

Balancedness and coboundaries in symbolic systems

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*This paper is dedicated to the memory of our friend and colleague Maurice Nivat,
for his constant and encouraging support, and for all of his ideas
which have nourished the present work.*

Abstract

This paper studies balancedness for infinite words and subshifts, both for letters and factors. Balancedness is a measure of disorder that amounts to strong convergence properties for frequencies. It measures the difference between the numbers of occurrences of a given word in factors of the same length. We focus on two families of words, namely dendric words and words generated by substitutions. The family of dendric words includes Sturmian and Arnoux-Rauzy words, as well as codings of regular interval exchanges. We prove that dendric words are balanced on letters if and only if they are balanced on words. In the substitutive case, we stress the role played by the existence of coboundaries taking rational values and show simple criteria when frequencies take rational values for exhibiting imbalancedness.

Keywords Balancedness; Dendric subshifts; Substitutions; Coboundaries; Word combinatorics.

1 Introduction

Balancedness for words or subshifts is a measure of disorder that provides strong convergence properties for frequencies of letters and words. In ergodic terms, balancedness can be interpreted as an optimal speed of convergence of Birkhoff sums toward frequencies of words. In combinatorial terms, given a finite alphabet \mathcal{A} , a word $u \in \mathcal{A}^{\mathbb{Z}}$ is said to be *balanced* on the finite word $v \in \mathcal{A}^*$ if there exists a constant C_v such that for every pair (w, w') of factors of u of

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the same length, the difference between the number of occurrences of v in each word w and w' differs by at most C_v , that is, $||w|_v - |w'|_v| \leq C_v$, for $|w| = |w'|$, where the notation $|w|_v$ stands for the number of occurrences of v , and $|w|$, for its length.

The study of balancedness belongs to the general domain of aperiodic order (see e.g. [BG13]), and is considered for words, as well as for tilings and Delone sets in the context of quasicrystals, with balancedness being closely related to the notion of bounded distance equivalence to lattice [Lac92, HKK17, FG18]. Balancedness first occurred in the form of 1-balance for letters for infinite words defined over a two-letter alphabet¹ in the seminal papers by Morse and Hedlund [MH38, MH40] who laid the groundwork for the study of symbolic dynamics and Sturmian words. The infinite words that are 1-balanced over a two-letter alphabet are indeed exactly the Sturmian words.

The notion of balancedness was then considered for larger alphabets, for C -balance, with $C > 1$, and for factors, instead of letters. Words over a larger alphabet that are 1-balanced have been characterized in [Hub00] and shown to be closely related to Sturmian words. Moreover, Sturmian words have been proved to be balanced on their factors in [FV02]. Note that the number of 1-balanced words of length n is polynomial [Lip82, Mig91] while the number of C -balanced words of length n for $C > 1$ is exponential [Lip82] and, therefore, being C -balanced is relatively common.

This notion is natural and has thus been widely studied from many viewpoints, for instance in ergodic theory [CFZ00, CFM08] to prove absence of weak mixing properties, in number theory in connection with Fraenkel's conjecture [Fra73, Tij00] (see also [Tij00] and the survey on balanced words [Vui03]), or else, in operations research, for optimal routing and scheduling; see e.g. [AGH00, BC04, BJ08, Tij80]. Balancedness is also closely related to symbolic discrepancy, as investigated in [Ada03, Ada04].

In the multidimensional framework, balancedness has been considered both for multidimensional words [BT02] and for tilings [Sad16]. This notion has to be compared with homogeneity (related to 0-balance) such as introduced by M. Nivat in [Niv02]. A binary two-dimensional word U in $\{0, 1\}^{\mathbb{Z}^2}$ is k -homogeneous for a finite subset F if, whatever the position of the window F in U , exactly k ones appear in the window. M. Nivat proved in [Niv02] that a two-dimensional word is 1-homogeneous for a window F if and only if F tiles the plane; recall that this latter property has been characterized in [BN91]. Providing measures of order for words or tilings has constantly been in M. Nivat's research interests; let us quote e.g. the paper [BHN04] which has opened the way to the study of palindromic defects and complexity, and also Nivat's conjecture which states that if a two-dimensional word admits at most mn rectangular factors of size (m, n) , then it admits at least one direction of periodicity. This elegant and apparently simple conjecture has been formulated by M. Nivat in 1997 during an invited talk at ICALP and has lead to various approaches and numerous results. See for instance [CK15, KM18, KS15] for further references and examples of the latests developments on this conjecture.

In the case of primitive substitutions, if balancedness is known to be closely related to the eigenvalues of the substitution matrix (see [HZ98, Ada03, Ada04] and Section 2.3), it is not completely characterized in purely linear terms (see Remark 2.6). Note that there is an important literature devoted to the measure of imbalancedness, considered as deviations for Birkhoff sums, for substitutions acting on words or, in the higher-dimensional framework, for tiling substitutions. See in particular [BBH14, PS17] where central limit theorems are considered for deviations of ergodic sums for substitution subshifts. In [BS13], deviation of ergodic averages for \mathbb{R}^d -actions by translation associated with self-similar tilings are also investigated with new

¹A binary infinite words is said 1-balanced if the difference between numbers of occurrences of a given letter in factors of the same length is bounded by 1.

phenomena due to the dimension; this is the so-called boundary effect expressed in terms of the eigenvalues of the underlying substitution matrix. See also [Sad11] based on the use of cohomology for tiling spaces. Similarly, a criterion for balancedness for S -adic words (obtained by iterating several substitutions and not just one) is given in [BD14, Theorem 5.8] in terms of convergence of products of matrices; see [DHS13, BST] for an example of application for Brun substitutions. We discuss further S -adic examples in Section 3.3.

In this paper, we study balancedness for some families of words of low factor complexity², not only for letters, but also for factors. We focus on two families of words, namely dendric words in Section 3 and fixed points of primitive substitutions having rational frequencies in Section 4. Dendric words are also called tree words (see e.g. [BDFD⁺15b, BDFD⁺15c, BDFD⁺15d, BDFD⁺15a, BD⁺17]). These two families behave in different ways with respect to balancedness. Frequencies of factors of dendric words are irrational [BD⁺17] and balancedness for letters is equivalent to balancedness for factors (Theorem 1.1). In the case of substitutions, the example of the Thue–Morse substitution (handled in Corollary 4.10) illustrates the fact that one can have balancedness on letters and imbalancedness on factors. Our approach does not rely on techniques of linear algebra using the substitution matrix, or, in the dendric case, on the underlying S -adic expansion. It also allows us to prove balancedness results for specific factors. We exploit ideas issued from topological dynamics (and in particular, the notion of coboundary) for the study of balancedness of fixed points of primitive substitutions words with rational frequencies in Section 4 and, conversely, we use balancedness to deduce spectral properties in Section 3, which uses purely combinatorial methods. Our main results are the following.

Theorem 1.1 *Let (X, T) be a minimal dendric subshift. Then (X, T) is balanced on letters if and only if it is balanced on factors. In particular, if (X, T) is balanced, then all the frequencies of factors are additive topological eigenvalues and all cylinders are bounded remainder sets.*

Theorem 1.2 *Let σ be a primitive substitution over the alphabet \mathcal{A} . Let $\mathcal{L}(X_\sigma)$ denote the language of σ . Let v be in $\mathcal{L}(X_\sigma)$, and suppose that it has a rational frequency $\mu_v = p_v/q_v$ written in irreducible form. Suppose that the associated subshift (X_σ, T) is balanced on v . Then, we have the following.*

1. *For each $a \in \mathcal{A}$ and each return word w to a , q_v divides $|\sigma^n(w)|$ for all n large. In particular, if $aa \in \mathcal{L}_2(X_\sigma)$, then q_v divides $|\sigma^n(a)|$ for all n large.*
2. *Let $a \in \mathcal{A}$ and suppose that there exist $b, c \in \mathcal{A}$ such that $bac \in \mathcal{L}(X_\sigma)$ and $bc \in \mathcal{L}(X_\sigma)$. Then q_v divides $|\sigma^n(a)|$ for all n large.*

We briefly describe the contents of this paper. Basic definitions are recalled in Section 2. In particular, we stress the relations between balancedness and discrepancy in Section 2.2, we recall balancedness results for substitutions in Section 2.3, and highlight the connections with coboundaries and spectral eigenvalues in Section 2.4. The approach of Section 3 is combinatorial and applies to a wide family of infinite words and subshifts, namely the family of dendric words. The case of Arnoux-Rauzy words is discussed in Section 3.3. In Section 4, a topological dynamics approach is developed for infinite words for which frequencies do exist and are rational, based on the existence of coboundaries taking rational values. Some examples are discussed in Section 4.3.

²Factor complexity counts the number of factors of a given size.

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2 Basic definitions

2.1 Words and symbolic dynamical systems

Let \mathcal{A} be a finite non-empty alphabet of cardinality d . Let us denote by ε the empty word of the free monoid \mathcal{A}^* , and by $\mathcal{A}^{\mathbb{Z}}$ the set of bi-infinite words over \mathcal{A} . For $a \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, $|w|_a$ stands for the number of occurrences of the letter a in the word w , and $|w|$ stands for the length of w . The i th letter of w is denoted as w_i by labelling indices from 0, i.e., $w = w_0 \cdots w_{|w|-1}$. The notation $w_{[i,j]}$ stands for the word $w_i \cdots w_j$, for i, j non-negative integers with $i \leq j$, and $w_{[i,j]}$ stands for $w_i \cdots w_{j-1}$, with $i < j$. A *factor* of a (finite or infinite) word u is defined as the concatenation of consecutive letters occurring in u . We use the notation $w \prec u$ for w a factor of u . The set of factors $\mathcal{L}(u)$ of an infinite word u is called its *language*. The *factor complexity* of the bi-infinite word u counts the number of factors of u a given length. The bi-infinite word $u = (u_n)_n \in \mathcal{A}^{\mathbb{Z}}$ is said to be *uniformly recurrent* if every word occurring in u occurs in an infinite number of positions with bounded gaps, that is, for every factor w , there exists s such that, for every n , w is a factor of $u_n \dots u_{n+s-1}$. Let u be a uniformly recurrent bi-infinite word in $\mathcal{A}^{\mathbb{Z}}$. Let $v \in \mathcal{A}^*$ be a factor of u . We say that a word w with wv in $\mathcal{L}(u)$ is a *first return word* to v in u if v is a prefix of wv and there are exactly two occurrences of v in wv . We say that it is just a *return word* to v in u if v is a prefix of wv .

Symbolic dynamical systems. Let T stand for the *shift* acting on $\mathcal{A}^{\mathbb{Z}}$, that is, $T((u_n)_{n \in \mathbb{Z}}) = (u_{n+1})_{n \in \mathbb{Z}}$. A *subshift* (also called a *shift*) is a pair (X, T) where X is a closed shift-invariant subset of some $\mathcal{A}^{\mathbb{Z}}$. Here the set $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology of the discrete topology on each copy of \mathcal{A} . One associates with any bi-infinite word u in $\mathcal{A}^{\mathbb{Z}}$ the symbolic dynamical system (X_u, T) , where the subshift $X_u \subset \mathcal{A}^{\mathbb{Z}}$ is defined as $X_u = \{v \in \mathcal{A}^{\mathbb{Z}} : \forall w, w \prec v \Rightarrow w \prec u\}$. A subshift X is said to be *minimal* if it admits no non-trivial closed and shift-invariant subset. This is equivalent to the fact that every bi-infinite word u in X is uniformly recurrent. If X is a subshift, then its *language* $\mathcal{L}(X)$ is defined as the set of factors of elements of X . For any $n \geq 1$, we let denote by $\mathcal{L}_n(X)$ the set of factors of length n of elements in X .

Substitution dynamical systems. A *substitution* σ defined on the alphabet \mathcal{A} is a non-erasing morphism of the free monoid \mathcal{A}^* , i.e., there is no letter in \mathcal{A} whose image under σ is the empty word. If there exists a letter a such that $\sigma(a)$ admits the letter a as a strict prefix, then there exists an infinite word $u = \sigma^\omega(a)$ such that $\sigma(u) = u$. Moreover, if $\sigma(b)$ admits the letter b as a strict suffix for some letter b , then there exists a bi-infinite word $v = \sigma^\omega(b) \cdot \sigma^\omega(a)$ such that $\sigma(v) = v$, where the dot is located between the letters of index -1 and 0 . Such infinite and bi-infinite words are said to be *fixed points* of the substitution σ .

