subset B^\omega$. The Wadge game $G(X, Y)$ is a game on $A \cup B$ defined as follows. Player I first chooses $a_0 \in A$. Then Player II chooses $b_0 \in B$. Player I chooses $a_1 \in A$, and so on. Thus a play in this game is a sequence $a_0 b_0 a_1 b_1 \cdots \in (A \cup B)^\omega$ which is the interleaving of the two sequences $x = a_0 a_1 \cdots$ and $y = b_0 b_1 \cdots \in B^\omega$ played by each player. Player II wins if either ($x \in X$ and $y \in Y$) or ($x \notin X$ and $y \notin Y$).

Such a game can be viewed as a game on the alphabet $A \cup B$ with a rule forcing Player I to choose a symbol from A and Player II a symbol from B. Observe that, if X and Y are Borel sets, then so is the winning set $Z \subset (A \cup B)^\omega$.

These games are strongly related with the following notion. We say that $X \subset A^\omega$ *Wadge reduces* or simply *reduces* to $Y \subset B^\omega$, denoted $(X, A^\omega) \leq_W (Y, B^\omega)$ or simply $X \leq_W Y$ if there exists a continuous function $f : A^\omega \to B^\omega$ such that

$$X = f^{-1}(Y)$$

The function f is called a *reduction*. It is important that the definition of the reduction is relative to the embedding of X in A^ω (see Example 5.2 below).

Obviously, $X \leq_W Y$ if and only if $X^c \leq_W Y^c$. It is possible to have simultaneously $X \leq_W Y$ and $Y \leq_W X$, in which case X and Y are called *Wadge equivalent*, denoted $X \equiv_W Y$. The $\equiv_W$-class of X is called the *Wadge class* of X.

The class of X^c and the class of X are called *dual*. The class of X is called self-dual if $X \equiv_W X^c$.