Let $|\mathcal{A}|$ stand for the cardinality of \mathcal{A} . The *substitution matrix* (or incidence matrix) M_σ of σ is the $|\mathcal{A}| \times |\mathcal{A}|$ -matrix whose coefficients are $M_\sigma(a, b) = |\sigma(b)|_a$. The substitution σ is said to be *primitive* if there exists a power of M_σ which is positive. Given a primitive substitution σ on the finite alphabet \mathcal{A} , the symbolic system associated with σ is the pair (X_σ, T) , where

$$X_\sigma := \{x \in \mathcal{A}^{\mathbb{Z}} : \forall w, w \prec x \Rightarrow \exists a \in \mathcal{A}, \exists n \in \mathbb{N} \text{ s.t. } w \prec \sigma^n(a)\}.$$

Note that the subshift X_σ is generated by any bi-infinite word v such that $\sigma^k(v) = v$ for some positive k , i.e., $X_\sigma = X_v$. The *language* of σ is defined as $\mathcal{L}(X_\sigma)$. Primitive substitutions

are known to be *recognizable*, i.e., they can be uniquely desubstituted [Mos92, Mos96]. More precisely, for any $x \in X_\sigma$, there exists a unique pair (y, k) with $y \in X_\sigma$ and $0 \leq k < |\sigma(y_0)|$ such that $x = T^k \sigma(y)$.

According to Perron's theorem, if a substitution is primitive, then its substitution matrix admits a dominant eigenvalue (it dominates strictly in modulus the other eigenvalues) that is positive. It is called its *Perron eigenvalue*, or its *expansion factor*. *Pisot substitutions* are primitive substitutions such that the dominant Perron eigenvalue of their substitution matrix is a *Pisot number*, that is, an algebraic integer whose conjugates lie strictly inside the unit disk. Note that Pisot irreducible substitutions, that is, Pisot substitutions for which the characteristic polynomial of their substitution matrix is irreducible, are conjectured to have pure discrete spectrum in the measure-theoretic sense. This is called the Pisot substitution conjecture (see e.g. the survey [ABB⁺15]).

Frequencies and invariant measures. Let u be an bi-infinite word in $\mathcal{A}^{\mathbb{Z}}$. The *frequency* μ_v of a finite word $v \in \mathcal{A}^*$ is defined as the limit, when n tends toward infinity, if it exists, of the number of occurrences of v in $u_{-n} \cdots u_{-1} u_0 u_1 \cdots u_{n-1}$ divided by $2n + 1$, i.e.,

$$\mu_v = \lim_{n \rightarrow +\infty} \frac{|u_{-n} \cdots u_0 \cdots u_n|_v}{2n + 1}.$$

Assume that the frequencies of the factors of u all exist. The infinite word u is said to have *uniform frequencies* if, for every factor v of u , the convergence toward μ_v of $\frac{|u_k \cdots u_{k+2n}|_v}{2n+1}$ is uniform in k , when n tends to infinity.

Let (X, T) be a subshift with $X \subset \mathcal{A}^{\mathbb{Z}}$. A probability measure μ on X is said to be *T-invariant* if $\mu(T^{-1}A) = \mu(A)$ for every measurable set $A \subset X$. The subshift (X, T) is *uniquely ergodic* if there exists a unique shift-invariant probability measure on X . The subshift (X, T) is uniquely ergodic if and only if every bi-infinite word u in X has uniform factor frequencies. In that case, one recovers the frequency μ_v of a factor $v = v_0 \cdots v_n$ as $\mu_v = \mu([v])$, where the *cylinder*

$$[v] := \{u \in X; u_0 \dots u_n = v\}.$$

For more on invariant measures and ergodicity, we refer to [Que10] and [BR10, Chap. 7]. We also recall that the substitutive subshift X_σ determined by a primitive substitution σ is uniquely ergodic and minimal (see e.g. [Que10]).

2.2 Balancedness and discrepancy

A bi-infinite word $u \in \mathcal{A}^{\mathbb{Z}}$ is said to be *balanced* on the factor $v \in \mathcal{L}(u)$ if there exists a constant C_v such that for every pair (w, w') of factors of u , if $|w| = |w'|$, then

$$||w|_v - |w'|_v| \leq C_v.$$

It is *balanced on letters* if it is balanced on each letter in \mathcal{A} , it is *balanced on factors* if it is balanced on all its factors, and lastly, it is *balanced on factors of length n* if it is balanced on all its factors of length n . Similarly, a subshift (X, T) with $X \subset \mathcal{A}^{\mathbb{Z}}$ is said to be *balanced* on the factor $v \in \mathcal{L}(X)$ if there exists a constant C_v such that for all w, w' in $\mathcal{L}(X)$ with $|w| = |w'|$, then $||w|_v - |w'|_v| \leq C_v$. The notions of balancedness for letters, words or words of a given length extend similarly to subshifts. Note that in the first papers devoted to balancedness, balance was used to refer to 1-balance for letters (see e.g. [Lot02]).

Proposition 2.1 below (which is a rephrasing of [Ada03, Lemma 23]) states that balancedness is preserved when decreasing the length of factors. It is thus sufficient to prove that balancedness

does not hold for some length to obtain that it does not hold for all larger lengths. This will be used in the examples of Section 4.3.

Proposition 2.1 [Ada03, Lemma 23] *If a bi-infinite word u is balanced on some factor v , then it is balanced on the prefix of v of length $|v| - 1$. If a bi-infinite word u or a subshift (X, T) is balanced on factors of length $n + 1$, then it is balanced on factors of length n .*

Proof. Let $u \in \mathcal{A}^{\mathbb{Z}}$. For every n , we consider an alphabet \mathcal{A}_n and a bijection $\theta_n: \mathcal{A}_n \rightarrow \mathcal{L}_n(u)$. The word $u^{(n)} := \theta_n(u)$, defined over the alphabet \mathcal{A}_n , codes factors of length n according to the bijection θ_n in the same order as in u with overlaps and without repetition. The map $\theta_n \circ \pi_n \circ \theta_{n+1}^{-1}$ is a morphism from the monoid \mathcal{A}_{n+1}^* to \mathcal{A}_n^* that maps letters to letters: it maps the coding of a block of length $n+1$ to the coding of its prefix of length n . The word $u^{(n)}$ is thus the image by a letter-to-letter substitution of the word $u^{(n+1)}$; indeed $u^{(n)} = \theta_n \circ \pi_n \circ \theta_{n+1}^{-1}(u^{(n+1)})$. We conclude by noticing that the action of a letter-to-letter substitution preserves balancedness. ■

Example 2.2 We consider the Thue–Morse substitution σ_{TM} defined over $\{0, 1\}$ as $\sigma_{TM}: 0 \mapsto 01, 1 \mapsto 10$. One has $\mathcal{L}_2(\sigma_{TM}) = \{00, 01, 10, 11\}$. Let $a = 00, b = 01, c = 10, d = 11$. The central letters of the two-sided word $u = (\sigma^2)^\omega(0) \cdot (\sigma^2)^\omega(0)$ are

$$\cdots 0110100110010110 \cdot 0110100110010110 \cdots$$

coded by the central letters of $u^{(2)}$

$$\cdots bdcbcabdcabcbdcba \cdot bdcbcabdcabcbdcba \cdots$$

Discrepancy. Let $u \in \mathcal{A}^{\mathbb{Z}}$ be a bi-infinite word and assume that each factor $v \in \mathcal{L}(u)$ admits a frequency μ_v in u . The *discrepancy* $\Delta_v(u)$ of u with respect to v is defined as

$$\Delta_v(u) = \sup_{n \in \mathbb{N}} ||u_{-n} \cdots u_0 \cdots u_n|_v - (2n + 1)\mu_v|.$$

The quantity $\Delta_v(u)$ is considered e.g. in [Ada03, Ada04]. One easily checks that u is balanced on the factor v if and only if its discrepancy $\Delta_v(u)$ is finite. If $\Delta_v(u)$ is finite, then the cylinder $[v]$ is said to be a *bounded remainder set*, according to the terminology developed in classical discrepancy theory. These definitions extend to any subshift (X, T) in a straightforward way.

2.3 Balancedness and substitutions

We now consider the special case of substitutions. We follow here [Que10]. Let σ be a primitive substitution. Letter frequencies for a primitive substitution σ are known to be provided by the normalized positive eigenvector (whose sum of coordinates equal 1) associated with the Perron eigenvalue of the substitution matrix M_σ . We call it the *renormalized Perron eigenvector*. In other words, the frequency μ_a of the letter a is the a th entry of this vector. Similarly, frequencies of factors in $\mathcal{L}(X_\sigma)$ (not only letters) are known to be provided by eigenvectors of a matrix, namely the two-letter substitution matrix that is associated with σ as follows. This construction relies on the coding by k -letter blocks described in the proof of Proposition 2.1.

Two-letter factor substitutions. Given a substitution σ , consider the finite set $\mathcal{L}_2(X_\sigma)$ as an alphabet and define the *two-letter factor substitution* σ_2 on $\mathcal{L}_2(X_\sigma)$ as follows (it is also called *induced substitution* in [BG13]): for every $u = ab \in \mathcal{L}_2(X_\sigma)$, $\sigma_2(u)$ is the word over the

alphabet $\mathcal{L}_2(X_\sigma)$ made of the first $|\sigma(a)|$ factors of length 2 in $\sigma(u)$. For instance, if $ab \in \mathcal{L}_2(X)$ with $\sigma(a) = a_0 \cdots a_r$, $\sigma(b) = b_0 \cdots b_s$, then

$$\sigma_2(ab) = (a_0a_1)(a_1a_2) \cdots (a_{r-1}a_r)(a_rb_0).$$

We recall from [Que10, Lemma 5.3–5.4] that if the substitution σ is primitive, then σ_2 is also primitive, and σ_2 has the same Perron eigenvalue as σ . Frequencies of factors of length 2 are provided by the renormalized Perron eigenvector of M_{σ_2} .

One then checks that the substitution σ_2 admits a bi-infinite fixed point that is composed by all the factors of length 2 of v without repetition and in the same order as in v .

Example 2.3 We continue Example 2.2 and consider the Thue–Morse substitution σ_{TM} defined over $\{0, 1\}$ as $\sigma_{TM}: 0 \mapsto 01$ and $\sigma_{TM}: 1 \mapsto 10$. One has $\mathcal{L}_2(\sigma_{TM}) = \{00, 01, 10, 11\}$. One has in particular $\sigma(00) = 0101$ and $\sigma_2(00) = (01)(10)$. One checks that $\sigma^{(2)}(a) = bc$, $\sigma^{(2)}(b) = bd$, $\sigma^{(2)}(c) = ca$, $\sigma^{(2)}(d) = cb$, by setting $a = 00$, $b = 01$, $c = 10$, $d = 11$. Observe that the image of 0 by σ^2 begins and ends with 0. We thus can consider the bi-infinite word $(\sigma^2)^\omega(0) \cdot (\sigma^2)^\omega(0)$. Similarly, the image of b by σ_2^2 begins with b , the image of a by σ_2^2 ends with a , and we can consider the bi-infinite word $(\sigma_2^2)^\omega(a) \cdot (\sigma_2^2)^\omega(b)$. Note that powers of the substitution σ generate the same subshift as σ . The central letters of the two-sided word $(\sigma^2)^\omega(0) \cdot (\sigma^2)^\omega(0)$ are

$$\cdots 0110100110010110 \cdot 0110 1001 1001 0110 \cdots$$

whose coding is provided by the central letters of the two-sided word $(\sigma_2^2)^\omega(a) \cdot (\sigma_2^2)^\omega(b)$

$$\cdots bdc b cabd cab c bdc a \cdot bdc b cabd cab c bdc b \cdots$$

The eigenvalues of M_σ are 2 and 0, and the eigenvalues of M_{σ_2} are 0, 1, -1 and 2.

We can similarly define a notion of higher-order factor substitution σ_k with respect to factors of length k . From M_σ to M_{σ_2} , eigenvalues of modulus 1 can be added, such as illustrated by the example of the Thue–Morse substitution (see Example 2.3). One crucial property is that the set of eigenvalues of M_{σ_2} is a subset of the spectrum of M_{σ_k} for $k \geq 2$; only the eigenvalue 0 is added, according to [Que10, Corollary 5.5].

Example 2.4 We continue Example 2.2 and 2.3 with the Thue–Morse substitution σ_{TM} . One has $\mathcal{L}_3(\sigma_{TM}) = \{001, 010, 011, 100, 101, 110\}$. One checks that $\sigma^{(3)}(a) = be$, $\sigma^{(3)}(b) = cf$, $\sigma^{(3)}(c) = cf$, $\sigma^{(3)}(d) = da$, $\sigma^{(3)}(e) = da$, $\sigma^{(3)}(f) = eb$, by setting $a = 001$, $b = 010$, $c = 011$, $d = 100$, $e = 101$, $f = 110$. The eigenvalues of M_{σ_3} are 0, 1, -1 and 2, and are the same as for M_{σ_2} ; the eigenvalue 0 has multiplicity 3 for M_{σ_3} and 1 for M_{σ_2} , and the other eigenvalues have the same multiplicity for both matrices.

Moreover, we can compute the frequency of any factor thanks to M_σ and M_{σ_2} . Roughly, given a factor, it occurs in the image by some power σ^n of σ of either a letter or of a factor of length 2. It thus can be determined by the normalized Perron eigenvectors of M_σ and M_{σ_2} . As a consequence, balancedness for factors can be described in terms of the spectrum of the substitution matrices M_σ and M_{σ_2} , which even provide estimates on the symbolic discrepancy and on the balance function $B_n(u)$ [Ada03, Ada04].

Theorem 2.5 ([Ada03, Ada04]) *Let σ be a primitive substitution. If σ (resp. σ_2) is a Pisot substitution, then the subshift X_σ is balanced on letters (resp. on factors). Conversely, if X_σ is balanced on letters (resp. on factors), then the Perron eigenvalue of M_σ (resp. M_{σ_2}) is the unique eigenvalue of M_σ (resp. M_{σ_2}) that is larger than 1 in modulus, and all possible eigenvalues of modulus one of M_σ (resp. M_{σ_2}) are roots of unity.*

Remark 2.6 In the case where the matrix M_σ admits a root of unity as an eigenvalue, we cannot decide whether balancedness holds or not just by inspecting the matrix M_σ . An example of a primitive and aperiodic substitution σ that is balanced on factors is given in Example 2.7 having 1 as an eigenvalue (aperiodic means that X_σ contains no periodic word). Note also that a substitution can be balanced on letters but unbalanced on factors if there exists an eigenvalue of modulus 1 that occurs in the spectrum of M_{σ_2} . This is the case of the Thue–Morse substitution (see Example 2.3 and Corollary 4.10). We also deduce from Theorem 2.5 that primitive Pisot substitutions have finite discrepancy with respect to letters.

Example 2.7 This example has been communicated by J. Cassaigne and M. Minervino. We consider the substitution σ over the alphabet $\{1, 2, 3\}$ defined by $\sigma: 1 \mapsto 121, 2 \mapsto 32, 3 \mapsto 321$. Its spectrum is $\{1, \frac{3 \pm \sqrt{5}}{2}\}$ and it is balanced on factors. This thus provides an example of a substitution that admits in the spectrum of its matrix the eigenvalue 1 and that is balanced on factors. Indeed, consider the Sturmian substitution $\tau: 3 \mapsto 30, 0 \mapsto 300$. The subshift (X_σ, T) is deduced from the Sturmian shift (X_τ, T) (which is balanced on factors by [FV02]) by applying the substitution $\varphi: 0 \mapsto 21, 3 \mapsto 3$, which preserves balancedness.

2.4 Coboundaries and topological eigenvalues

We now recall a convenient topological interpretation of the notion of balancedness in terms of coboundaries and topological eigenvalues.

Let (X, T) be a topological dynamical system, that is, T is a homeomorphism acting on the compact space X . Consider e.g. (X_σ, T) for some primitive substitution σ or some subshift (X, T) . Recall that it is said to be *minimal* if every non-empty closed T -invariant subset of X is equal to X . We let denote by $C(X, \mathbb{R})$ the additive group of continuous maps from X to \mathbb{R} and by $C(X, \mathbb{Z})$ the additive group of continuous maps from X to \mathbb{Z} . We take \mathbb{Z} as a topological space with the discrete topology, so that any $f \in C(X, \mathbb{Z})$ is *locally constant*: for all $x \in X$, there is an open neighborhood U of x such that $y \in U$ implies $f(y) = f(x)$. Since X is a compact space, any $f \in C(X, \mathbb{Z})$ takes a finite number of values. These two conditions imply that, for any $f \in C(X, \mathbb{Z})$, there exists $R > 0$ such that for all $x, y \in X$, $d(x, y) \leq R$ implies $f(x) = f(y)$. If (X, T) is a subshift, then, for all $f \in C(X, \mathbb{Z})$, there exists a positive integer k such that $x_{[-k, k]} = y_{[-k, k]}$ implies $f(x) = f(y)$.

For every $g \in C(X, \mathbb{R})$, we define the *coboundary* of g by

$$\partial g = g \circ T - g. \tag{1}$$

The map $g \mapsto \partial g$ is an endomorphism of $C(X, \mathbb{R})$. When an element f belongs to $\partial C(X, \mathbb{R})$, we say that f is a *coboundary*. Two functions f, g are said to be *cohomologous* if $f - g$ is a coboundary. For any non-negative integer n , $f^{(n)}$ stands for the map in $C(X, \mathbb{R})$ defined for any $x \in X$ as

$$f^{(n)}(x) := f(x) + f \circ T(x) + \cdots + f \circ T^j(x) + \cdots + f \circ T^{n-1}(x).$$

The family of maps $(f^{(n)})_{n \in \mathbb{N}}$ is called the *cocycle* of f . The following theorem states that being a coboundary means having a bounded cocycle.

Theorem 2.8 (Gotshalk-Hedlund’s Theorem [GH55]) *Let (X, T) be a minimal topological dynamical system. The map $f \in C(X, \mathbb{R})$ is a coboundary if and only if there exists $x_0 \in X$ such that the sequence $(f^{(n)}(x_0))_{n \in \mathbb{N}}$ is bounded.*

Let (X, T) be a topological dynamical system, where T is a homeomorphism. A non-zero complex-valued continuous function $f \in C(X, \mathbb{C})$ is an *eigenfunction* for (X, T) if there exists $\lambda \in \mathbb{C}$ such that $\forall x \in X, f(Tx) = \lambda f(x)$. The eigenvalues corresponding to those eigenfunctions are called the *continuous eigenvalues* of (X, T) . If θ is such that $e^{2i\pi\theta}$ is an eigenvalue, θ is said to be an *additive topological eigenvalue*.

As a direct consequence of Theorem 2.8, we now can reformulate balancedness in spectral terms.

Proposition 2.9 *Let (X, T) be a minimal and uniquely ergodic subshift and let μ stand for its invariant measure. Given a factor $v \in \mathcal{L}_X$, define*

$$f_v = \chi_{[v]} - \mu([v]) \in C(X, \mathbb{R}), \quad (2)$$

where $\chi_{[v]}$ stands for the characteristic function of the cylinder $[v]$. Then, (X, T) is balanced on the factor v if and only if the map f_v is a coboundary. If σ is balanced on the factor v , then $\mu[v]$ is an additive topological eigenvalue of (X, T) .

Proof. We assume that X is balanced on the factor v . By Theorem 2.8, there exists $g \in C(X, \mathbb{R})$ such that $f_v = g \circ T - g$. Note that $e^{2i\pi\chi_{[v]}(u)} = 1$ for any $u \in X$. This yields

$$\exp^{2i\pi g \circ T} = \exp^{2i\pi\mu_v} \exp^{2i\pi g}.$$

Hence $\exp^{2i\pi g}$ is a topological eigenfunction associated with the additive topological eigenvalue μ_v . ■

We will also need the following statement in Section 4.

Proposition 2.10 *Let (X, T) be a minimal topological dynamical system. If $f \in C(X, \mathbb{Z})$ is a coboundary of some function $g \in C(X, \mathbb{R})$, then it is the coboundary of some function $h \in C(X, \mathbb{Z})$.*

Proof. We recall the proof of [Hos95, DHP18]. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and $\pi : \mathbb{R} \rightarrow \mathbb{T}$ the canonical projection. Let $\tilde{\partial}$ denote the coboundary map defined on $C(X, \mathbb{T})$ in the same way as in (1). Note first that if $\gamma \in C(X, \mathbb{T})$ and $\tilde{\partial}\gamma = 0$, then γ is constant. Indeed, let $\tilde{c} \in \mathbb{T}$ and set $Y = \gamma^{-1}(\{\tilde{c}\})$. The subset Y is closed since γ is continuous and it is T -invariant since $\tilde{\partial}\gamma = 0$. The system being minimal, if Y is nonempty, it is necessarily the whole space X . Suppose $f \in C(X, \mathbb{Z})$ is the coboundary of some $g \in C(X, \mathbb{R})$. Then, $g \circ T(x) - g(x) \in \mathbb{Z}$ for all $x \in X$. This implies that $\tilde{\partial}(\pi \circ g) = 0$ and then there exists $\tilde{c} \in \mathbb{T}$ such that $\pi \circ g(x) = \tilde{c}$ for all $x \in X$. Let c be any element in $\pi^{-1}(\{\tilde{c}\})$ and define $h(x) := g(x) - c$. Since $\pi \circ h = 0$, $h \in C(X, \mathbb{Z})$, and it is clear that $\partial h = \partial g = f$. ■

3 Balancedness of dendric words

Dendric subshifts are minimal subshifts defined with respect to combinatorial properties of their language expressed in terms of extension graphs, such as recalled in Section 3.1. Elements of dendric subshifts are also called tree words (see e.g. [BDFD⁺15b, BDFD⁺15c, BDFD⁺15d, BDFD⁺15a, BD⁺17]). We use the terminology dendric subshifts in order to avoid any ambiguity with respect to the notion of tree shift that refers to shifts defined on trees (see e.g. [AB12]). We consider balancedness for dendric subshifts in Section 3.2 and prove Theorem 1.1. This class of subshifts encompasses subshifts generated by interval exchanges, as well as Sturmian and Arnoux–Rauzy subshifts discussed in Section 3.3. They have linear factor complexity.

3.1 Dendric subshifts

Extension graphs are bipartite graphs that describe the left and right extensions of factors and dendric subshifts are such that all their extension graphs are trees. More precisely, let (X, T) be a subshift on the alphabet \mathcal{A} . For $w \in \mathcal{L}(X)$, we let denote as

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in \mathcal{L}(X)\}, \\ R(w) &= \{a \in \mathcal{A} \mid wa \in \mathcal{L}(X)\}, \\ E(w) &= \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}(X)\}. \end{aligned}$$

For a word $w \in F$, we consider the undirected bipartite graph $\mathcal{E}(w)$, called its *extension graph*, defined as follows: its set of vertices is the disjoint union of $L(w)$ and $R(w)$ and its edges are the pairs $(a, b) \in E(w)$. For an illustration, see Example 3.1 below. A minimal subshift (X, T) is a *dendric subshift* if, for every word $w \in \mathcal{L}(X)$, the graph $\mathcal{E}(w)$ is a tree.

Example 3.1 Let σ_F be the Fibonacci substitution defined over $\{0, 1\}$ by $\sigma_F: 0 \mapsto 01, 1 \mapsto 0$. The extension graphs of the empty word and of the letters 0 and 1 are depicted in Figure 1.

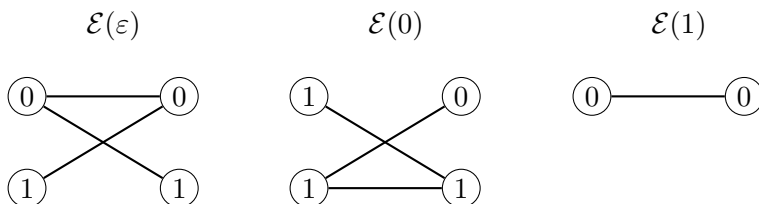


Figure 1: The extension graphs of ε (on the left), 0 (on the center) and 1 (on the right) are trees.

3.2 Balancedness for dendric subshifts

The main result of this section is Theorem 1.1, whose proof relies on Lemmas 3.2 and 3.3 stated and proved below.

Lemma 3.2 Let \mathcal{T} be a finite tree, with a bipartition X and Y of its set of vertices, with $\text{Card}(X), \text{Card}(Y) \geq 2$. Let E stand for its set of edges. For all $x \in X, y \in Y$, define

$$Y_x := \{y \in Y : (x, y) \in E\} \quad X_y := \{x \in X : (x, y) \in E\}.$$

Let $(G, +)$ be an abelian group and H a subgroup of G . Suppose that there exists a function $g : X \cup Y \cup E \rightarrow G$ satisfying the following conditions:

- (1) $g(X \cup Y) \subseteq H$;
- (2) for all $x \in X, g(x) = \sum_{y \in Y_x} g(x, y)$, and for all $y \in Y, g(y) = \sum_{x \in X_y} g(x, y)$.

Then, for all $(x, y) \in E, g(x, y) \in H$.

Proof. Observe first that Conditions (1) and (2) imply that the image under g of any edge connected to a leaf belongs to H .

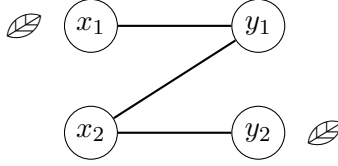


Figure 2: The tree \mathcal{T} when $k = 2$.

We proceed by induction on $k := \max\{\text{Card}(X), \text{Card}(Y)\}$ and we first assume $k = 2$. Such as illustrated in Figure 2, there is only one possibility for the graph \mathcal{T} (modulo a relabeling of the vertices), since \mathcal{T} is connected and has no cycles, which is

$$X = \{x_1, x_2\}, Y = \{y_1, y_2\}, E = \{(x_1, y_1), (x_2, y_1), (x_2, y_2)\}.$$

Both $g(x_1, y_1)$ and $g(x_2, y_2)$ are in H because x_1 and y_2 are leaves. By Condition (2), one has $g(x_2) = g(x_2, y_1) + g(x_2, y_2)$, and then $g(x_2, y_1) = g(x_2) - g(x_2, y_2)$. Since $g(x_2) \in H$ by Condition (1) and H is a group, then $g(x_2, y_1) \in H$.

We now assume $k > 2$ and that the induction hypothesis holds for $k - 1$. Suppose also *wlog* that $\text{Card}(X) \geq \text{Card}(Y)$. Note that in this case there exists a leaf in X . Indeed, if all vertices in X have degree at least 2, then

$$\text{Card}(E) = \sum_{x \in X} \deg(x) \geq 2 \text{Card}(X)$$

because \mathcal{T} is a bipartite graph. On the other hand, since \mathcal{T} is a tree,

$$\text{Card}(E) = \text{Card}(X) + \text{Card}(Y) - 1 < \text{Card}(X) + \text{Card}(Y) \leq 2 \text{Card}(X)$$

which yields the desired contradiction. The same argument shows that if X and Y have the same cardinality, then both X and Y have at least one leaf. We distinguish two cases, namely $\text{Card}(X) > \text{Card}(Y)$ and $\text{Card}(X) = \text{Card}(Y)$.

We first assume that $\text{Card}(X) > \text{Card}(Y)$. Take a leaf in X , and call it x_0 . Consider the graph $\tilde{\mathcal{T}}$ obtained from \mathcal{T} by removing the vertex x_0 and the edge (x_0, y_0) , where y_0 is the only vertex in Y connected with x_0 . This new graph is also a tree, with bipartition of vertices $\tilde{X} = X - \{x_0\}$, $\tilde{Y} = Y$, and set of edges $\tilde{E} = E - \{(x_0, y_0)\}$. Since $\text{Card}(\tilde{X}) = k - 1$ and $\text{Card}(\tilde{Y}) = \text{Card}(Y)$, then $\max\{\text{Card}(\tilde{X}), \text{Card}(\tilde{Y})\} = k - 1$.

We define \tilde{g} in $\tilde{X} \cup \tilde{Y} \cup \tilde{E}$ as follows. On $(\tilde{X} \cup \tilde{Y} \cup \tilde{E}) - \{y_0\}$, $\tilde{g} = g$; on y_0 , define $\tilde{g}(y_0) = g(y_0) - g(x_0, y_0)$. Let us verify that \tilde{g} satisfies Conditions (1) and (2) with respect to $\tilde{\mathcal{T}}$.

- (1) If $x \in \tilde{X}$, $\tilde{g}(x) = g(x) \in H$. If $y \in \tilde{Y}$ and $y \neq y_0$, $\tilde{g}(y) = g(y) \in H$. If $y = y_0$, then $\tilde{g}(y_0) = g(y_0) - g(x_0, y_0)$, but both $g(y_0)$ and $g(x_0, y_0)$ are in H , since g satisfies Conditions (1) and (2), and x_0 is a leaf. Therefore, the image under \tilde{g} of any vertex of $\tilde{\mathcal{T}}$ is in H .
- (2) We need a more precise notation here. For a vertex $x \in \tilde{X}$, we define $Y_x^{\tilde{\mathcal{T}}} := \{y \in Y : (x, y) \in E\}$ and $Y_x^{\tilde{\mathcal{T}}} := \{y \in \tilde{Y} : (x, y) \in \tilde{E}\}$. If $x \in \tilde{X}$, then $Y_x^{\tilde{\mathcal{T}}} = Y_x^{\tilde{\mathcal{T}}}$, and for all $y \in Y_x^{\tilde{\mathcal{T}}}$, $\tilde{g}(x, y) = g(x, y)$. Therefore,

$$\tilde{g}(x) = g(x) = \sum_{y \in Y_x^{\tilde{\mathcal{T}}}} g(x, y) = \sum_{y \in Y_x^{\tilde{\mathcal{T}}}} \tilde{g}(x, y).$$

We use analogously the notation $X_y^{\mathcal{T}}$ and $X_y^{\tilde{\mathcal{T}}}$ for a vertex $y \in \tilde{Y}$. Let be $y \in \tilde{Y}$. If $y \neq y_0$, then $X_y^{\mathcal{T}} = X_y^{\tilde{\mathcal{T}}}$ and for all $x \in X_y^{\mathcal{T}}$, $\tilde{g}(x, y) = g(x, y)$. Hence,

$$\tilde{g}(y) = g(y) = \sum_{x \in X_y^{\mathcal{T}}} g(x, y) = \sum_{x \in X_y^{\tilde{\mathcal{T}}}} \tilde{g}(x, y).$$

Finally, if $y \in \tilde{Y}$ and $y = y_0$, then $X_y^{\mathcal{T}} = X_y^{\tilde{\mathcal{T}}} \cup \{x_0\}$. We thus have

$$\begin{aligned} \tilde{g}(y) = g(y_0) - g(x_0, y_0) &= -g(x_0, y_0) + \sum_{x \in X_y^{\mathcal{T}}} g(x, y) \\ &= -g(x_0, y_0) + g(x_0, y_0) + \sum_{x \in X_y^{\tilde{\mathcal{T}}}} \tilde{g}(x, y) = \sum_{x \in X_y^{\tilde{\mathcal{T}}}} \tilde{g}(x, y), \end{aligned}$$

which ends the proof of the fact that \tilde{g} satisfies Conditions (1) and (2).

Hence, by induction, for all $(x, y) \in \tilde{E}$, $\tilde{g}(x, y) \in H$. But in \tilde{E} one has $\tilde{g} = g$, which implies that for all $(x, y) \in \tilde{E}$, $g(x, y) \in H$. Since x_0 is a leaf in X , $g(x_0, y_0) \in H$, and then for all $(x, y) \in E$, $g(x, y) \in H$. This ends the case $\text{Card}(X) > \text{Card}(Y)$.

We now assume that $\text{Card}(X) = \text{Card}(Y)$. Then, both X and Y have at least one leaf; let us call them x_0 and y_0 , respectively. Let x_{y_0} and y_{x_0} denote the only vertices connected with x_0 and y_0 , respectively. It is not difficult to see that $y_0 \neq y_{x_0}$ and $x_0 \neq x_{y_0}$, since \mathcal{T} is connected and has no cycles.

Consider the graph $\tilde{\mathcal{T}}$ obtained from \mathcal{T} by removing the vertices x_0 and y_0 , and the edges (x_0, y_{x_0}) and (x_{y_0}, y_0) . This new graph is again a tree, with bipartition of vertices $\tilde{X} = X - \{x_0\}$, $\tilde{Y} = Y - \{y_0\}$, and set of edges $\tilde{E} = E - \{(x_0, y_{x_0}), (x_{y_0}, y_0)\}$. Since $\text{Card}(\tilde{X}) = k - 1$ and $\text{Card}(\tilde{Y}) = k - 1$, then $\max\{\text{Card}(\tilde{X}), \text{Card}(\tilde{Y})\} = k - 1$.

On the new set $\tilde{X} \cup \tilde{Y} \cup \tilde{E}$, define the function \tilde{g} as follows. On $(\tilde{X} \cup \tilde{Y} \cup \tilde{E}) - \{x_{y_0}, y_{x_0}\}$, $\tilde{g} = g$; on x_{y_0} , define $\tilde{g}(x_{y_0}) = g(x_{y_0}) - g(x_{y_0}, y_0)$, and on y_{x_0} , $\tilde{g}(y_{x_0}) = g(y_{x_0}) - g(x_0, y_{x_0})$.

Following the same strategy as in the case $\text{Card}(X) > \text{Card}(Y)$, one can see that \tilde{g} satisfies Conditions (1) and (2) in $\tilde{\mathcal{T}}$, and since $\max\{\text{Card}(\tilde{X}), \text{Card}(\tilde{Y})\} = k - 1$, we conclude by induction that for any edge $(x, y) \in \tilde{E}$, $\tilde{g}(x, y)$ belongs to H , which implies that $g(x, y) \in H$. Since x_0 and y_0 are leaves in X and Y , $g(x_0, y_{x_0}), g(x_{y_0}, y_0) \in H$. We conclude that for all $(x, y) \in E$, $g(x, y) \in H$. \blacksquare

Lemma 3.3 *Let (X, T) be a minimal dendric subshift. Let H be the following subset of $C(X, \mathbb{Z})$:*

$$H = \left\{ \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \chi_{T^k(\{a\})} : K_a \subseteq \mathbb{Z}, |K_a| < \infty, \alpha(a, k) \in \mathbb{Z} \right\},$$

where χ_A denotes the characteristic function of the set A , for all $A \subseteq X$. Then, for all $v \in \mathcal{L}(X)$, the characteristic function $\chi_{[v]}$ belongs to H .

Proof. One first easily checks that H is a subgroup. We now proceed by induction on the length of v . The claim is clearly true if $|v| = 1$, that is, when v is a letter of \mathcal{A} , by setting $K_a = \{0\}$ and $\alpha(a, k) = 1$ if $a = v$, 0 otherwise. Now suppose that for all $u \in \mathcal{L}(X)$ with $|u| \leq n$, one has $\chi_{[u]} \in H$. Let v be a word of length $n + 1$. We write

$$v = v_0 \cdots v_n \text{ and define } \tilde{v} = v_1 \cdots v_n, v' = v_0 \cdots v_{n-1}, \bar{v} = v_1 \cdots v_{n-1}.$$

We analyze separately three cases depending on the right/left extensions of \tilde{v} and v' , namely $l(\tilde{v}) = 1$, $r(v') = 1$, and then, finally, $l(\tilde{v}) \geq 2$ and $r(v') \geq 2$, with this latter case being handled thanks to Lemma 3.2.

- Suppose first that $l(\tilde{v}) = 1$. The only left extension of \tilde{v} is v_0 , and thus, for all $x \in X$, $\chi_{[v]}(x) = \chi_{[\tilde{v}]}(Tx)$. By induction hypothesis we have that $\chi_{[\tilde{v}]}$ belongs to H , so we obtain that for all $x \in X$,

$$\chi_{[v]}(x) = \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \chi_{T^{k-1}([a])}(x).$$

Defining $K'_a := \{k - 1 : k \in K_a\}$ for all $a \in \mathcal{A}$, and $\beta(a, k) = \alpha(a, k + 1)$ for all $k \in K'_a$, we conclude that, for all $x \in X$,

$$\chi_{[v]}(x) = \sum_{a \in \mathcal{A}} \sum_{k \in K'_a} \beta(a, k) \chi_{T^k([a])},$$

and then $\chi_{[v]}$ belongs to H .

- Now suppose that $r(v') = 1$. The only right extension of v' is v_n , and thus, for all $x \in X$, $\chi_{[v]}(x) = \chi_{[v']}(x)$. We conclude by applying the induction hypothesis.
- Finally, we assume $l(\tilde{v}) \geq 2$ and $r(v') \geq 2$. Let $\mathcal{E}(\bar{v})$ be the extension graph of \bar{v} (as defined in Section 3.1). It is a tree by definition, and each of the sets in its bipartition of vertices has cardinality at least two.

Define $g : L(\bar{v}) \cup R(\bar{v}) \cup E(\bar{v}) \rightarrow G$ as follows. For $a \in L(\bar{v})$, $g(a) = \chi_{[a\bar{v}]}$, for $b \in R(\bar{v})$, $g(b) = \chi_{T^{-1}[\bar{v}b]}$, and for $(a, b) \in E(\bar{v})$, $g(a, b) = \chi_{[a\bar{v}b]}$. Condition (1) of Lemma 3.2 holds by induction hypothesis.

Let us check that (2) holds. Let $a \in L(\bar{v})$. One has

$$\chi_{[a\bar{v}]} = \sum_{b \in R(\bar{v}), (a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x) \quad \text{and thus} \quad g(a) = \sum_{b \in R(\bar{v}), (a,b) \in E(\bar{v})} g(a, b).$$

Similarly, let $b \in R(\bar{v})$ and $x \in X$. One has

$$\chi_{T^{-1}[\bar{v}b]}(x) = \chi_{[\bar{v}b]}(Tx) = \sum_{a \in L(\bar{v}), (a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x).$$

We conclude that for all $b \in R(\bar{v})$, $g(b) = \sum_{a \in L(\bar{v}), (a,b) \in E(\bar{v})} g(a, b)$. We now can apply Lemma 3.2 which yields that $\chi_{[a\bar{v}b]} \in H$, for any biextension $a\bar{v}b$ of \bar{v} . In particular, since $(v_0, v_n) \in E(\bar{v})$, then $\chi_{[v]} \in H$. ■

Proof. [Proof of Theorem 1.1] We assume that the dendric subshift (X, T) is balanced on the letters. Let $v \in \mathcal{L}(X)$. Let C be a constant of balancedness for the letters. Let n be a positive integer and let u, w be two factors of \mathcal{L}_X of length $n - 1$ with $n - 1 > |v|$. Pick a bi-infinite word $x \in X$ such that $u = x_{[i, i+n]}$ and $w = x_{[j, j+n]}$ for some indices $i, j \in \mathbb{Z}$. We have

$$||u|_v - |w|_v| = \left| \sum_{l=i}^{i+n-1-|v|} \chi_{[v]}(T^l x) - \sum_{l=j}^{j+n-1-|v|} \chi_{[v]}(T^l y) \right|.$$

Now, according to Lemma 3.3, for all $a \in \mathcal{A}$, let K_a be a finite subset of \mathbb{Z} such that, for all $k \in K_a$, there exists $\alpha(a, k) \in \mathbb{Z}$ verifying

$$\chi_{[v]} = \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \chi_{T^k([a])}.$$

Then,

$$\begin{aligned}
||u|_v - |w|_v| &= \left| \sum_{l=i}^{i+n-1-|v|} \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \chi_{T^k[a]}(T^l x) - \sum_{l=j}^{j+n-1-|v|} \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \chi_{T^k[a]}(T^l x) \right| \\
&= \left| \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a, k) \left(\sum_{l=i}^{i+n-1-|v|} \chi_{T^k[a]}(T^l x) - \sum_{l=j}^{j+n-1-|v|} \chi_{T^k[a]}(T^l x) \right) \right| \\
&\leq \sum_{a \in \mathcal{A}} \sum_{k \in K_a} |\alpha(a, k)| \left| \sum_{l=i}^{i+n-1-|v|} \chi_{T^k[a]}(T^l x) - \sum_{l=j}^{j+n-1-|v|} \chi_{T^k[a]}(T^l x) \right| \\
&= \sum_{a \in \mathcal{A}} \sum_{k \in K_a} |\alpha(a, k)| \left| \sum_{l=i}^{i+n-1-|v|} \chi_{[a]}(T^l(T^{-k}x)) - \sum_{l=j}^{j+n-1-|v|} \chi_{[a]}(T^l(T^{-k}x)) \right| \\
&= \sum_{a \in \mathcal{A}} \sum_{k \in K_a} |\alpha(a, k)| \cdot ||(T^{-k}x)_{[i, i+n-|v|]}|_a - |(T^{-k}x)_{[j, j+n-|v|]}|_a|.
\end{aligned}$$

Note that $(T^{-k}x)_{[i, i+n-|v|]}$ and $(T^{-k}y)_{[j, j+n-|v|]}$ are two factors of length $n - 1 - |v|$ belonging to $\mathcal{L}(X)$, and then by balancedness on the letters, for all $a \in \mathcal{A}$

$$|| (T^{-k}x)_{[i, i+n-|v|]}|_a - |(T^{-k}y)_{[j, j+n-|v|]}|_a | \leq C.$$

We obtain that $||u|_v - |w|_v| \leq |\mathcal{A}|KC$, where $K = \max_{a \in \mathcal{A}} \{ \sum_{k \in K_a} |\alpha(a, k)| \}$, which ends the proof of the balancedness on v .

Lastly, the result on additive topological eigenvalues comes from Theorem 2.8. \blacksquare

3.3 Examples

Sturmian words, Arnoux-Rauzy words (introduced in [AR91] and also called episturmian words), and codings of regular interval exchanges are typical examples of dendric words (see [BDFD⁺15a]). See also as an interesting family of dendric words, the words produced by the Cassaigne–Selmer multidimensional continued fraction algorithm [CLL17]. Note that dendric words have factor complexity $(d - 1)n + 1$ when defined over an alphabet of cardinality d (see [BDFD⁺15a]). We recall that the factor complexity of a bi-infinite word u counts the number of factors of u a given length.

Recall that Sturmian words are known to be 1-balanced on letters [Lot02]. They are also known to be balanced on factors [FV02]. Note that we also recover this property as a direct consequence of Theorem 1.1. It was believed that Arnoux-Rauzy words would be 2-balanced on letters, as generalizations of Sturmian words. But there exist Arnoux-Rauzy words that are not balanced on letters, such as proved in [CFZ00], also see [CFM08].

More precisely, Arnoux-Rauzy words are uniformly recurrent dendric words that can be expressed as S -adic words as follows. Let $\mathcal{A} = \{1, 2, \dots, d\}$. We define the set \mathcal{S}_{AR} of substitutions defined as $\mathcal{S}_{AR} = \{\sigma_i : i \in \mathcal{A}\}$, with $\sigma_i : i \mapsto i, j \mapsto ji$ for $j \in \mathcal{A} \setminus \{i\}$. A bi-infinite word $u \in \mathcal{A}^{\mathbb{Z}}$ is an *Arnoux-Rauzy word* if its language coincides with the language of a word of the form $\lim_{n \rightarrow \infty} \sigma_{i_0} \sigma_{i_1} \dots \sigma_{i_n}(1)$, where the sequence $\mathbf{i} = (i_n)_{n \geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in \mathcal{A} occurs infinitely often in $\mathbf{i} = (i_n)_{n \geq 0}$. In this latter case, the infinite word u is uniformly recurrent and we can associate with it a two-sided subshift $(X_{\mathbf{i}}, T)$ which contains all the bi-infinite words having the same language as u . Furthermore, such a sequence $\mathbf{i} = (i_n)_{n \geq 0} \in \mathcal{A}^{\mathbb{N}}$ is uniquely defined for a given u . For any given Arnoux-Rauzy word, the sequence $\mathbf{i} = (i_n)_{n \geq 0}$ is

called the \mathcal{S}_{AR} -directive word of u . All the Arnoux-Rauzy words that belong to the dynamical system $(X_{\mathbf{i}}, T)$ have the same \mathcal{S}_{AR} -directive word. An *Arnoux-Rauzy substitution* is a finite product of substitutions in \mathcal{S}_{AR} . The following statement is deduced from Theorem 1.1.

Corollary 3.4 *Let σ be a primitive Arnoux-Rauzy substitution. Then, (X_{σ}, T) is balanced on factors.*

Let $(X_{\mathbf{i}}, T)$ be an Arnoux-Rauzy subshift on a three-letter alphabet with \mathcal{S}_{AR} -directive sequence $\mathbf{i} = (i_n)_{n \geq 0}$. If there exists some constant h such that we do not have $i_n = i_{n+1} = \dots = i_{n+h}$ for any $n \geq 0$, then $(X_{\mathbf{i}}, T)$ is balanced on factors.

In both cases, frequencies of factors are additive topological eigenvalues and cylinders are bounded remainder sets.

Proof. Arnoux-Rauzy substitutions are known to be Pisot [AI01, AD15] and thus to generate Arnoux-Rauzy words that are balanced on letters by Theorem 2.5, and consequently on factors by Theorem 1.1. The condition of the second statement is proved in [BCS13] to imply that $(X_{\mathbf{i}}, T)$ is $(2h+1)$ -balanced on letters. We again conclude thanks to Theorem 1.1. ■

It is proved in [BST] that on a three-letter alphabet, a.e. Arnoux-Rauzy subshift is balanced on letters and has pure discrete spectrum in the measure-theoretic sense, that is, it is measurably conjugate to a translation on the torus \mathbb{T}^2 . Here almost everywhere (a.e.) refers to some invariant measure; as an example of such a measure, consider the measure of maximal entropy for the suspension flow of the Rauzy gasket constructed in [AHS16a] (see also [AHS16b]). As an application of Theorem 1.1, we deduce that in case of pure discrete spectrum, all cylinders provide bounded remainder sets for the underlying toral translation and that for a.e. Arnoux-Rauzy subshift, frequencies of factors are additive topological eigenvalues.

4 Balancedness for substitutions with rational frequencies

Through this section, σ will be a primitive substitution on the alphabet \mathcal{A} and (X_{σ}, T) the minimal, uniquely ergodic subshift generated by σ . Let μ denote the unique invariant probability measure on X_{σ} . We first introduce a suitable partition in towers for substitutions in Section 4.1, we then provide criteria for producing imbalancedness in Section 4.2, and lastly, we discuss several examples in Section 4.3.

4.1 Two-letter towers

Let (X, T) be a subshift. A *partition in towers* of (X, T) is a partition of the space X of the form

$$\mathcal{P} = \{T^j B_i : 1 \leq i \leq m, 0 \leq j < h_i\}$$

where the B_i 's are clopen sets (i.e., closed and open sets) and non-empty. The number m is the *number of towers* of \mathcal{P} . For all $1 \leq i \leq m$, the subset $\{T^j B_i : 0 \leq j < h_i\}$ is called the *i -th tower* of \mathcal{P} ; h_i is its *height* and B_i its *base*. The elements of the partition are called *atoms*. For an illustration, see Figure 3.

We recall below a classical description of the subshift (X_{σ}, T) in terms of Kakutani-Rohlin partitions provided by the substitution σ which will play a crucial role in the following (in particular for Proposition 4.6). We provide the proof of the following folklore result for the sake of self-containedness. An illustration of the partition \mathcal{P}_n defined below is provided in Figure 4.

Lemma 4.1 *Let σ be a primitive substitution. For all $n \in \mathbb{N}$, define*

$$\mathcal{P}_n = \{T^j \sigma^n([ab]) : ab \in \mathcal{L}_2(X), 0 \leq j < |\sigma^n(a)|\}. \quad (3)$$

The sequence $(\mathcal{P}_n)_{n \in \mathbb{N}}$ is a nested sequence of partitions in towers of (X_σ, T) , i.e., for all $n \in \mathbb{N}$, \mathcal{P}_{n+1} is finer than \mathcal{P}_n and $\bigcup_{n, ab \in \mathcal{L}_2(X_\sigma)} \sigma^{n+1}([ab]) \subset \bigcup_{n, ab \in \mathcal{L}_2(X_\sigma)} \sigma^n([ab])$.

Proof. First we show that for all $n \geq 1$, \mathcal{P}_n covers X_σ . We fix $n \geq 1$ and let $x \in X_\sigma$. By definition of X_σ , for all $\ell \geq 1$, there exist $N \geq 1$ and $a \in \mathcal{A}$ such that³ $x_{[-\ell, \ell]} \prec \sigma^N(a)$. For all ℓ large, one has $N > n$, and then $x_{[-\ell, \ell]} \prec \sigma^n(w)$ for some $w \in \mathcal{L}(X_\sigma)$. This implies that there exist $0 \leq j < |w|$ and $0 \leq k, k' \leq \max_{a \in \mathcal{A}} \{|\sigma^n(a)|\}$ such that $x_{[-\ell+k, \ell-k']} = \sigma^n(w_j)$. Since $|w| \rightarrow \infty$ as $\ell \rightarrow \infty$, a Cantor diagonal argument provides a word $y \in X_\sigma$ and an integer k with $0 \leq k < |\sigma^n(y_0)|$ such that $x = T^k \sigma^n(y)$. Setting $ab = y_0 y_1$, we have that $ab \in \mathcal{L}_2(X_\sigma)$ and $x \in T^k \sigma^n([ab])$.

We now prove that \mathcal{P}_n is a partition. Suppose that there exist $ab, cd \in \mathcal{L}_2(X_\sigma)$, with $0 \leq j < |\sigma^n(a)|$ and $0 \leq k < |\sigma^n(c)|$, such that $x \in T^j \sigma^n([ab]) \cap T^k \sigma^n([cd])$. Then, $x = T^j \sigma^n(y_1) = T^k \sigma^n(y_2)$, where $y_1 \in [ab]$, $y_2 \in [cd]$. By recognizability (we use the fact that σ is primitive), $j = k$ and $ac = bd$, so in fact $T^j \sigma^n([ab]) = T^k \sigma^n([cd])$.

Finally, let us show that \mathcal{P}_{n+1} is finer than \mathcal{P}_n . Let $T^k \sigma^{n+1}([ab])$ be an atom of \mathcal{P}_{n+1} , and let x belong to it. There exists $y \in [ab]$ such that $x = T^k \sigma^{n+1}(y)$, and therefore x belongs also to $T^k \sigma^n([cd])$, where $c = \sigma(y)_0$, $d = \sigma(y)_1$. By definition of \mathcal{P}_{n+1} , one has $0 \leq k < |\sigma^{n+1}(a)|$. If $0 \leq k < |\sigma^n(\sigma(a)_0)| = |\sigma^n(c)|$, then $T^k \sigma^n([cd])$ is an atom of \mathcal{P}_n and we conclude that $T^k \sigma^{n+1}([ab])$ is contained in an atom of \mathcal{P}_n . If $|\sigma^n(\sigma(a)_0)| \leq k < |\sigma^{n+1}(a)|$, then there is a unique j with $1 \leq j < |\sigma(a)|$ such that

$$|\sigma^n(\sigma(a)_{[0, j]})| \leq k < |\sigma^n(\sigma(a)_{[0, j+1]})|.$$

Define $m = |\sigma^n(\sigma(y)_{[0, j]})|$. We know that $T^m \sigma^n(\sigma(y)) = \sigma^n T^j(\sigma(y))$, so we conclude that

$$x = T^k \sigma^n \sigma(y) = T^{k-m} \sigma^n T^j \sigma(y) \in T^{k-m} \sigma^n([\sigma(y)_j \sigma(y)_{j+1}]).$$

Since $0 \leq k - m < |\sigma^n(\sigma(y)_j)|$ and $\sigma(y)_j \sigma(y)_{j+1} \in \mathcal{L}_2(X_\sigma)$, the subset $T^{k-m} \sigma^n([\sigma(y)_j \sigma(y)_{j+1}])$ is an atom of \mathcal{P}_n , so we conclude again that $T^k \sigma^{n+1}([ab])$ is contained in an atom of \mathcal{P}_n . ■

Remark 4.2 Note that we could have used a very similar proof to show that the sequence of partitions $(\mathcal{Q}_n)_{n \in \mathbb{N}}$ with

$$\mathcal{Q}_n = \{T^j \sigma^n([a]) : a \in \mathcal{A}, 0 \leq j < |\sigma^n(a)|\}$$

is a nested sequence of partitions in towers of (X_σ, T) , see e.g. [DHS99, Proposition 14]. However, we are not able to ensure that for every factor $v \in \mathcal{L}(X_\sigma)$, the function $f_v = \chi_{[v]} - \mu[v] \in C(X_\sigma, \mathbb{R})$, as defined in Lemma 2.9, will be constant in the atoms of \mathcal{Q}_n for all n large. Indeed, for all $n \geq 1$, the last level of any tower of \mathcal{Q}_n determines only the first letter of its elements, unless we put some additional condition on σ , like being *proper* (see [DHS99] for details). We will see in Section 4.2 that strategies to provide imbalancedness criteria relies on the fact that for any factor v we can always find a positive integer n such that f_v is constant in the atoms of \mathcal{P}_n .

Example 4.3 (Thue–Morse) We continue Example 2.3. Let σ_{TM} be the Thue–Morse substitution on $\{0, 1\}$ given by $\sigma_{TM}: 0 \mapsto 01, 1 \mapsto 10$. Let $(\mathcal{P}_n)_{n \geq 1}$ the sequence of partitions in towers defined in (3). Each \mathcal{P}_n has four towers with 2^n levels. Two elements in the same atom of \mathcal{P}_n share at least their first $2^n + 1$ letters. The partition \mathcal{P}_1 is depicted in Figure 5.

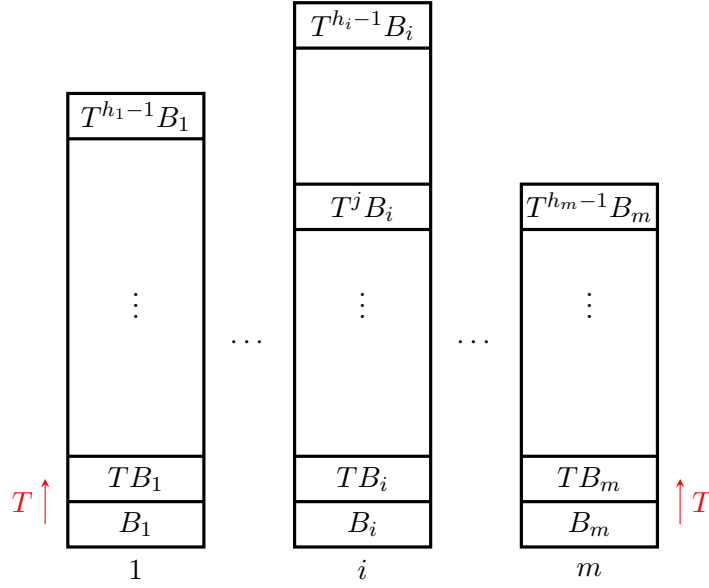


Figure 3: A partition in towers.

We end this section with the following lemma which provides a convenient expression for the entry (ab, cd) of the two-block matrix M_{σ_2} .

Lemma 4.4 *Let σ be a primitive substitution. Then, for all $ab, cd \in \mathcal{L}_2(X_\sigma)$, and for all $n \geq 1$,*

$${}^t M_{\sigma_2}(ab, cd) = \text{Card}\{0 \leq k < |\sigma^{n+1}(a)| : T^k \sigma^{n+1}([ab]) \subseteq \sigma^n([cd])\}. \quad (4)$$

Remark 4.5 Lemma 4.4 means that the matrix M_{σ_2} contains all the information for describing the transition from \mathcal{P}_n to \mathcal{P}_{n+1} , for all n . This corresponds to applying σ once. It follows easily by induction that, for all $r \geq 0$, $M_{\sigma_2}^r$ codes the transition from \mathcal{P}_n to \mathcal{P}_{n+r} and we have

$$({}^t M_2)^r(ab, cd) = \text{Card}\{0 \leq k < |\sigma^{n+r}(a)| : T^k \sigma^{n+r}([ab]) \subseteq \sigma^n([cd])\}.$$

Proof. Let $ab, cd \in \mathcal{L}_2(X_\sigma)$. We fix $n \geq 1$. We recall that $\sigma_2(ab)_j$ stands for the j th letter of $\sigma_2(ab)$ on the alphabet $\mathcal{L}_2(X_\sigma)$, with the first letter being indexed by 0. By definition, one has

$${}^t M_{\sigma_2}(ab, cd) = \text{Card}\{0 \leq j < |\sigma(a)| : \sigma_2(ab)_j = cd\} = \text{Card}\{0 \leq j < |\sigma(a)| : T^j \sigma([ab]) \subseteq [cd]\}.$$

We thus want to show that

$$\text{Card}\{0 \leq j < |\sigma(a)| : T^j \sigma([ab]) \subseteq [cd]\} = \text{Card}\{0 \leq k < |\sigma^{n+1}(a)| : T^k \sigma^{n+1}([ab]) \subseteq \sigma^n([cd])\}.$$

Suppose that there exists $0 \leq j < |\sigma(a)|$ such that $T^j \sigma([ab]) \subseteq [cd]$. If $j = 0$, $\sigma([ab]) \subseteq [cd]$ and therefore $\sigma^{n+1}([ab]) \subseteq \sigma^n([cd])$ ($k = 0$). If $1 \leq j < |\sigma(a)|$, set $k = |\sigma^n(\sigma(a)_{[0,j]})|$. One has $k < |\sigma^{n+1}(a)|$. Now take $x \in [ab]$. We have $T^k \sigma^n(\sigma(x)) = \sigma^n T^j(\sigma(x))$. By hypothesis, $T^j(\sigma(x)) \in [cd]$, and then $T^k \sigma^{n+1}(x) \in \sigma^n([cd])$. Note also that by definition the k associated with a given j is unique, so we conclude that

$$\text{Card}\{0 \leq j < |\sigma(a)| : T^j \sigma([ab]) \subseteq [cd]\} \geq \text{Card}\{0 \leq k < |\sigma^{n+1}(a)| : T^k \sigma^{n+1}([ab]) \subseteq \sigma^n([cd])\}.$$

³We recall that the notation $u \prec v$ stands for u being a factor of v .

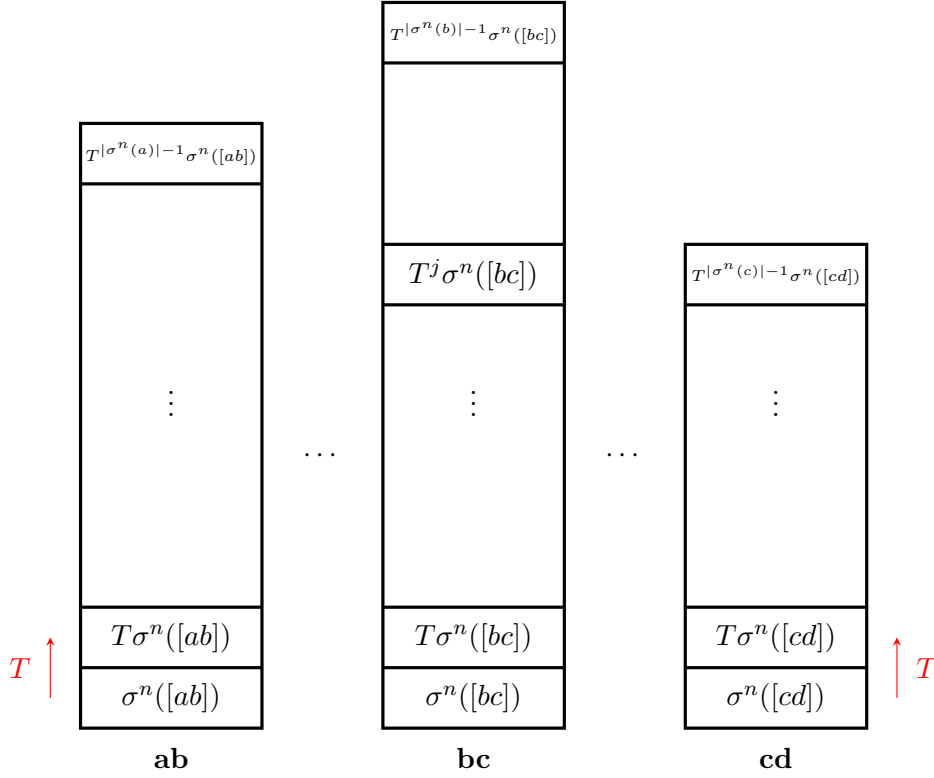


Figure 4: Partition \mathcal{P}_n for a primitive substitution σ . The last levels are mapped by T on some atoms of the base by injectivity. Note that the elements of a same last level are not necessarily mapped to the same atom of the base.

Conversely, suppose that there exists $0 \leq k < |\sigma^{n+1}(a)|$ such that $T^k \sigma^{n+1}([ab])$ is included in $\sigma^n([cd])$. Let $x \in [ab]$ and let $y = T^k \sigma^{n+1}(x)$.

We first assume $0 \leq k < |\sigma^n(\sigma(a)_0)|$. By hypothesis, there exists $z \in [cd]$ such that $y = \sigma^n(z)$. By recognizability, $k = 0$ and $\sigma(x) = z$, and thus $\sigma(x) \in [cd]$. We conclude that $\sigma([ab]) \subseteq [cd]$.

Now we assume that $|\sigma^n(\sigma(a)_0)| \leq k < |\sigma^{n+1}(a)|$. There exists a unique j with $1 \leq j < |\sigma(a)|$ such that

$$|\sigma^n(\sigma(a)_{[0,j]})| \leq k < |\sigma^n(\sigma(a)_{[0,j+1]})|.$$

Let $m = |\sigma^n(\sigma(a)_{[0,j]})|$. One has $T^m \sigma^n(\sigma(x)) = \sigma^n T^j(\sigma(x))$, and thus $y = T^k \sigma^n(\sigma(x)) = T^{k-m} \sigma^n(T^j \sigma(x))$. On the other hand, $y = T^k \sigma^{n+1}(x) \in \sigma^n([cd])$, and then, there exists $z \in [cd]$ such that $y = \sigma^n(z)$. One has $0 \leq k - m < |\sigma^n(\sigma(a)_j)|$. By recognizability, $k - m = 0$ and $T^j \sigma(x) = z \in [cd]$. We conclude that $T^j \sigma([ab]) \subseteq [cd]$. Since the integer j associated with a given k is unique, we conclude that

$$\text{Card}\{0 \leq j < |\sigma(a)| : T^j \sigma([ab]) \subseteq [cd]\} \leq \text{Card}\{0 \leq k < |\sigma^{n+1}(a)| : T^k \sigma^{n+1}([ab]) \subseteq \sigma^n([cd])\}.$$

■

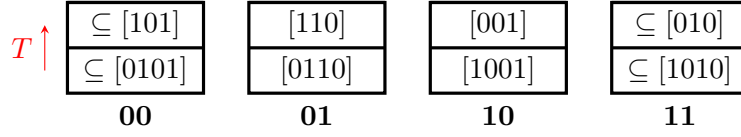


Figure 5: Partition \mathcal{P}_1 for the Thue–Morse substitution.

4.2 Some criteria for detecting imbalancedness

For any $n \geq 1$, let $R_n(X_\sigma)$ (resp. $Z_n(X_\sigma)$) be the set of maps from $\mathcal{L}_n(X_\sigma)$ to \mathbb{R} (resp. to \mathbb{Z}) and let β be the map defined as

$$\beta : R_1(X_\sigma) \rightarrow R_2(X_\sigma), \quad f \mapsto (\beta f)(ab) = f(b) - f(a) \text{ for all } ab \in \mathcal{L}_2(X_\sigma).$$

Our strategy works as follows. We consider the map $f_v = \chi_{[v]} - \mu([v]) = \chi_{[v]} - \mu_v$ such as defined in (2). We will use the fact that the map f_v is constant in the atoms of the two-letter partition \mathcal{P}_n (defined in (3)) for all n large, and associate with f_v a map $\phi_{v,n} \in R_2(X_\sigma)$, thus defined on $\mathcal{L}_2(X_\sigma)$. Proposition 4.6 first provides a convenient necessary condition on such a map, namely, it is proved to belong to $\beta(R_1(X_\sigma))$. This condition is translated in symbolic terms in Proposition 4.7, and then exploited in Theorem 1.2. Indeed, knowing that a map belongs to $\beta(R_1(X_\sigma))$ implies several convenient restrictions, for instance its coordinate on each factor of the form aa is equal to 0. The proof of Proposition 4.6 below closely follows the approach developed in [Hos95, Hos00, DHP18]. Corollary 4.10 illustrates how powerful this simple formulation can be.

Proposition 4.6 *Let σ be a primitive substitution. Let $f \in C(X_\sigma, \mathbb{Z})$ such that there exists $k \in \mathbb{N}$ for which f is constant in the atoms of \mathcal{P}_k . For all $n \geq k$, define $\phi_n \in \mathbb{R}^{\mathcal{L}_2(X_\sigma)}$ by*

$$\phi_n(ab) = \sum_{j=0}^{|\sigma^n(a)|-1} f|_{T^j\sigma^n([ab])} \quad \forall ab \in \mathcal{L}_2(X_\sigma). \quad (5)$$

Let $d = |\mathcal{L}_2(X_\sigma)|$. If f is a coboundary, then $\phi_n \in \beta(R_1(X_\sigma))$ for all $n \geq k + d$.

Proof. Let $f \in C(X_\sigma, \mathbb{Z})$ such that there exists $k \in \mathbb{N}$ for which f is constant in the atoms of \mathcal{P}_k . Suppose that f is a coboundary, that is, there exists $g \in C(X_\sigma, \mathbb{R})$ such that $f = g \circ T - g$. By Proposition 2.10, $g \in C(X_\sigma, \mathbb{Z})$, and then it is locally constant. We claim that there exists $\ell \geq k + d$ such that for all $ab \in \mathcal{L}_2(X_\sigma)$, g is constant on the set $\sigma^\ell([ab])$.

Indeed, let i be a positive integer such that for all $x \in X_\sigma$, g depends only on $x_{[-i,i]}$. Such an integer exists since g is locally constant, as observed in Section 2.4. Take ℓ large enough so that $\ell \geq k + d$ and $\min\{|\sigma^\ell(a)| : a \in A\} > i$. Since f is constant on the atoms of \mathcal{P}_k , so is it on those of \mathcal{P}_ℓ . Let $ab \in \mathcal{L}_2(X_\sigma)$ and $y, z \in \sigma^\ell([ab])$. Since $g(x)$ depends only on $x_{[-i,i]}$, $g \circ T^i(x)$ depends on $x_{[0,2i]}$. Since $y, z \in \sigma^\ell([ab])$ and $|\sigma^\ell(a)|, |\sigma^\ell(b)| > i$, y and z share the same $2i$ first coordinates and thus $g \circ T^i(y) = g \circ T^i(z)$. On the other hand, for all $0 \leq j < |\sigma^\ell(a)|$, $T^j(y)$ and $T^j(z)$ are in the same atom of \mathcal{P}_ℓ , so in particular for all $0 \leq j < i$, $T^j(y)$ and $T^j(z)$ are in the same atom of \mathcal{P}_ℓ . Since f is constant on the atoms of \mathcal{P}_ℓ , we obtain that $f^{(i)}(y) = f^{(i)}(z)$, by recalling that $f^{(i)}(x)$ stands for $f(x) + f \circ T(x) + \dots + f \circ T^j(x) + \dots + f \circ T^{i-1}(x)$. Finally, note that for all $x \in X_\sigma$ and for all $s \in \mathbb{N}$, $g(x) = g \circ T^s(x) - f^{(s)}(x)$, which implies that $g(y) = g(z)$, which ends the proof of the claim, that is, g is constant on each atom of the base.

We thus can define a map $\psi \in Z_2(X_\sigma)$ as $\psi(ab) = g(x)$ for $x \in \sigma^\ell([ab])$. Then, if $x \in \sigma^\ell([ab])$ and $T^{|\sigma^\ell(a)|}(x) \in \sigma^\ell([bc])$, we have

$$\psi(bc) - \psi(ab) = g \circ T^{|\sigma^\ell(a)|}(x) - g(x) = f^{(|\sigma^\ell(a)|)}(x) = \phi_\ell(ab). \quad (6)$$

The function ϕ_n defined in (5) can be seen as a vector in \mathbb{R}^d . In the following we refer to it indistinctly as a function belonging to $\mathbb{R}^{\mathcal{L}_2(X_\sigma)}$ or as a vector in \mathbb{R}^d .

Let $n \geq k + d$. We now want to prove that by multiplying ϕ_n by a suitable power of M_{σ_2} yields an element of $\beta(R_1(X_\sigma))$.

We recall that, given a $d \times d$ -matrix M , its eventual range \mathcal{R}_M and its eventual kernel \mathcal{K}_M are respectively

$$\mathcal{R}_M = \bigcap_{k \geq 1} M^k \mathbb{R}^d, \quad \mathcal{K}_M = \bigcup_{k \geq 1} \ker(M^k).$$

Note that $\mathbb{R}^d = \mathcal{R}_M \oplus \mathcal{K}_M$, $\mathcal{R}_M = M^d \mathbb{R}^d$ and $\mathcal{K}_M = \ker(M^d)$ (see e.g. [LM95, Chapter 7]). First observe that $\phi_n \in \mathcal{R}_{M_{\sigma_2}}$. Indeed, by following Remark 4.5, one can show that

$$\phi_n = M_{\sigma_2}^{n-k} \phi_k \in M_{\sigma_2}^{n-k} \mathbb{R}^{\mathcal{L}_2(X_\sigma)},$$

and since $n - k \geq d$ and $\mathcal{R}_{M_{\sigma_2}} = M_{\sigma_2}^d \mathbb{R}^d$, we conclude that $\phi_n \in \mathcal{R}_{M_{\sigma_2}}$.

Again thanks to Remark 4.5, and by assuming $\ell \geq n$, we obtain that, for every $abc \in \mathcal{L}_3(X_\sigma)$,

$$\psi(bc) - \psi(ab) = (M_{\sigma_2}^{\ell-n} \phi_n)(ab).$$

Choose m large enough so that $|\sigma^m(a)| \geq 2$ for every $a \in \mathcal{A}$, and define $\theta \in Z_1(X_\sigma)$ by $\theta(a) = \psi(a_1 a_2)$ for $a \in \mathcal{A}$ if $\sigma^m(a) = a_1 \cdots a_r$. If $ab \in \mathcal{L}_2(X_\sigma)$ with $\sigma^m(a) = a_1 \cdots a_r$ and $\sigma^m(b) = b_1 \cdots b_s$, we obtain

$$\begin{aligned} (M_{\sigma_2}^{\ell-n+m} \phi_n)(ab) &= (M_{\sigma_2}^{\ell-n} \phi_n)(a_1 a_2) + \cdots + (M_{\sigma_2}^{\ell-n} \phi_n)(a_r b_1) \\ &= \psi(b_1 b_2) - \psi(a_1 a_2) = \theta(b) - \theta(a) = (\beta\theta)(ab), \end{aligned}$$

and it follows that $M_{\sigma_2}^{\ell-n+m} \phi_n$ belongs to $\beta(Z_1(X_\sigma))$.

It remains to prove that $\phi_n \in \beta(R_1(X_\sigma))$. In particular, $M_{\sigma_2}^{\ell-n+m} \phi_n \in \beta(R_1(X_\sigma))$. Choosing m large enough, we can assume that $M_{\sigma_2}^{\ell-n+m} \phi_n \in \mathcal{R}_{M_{\sigma_2}}$. Since the subspace $\beta(R_1(X_\sigma))$ is invariant under M_{σ_2} and M_{σ_2} is an automorphism of $\mathcal{R}_{M_{\sigma_2}}$, we obtain that

$$M_{\sigma_2}^{\ell-n+m} : \mathcal{R}_{M_{\sigma_2}} \cap \beta(R_1(X_\sigma)) \rightarrow \mathcal{R}_{M_{\sigma_2}} \cap \beta(R_1(X_\sigma))$$

is a bijection. Therefore, there exists a unique $\varphi \in \mathcal{R}_{M_{\sigma_2}} \cap \beta(R_1(X_\sigma))$ such that

$$M_{\sigma_2}^{\ell-n+m} \phi_n = M_{\sigma_2}^{\ell-n+m} \varphi,$$

and then $\phi_n = (\phi_n - \varphi) + \varphi$ belongs to $\mathcal{K}_{M_{\sigma_2}} + \mathcal{R}_{M_{\sigma_2}} \cap \beta(R_1(X_\sigma))$. Finally, recall that $\mathbb{R}^d = \mathcal{R}_{M_{\sigma_2}} \oplus \mathcal{K}_{M_{\sigma_2}}$ and $\phi_n \in \mathcal{R}_{M_{\sigma_2}}$. This implies that $\phi_n - \varphi = 0$ and thus $\phi_n \in \mathcal{R}_{M_{\sigma_2}} \cap \beta(R_1(X_\sigma)) \subseteq \beta(R_1(X_\sigma))$. \blacksquare

We now translate the previous proposition in terms of balancedness for substitutive symbolic systems having rational frequencies.

Proposition 4.7 *Let σ be a primitive substitution. Let $v \in \mathcal{L}_\sigma$ having a rational frequency μ_v and $f_v = \chi_{[v]} - \mu_v \in C(X_\sigma, \mathbb{R})$. There exists $k \in \mathbb{N}$ be such that f_v is constant in the atoms of the two-letter partition \mathcal{P}_k . If (X_σ, T) is balanced on v , then $\phi_{v,b} \in \beta(R_1(X))$ for all $n \geq k + d$, where $d = \text{Card } \mathcal{L}_2(X)$ and $\phi_{v,n}$ is defined as in (5), i.e.,*

$$\phi_{v,n}(ab) = \sum_{j=0}^{|\sigma^n(a)|-1} f_v |T^j \sigma^n([ab])| \quad \forall ab \in \mathcal{L}_2(X_\sigma).$$

Proof. We write $\mu_v = p_v/q_v$ in irreducible form. For all $n \geq 0$, the two-letter partition \mathcal{P}_n (as defined in (3)) verifies that all elements in any atom of \mathcal{P}_n share at least their $L_n + 1$ letters, where $L_n = \min\{|\sigma^n(a)| : a \in \mathcal{A}\}$. Therefore, for all k large, f_v (and consequently $q_v \cdot f_v$) is constant in the atoms of \mathcal{P}_k . By Lemma 2.9, since (X_σ, T) is balanced in v , f_v is a coboundary, and then so is $q_v \cdot f_v$. By Proposition 4.6, $q_v \cdot \phi_n \in \beta(R_1(X))$ for all $n \geq k + d$, and consequently $\phi_n \in \beta(R_1(X))$ for all $n \geq k + d$. \blacksquare

We now deduce from Proposition 4.7 necessary conditions for balancedness. Let (X, T) be a minimal symbolic system on the alphabet \mathcal{A} and let $a \in \mathcal{A}$. We recall that a word w with $wa \in \mathcal{L}(X)$ is a *return word* to the letter a if a is a prefix of wa .

Lemma 4.8 *Let (X, T) be a minimal symbolic system defined on the alphabet \mathcal{A} , $a \in \mathcal{A}$ and $w = w_0 \cdots w_{|w|-1}$ be a return word to a . If $\phi \in \beta(R_1(X))$, then*

$$\phi(w_{|w|-1}a) + \sum_{i=1}^{|w|-1} \phi(w_{i-1}w_i) = 0.$$

Proof. One has $w_0w_1, w_1w_2, \dots, w_{|w|-2}w_{|w|-1}, w_{|w|-1}a \in \mathcal{L}_2(X)$. The result follows directly from the definition of return words and from the fact that there exists $\varphi \in R_1(X)$ such that $\phi = \beta\varphi$. \blacksquare

We now can prove Theorem 1.2.

Proof. By Proposition 4.7, $\phi_{v,n} \in \beta(R_1(X_\sigma))$ for all n large. For any $ab \in \mathcal{L}_2(X_\sigma)$

$$\phi_n(ab) = \alpha_{ab} \left(1 - \frac{p_v}{q_v}\right) - (|\sigma^n(a)| - \alpha_{ab}) \cdot \frac{p_v}{q_v}, \quad (7)$$

where

$$\alpha_{ab} = \text{Card}\{0 \leq j < |\sigma^n(a)| : T^j \sigma^n([ab]) \subseteq [v]\},$$

that is, α_{ab} is the number of levels in the ab -tower of \mathcal{P}_n in which all elements begin with the word v . Using Lemma 4.8 and (7), we obtain

$$\begin{aligned} 0 &= \alpha_{w_{|w|-1}a} (q_v - p_v) - (|\sigma^n(w_{|w|-1})| - \alpha_{w_{|w|-1}a}) \cdot p_v + \\ &\quad \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_i} (q_v - p_v) - (|\sigma^n(w_{i-1})| - \alpha_{w_{i-1}w_i}) \cdot p_v \end{aligned}$$

which implies

$$\begin{aligned} q_v \left(\alpha_{w_{|w|-1}a} + \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_i} \right) &= p_v \left(|\sigma^n(w_{|w|-1})| + \sum_{i=1}^{|w|-1} |\sigma^n(w_{i-1})| \right) \\ &= p_v |\sigma^n(w)|. \end{aligned}$$

The integers p_v and q_v being coprime, either

$$\left(\alpha_{w_{|w|-1}a} + \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_i} \right) = 0$$

or q_v divides $|\sigma^n(w)|$. Since $|\sigma^n(w)| \neq 0$, we conclude that q_v divides $|\sigma^n(w)|$, which ends the proof of the first assertion.

We now prove the second assertion of Theorem 1.2. Let $a \in \mathcal{A}$ and assume that there exist b, c such that bac belongs to $\mathcal{L}_3(X_\sigma)$ and $bc \in \mathcal{L}_2(X_\sigma)$. Since $\phi_{v,n} \in \beta(R_1(X_\sigma))$ and $ba, ac, bc \in \mathcal{L}_2(X_\sigma)$, one has $\phi_n(ba) + \phi_n(ac) = \phi_n(bc)$, that is,

$$\begin{aligned} 0 &= \alpha_{ba}(q_v - p_v) - p_v(|\sigma^n(b)| - \alpha_{ba}) + \alpha_{ac}(q_v - p_v) - p_v(|\sigma^n(a)| - \alpha_{ac}) \\ &\quad - \alpha_{bc}(q_v - p_v) + p_v(|\sigma^n(b)| - \alpha_{bc}) \\ &= (\alpha_{ba} + \alpha_{ac} - \alpha_{bc})q_v - p_v|\sigma^n(a)|. \end{aligned}$$

The integers p_v and q_v being coprime, either $\alpha_{ba} + \alpha_{ac} - \alpha_{bc} = 0$ or q_v divides $|\sigma^n(a)|$. Here again $\alpha_{ba} + \alpha_{ac} - \alpha_{bc} \neq 0$, since $|\sigma^n(a)| \neq 0$, hence q_v divides $|\sigma^n(a)|$. ■

Remark 4.9 Note that Proposition 4.7 gives us the smallest n for which the conclusions of both parts of Theorem 1.2 are always true. It corresponds to $n = k + d$ and thus it can be determined in an effective way. See Example 4.12 below for an application.

As a consequence of the previous theorem, we have the following corollary about the Thue–Morse substitution.

Corollary 4.10 (Thue–Morse substitution) *Let σ_{TM} be the Thue–Morse substitution on $\{0, 1\}$ given by $\sigma_{TM}: 0 \mapsto 01, 1 \mapsto 10$. The subshift $(X_{\sigma_{TM}}, T)$ is balanced on letters but it is unbalanced on any factor of length ℓ , with $\ell \geq 2$.*

Proof. From [Dek92, Theorem1], we know that the frequency μ_v of a factor v of length $\ell \geq 2$ verifies $\mu_v = \frac{1}{6}2^{-m}$ or $\mu_v = \frac{1}{3}2^{-m}$, where m is such that $2^m < \ell \leq 2^{m+1}$. Frequencies are then rational, $p_v = 1$, and $q_v \in \{3 \cdot 2^{m+1}, 3 \cdot 2^m\}$. Note that 00 belongs to $\mathcal{L}_2(X_{\sigma_{TM}})$. We then apply the first condition of Theorem 1.2. ■

We also deduce from Theorem 1.2 the following.

Corollary 4.11 *Let σ be primitive substitution of constant length ℓ over the alphabet \mathcal{A} of cardinality d such that its substitution matrix is symmetric and d is coprime with ℓ , or does not divide ℓ^n , for all n large. If there exists a letter a and a return word w to a such that d is coprime with $|w|$. Then, (X_σ, T) is not balanced on letters.*

Proof. The substitution matrix M_σ admits as left eigenvector (and thus as right eigenvector) associated with the eigenvalue ℓ the vector with coordinates all equal to 1. One thus has $\mu_a = 1/d$ for all a and we apply the first part of Theorem 1.2. ■

4.3 Examples

Example 4.12 (Chacon substitution) The primitive Chacon substitution σ_C is defined over the alphabet $\{1, 2, 3\}$ by $\sigma_C: 1 \mapsto 1123, 2 \mapsto 23, 3 \mapsto 123$. The spectrum of M_{σ_C} is $\{3, 1, 0\}$. We cannot apply directly Theorem 2.5. The letter frequency vector is $(1/3, 1/3, 1/3)$ and then

$q_1 = q_2 = q_3 = 3$. One has $11 \in \mathcal{L}_2(X_{\sigma_C})$, and then, for every $a \in \{1, 2, 3\}$, if the system is balanced on a , 3 divides $|\sigma_C^n(1)|$ for all $n \geq k + d$ (see Proposition 4.7 for notation and Remark 4.9). In this case, it is enough to take $k = 1$; moreover one has $d = 5$, so that 3 divides $|\sigma_C^6(1)|$. But $|\sigma_C^6(1)| = 1093$, which is not divisible by 3. We conclude that (X_{σ_C}, T) is neither balanced on letters, nor balanced on factors of a given size, by Proposition 2.1. In view of Theorem 1.1, this is consistent with the fact that (X_{σ_C}, T) is weakly mixing, that is, it admits no non-trivial topological eigenfunction [PF02, Lemma 5.5.1].

Example 4.13 (Toeplitz substitution) The Toeplitz substitution, also called Period doubling substitution, is defined over $\{0, 1\}$ as $\sigma_T: 0 \mapsto 01, 1 \mapsto 00$. The spectrum of M_{σ_T} is $\{2, -1\}$. We cannot apply directly Theorem 2.5. The frequencies of letters are $1/3$ and $2/3$ and then $q_0 = q_1 = 3$. One has $00 \in \mathcal{L}_2(X_{\sigma_C})$, so if the system is balanced on $a \in \{0, 1\}$, 3 divides $|\sigma_T^n(0)|$ for all n large. Since for all $n \geq 1$, $|\sigma_T^n(0)| = 2^n$, which is not divisible by 3, (X_{σ_T}, T) is neither balanced on letters, by applying the first condition of Theorem 1.2, nor balanced on factors of a given size, by Proposition 2.1.

Example 4.14 Consider the substitution $\sigma: 0 \mapsto 11, 1 \mapsto 21, 2 \mapsto 10$. The spectrum of M_σ is $\{2, -1/2(\sqrt{3}i + 1), 1/2(\sqrt{3}i - 1)\}$. Once again, we cannot apply directly Theorem 2.5. The frequencies of letters are $(1/7, 4/7, 2/7)$ and 11 is in its language. We apply the first condition of Theorem 1.2 to deduce that (X_{σ_T}, T) is not balanced.

Example 4.15 Consider the substitution $\sigma: 0 \mapsto 010, 1 \mapsto 102, 2 \mapsto 201$ from [Que10, Example 6.2]. The spectrum of M_σ is $\{3, 1, 0\}$. We can neither apply Theorem 2.5, nor the criteria of Theorem 1.2. The letter frequency vector is $(1/2, 1/3, 1/6)$ and $q_0 = 2, q_1 = 3, q_2 = 6$. No factor of the form aa appears in $\mathcal{L}(X_\sigma)$. For instance, the word $w = 01$ is a return word to 0, and $|\sigma^n(w)| = 2 \cdot 3^n$ for all n , which provides no contradiction with q_a divides $|\sigma^n(w)|$. In fact the substitution σ is balanced on letters. Indeed, consider the substitution τ on the alphabet $\{a, b\}$ defined by $\tau: a \mapsto aab, b \mapsto aba$, and the morphism φ from $\{a, b\}^*$ to $\{0, 1, 2\}^*$ defined by $\varphi(a) = 01$ and $\varphi(b) = 02$. One has $\sigma \circ \varphi = \varphi \circ \tau$ which implies that $\sigma^\infty(0) = \varphi(\tau^\infty(a))$. The substitution τ is balanced on letters since the spectrum of M_τ is $\{2, 0\}$ which implies that σ is balanced on letters.

Example 4.16 Consider the substitution $\sigma: 0 \mapsto 001, 1 \mapsto 101$. The spectrum of its substitution matrix is $\{1, 3\}$. The frequencies of its letters are $\mu_0 = \mu_1 = 1/2$ and 00 is a factor. We deduce from Corollary 4.11 that it is not balanced on letters.

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