On decision problems for substitutions in symbolic dynamics
⋆

Valérie Berthé[1] [0000−0001−5561−7882]

Université de Paris, IRIF, CNRS, F-75013 Paris, France
berthe@irif.fr

Abstract. In this survey, we discuss decidability issues for symbolic dynamical systems generated by substitutions. Symbolic dynamical systems are discrete dynamical systems made of infinite sequences of symbols, with the shift acting on them. Substitutions are simple rules that replace letters by string of letters and allow the generation of infinite words. We focus here on symbolic dynamical systems that are generated by infinite compositions of substitutions, allowing to go beyond the case of the iteration of a single substitution. This is the so-called S-adic framework. Motivated by decidabillty and ergodic questions, we focus on questions dealing with the convergence of products of nonnegative matrices and associated Lyapounov exponents.

Keywords: Substitutions · Symbolic dynamics · Decidability · Lyapunov exponents · Primitive matrix · Perron–Frobenius theorem

1 Introduction

Discrete dynamical systems Dynamical systems describe the evolution of systems over time: a dynamical system is endowed with an evolution rule that describes the time dependence of the state of the elements of the system, allowing to bring out a global average behaviour. One can thus model the evolution of a system whose components interact in a simple way. In physics, this is for instance a set of particles whose state obeys differential equations involving time derivatives, with past determining the future. Modeling using dynamical systems has largely proved its relevance in physics and engineering, and also for numerous phenomena from the digital world, in deep connection with algorithmics: dynamical systems can model the execution of an algorithm, as well as a loop in a program.

In a digital framework, discrete systems arise naturally. More precisely, a discrete (time) dynamical system is defined as the action of a map T acting on a phase space X (usually assumed to be compact), where the rule T governs the discrete time evolution of states in X. Note that the iterative nature of dynamical

⋆ This work was supported by the Agence Nationale de la Recherche through the project Codys (ANR-18-CE40-0007).

systems is particularly well adapted to model executions of algorithms. One then studies the evolution of the system in discrete time steps: at time n, we consider the nth iterate T^n. The evolution of the system when starting with the initial condition $x \in X$ is described by the *orbit* or *the trajectory* $(X, Tx, T^2, \cdots, T^n)$ of x. Discrete dynamical systems can be of a geometric nature (e.g., $X = [0, 1]$), or of a symbolic nature (e.g., $X = \{0, 1\}^{\mathbb{N}}$). Consider for instance cellular automata or, here, symbolic dynamical systems, with discrete dynamical systems made of infinite sequences of symbols.

Reachability vs. statistical properties of orbits Understanding the behavior of orbits allows the global understanding of a dynamical system (X, T). Two types of questions arise naturally in this context. Reachability problems deal with the question of knowing whether an orbit will enter a given subregion Z of X or even reach a given point, whereas the understanding of the long-term behavior allows one to answer the following: will a trajectory visit infinitely often Z (recurrence) and how long will it stay in the subregion Z?

This second type of questions is handled through the use of ergodic theory that describes the long-term statistical behaviour of dynamical systems. An ergodic system is such that the time spent by a trajectory in some region is proportional to the volume of this region; it has the same behavior averaged over time as averaged over all the space. (For a more precise statement, see Section 2.)

For linear dynamical systems given by the action of a square matrix with rational entries, reachability problems in the framework of the orbit problem are known to lead to famous number-theoretic undecidable problems, such as the Skolem problem (does a trajectory hit a hyperplane?), the Kannan-Lipton problem [27] (does it hit a point?) and its variants in terms of positivity and ultimate positivity. The synthesis of invariants providing certificates of non-reachability (i.e., invariant sets under the action of T that contain a point x and not y) has opened the way to a large set of applications in verification, control theory, program analysis, etc. See e.g. [22] for the synthesis of semialgebraic invariants.

Substitutions and positiveness We focus in the present survey on decision questions that fall within the scope of the ergodic study of the long-term behaviour of orbits. They will conduct us to questions that are related to the existence of occurrences of positive products of matrices.

Dynamically, our main object of study here are substitutions and the words and symbolic dynamical systems they generate. Symbolic dynamical systems are defined on sets of symbols and words with the shift acting on them: the shift is the operation that deletes the first letter of an infinite word. A substitution is a rule, either combinatorial or geometric, that replaces a letter by a string of letters on a finite alphabet, or a tile by a geometric pattern. Iterating substitutions enables the generation of hierarchically ordered structures (infinite words, subshifts, point sets, tilings) that display strong self-similarity properties; one of the best known examples is the Penrose tiling. Substitutions are generalized as S-adic dynamical systems via a nonstationary (i.e., time inhomogeneous) setting which consists in iterating sequences of substitutions, and not only a single one [8].

For the sake of clarity, we focus here on substitutions acting on words which are more elementary in nature. These simple algorithmic constructions thus produce infinite words and symbolic dynamical systems whose study involves combinatorics on words, ergodic theory, spectral theory, geometry of tilings, Diophantine approximation, number theory, aperiodic order, harmonic analysis, and so on. There exist analogue notions of substitutions and S-adic systems defined on tilings and point sets, and acting as inflation/subdivision rules; see e.g. [34].

Simple refers here to the combinatorial notion of factor complexity of an infinite word with values in a finite alphabet, which counts the number of factors (i.e., strings of consecutive letters that occur in this infinite word) of a given length. This gives an indication of the degree of randomness of this infinite word. The S-adic systems cover a wide class of symbolic dynamical systems with at most linear factor complexity. They were in fact introduced as models for such systems; this is the so-called S-adic conjecture [35,8].

A substitution can efficiently be understood through its incidence matrix (this is somehow the analogue of the incidence matrix of a graph or an automaton). This linear viewpoint enables to exploit a very fruitful dictionary between actions of linear maps and symbolic dynamics, through the matrix/substitution relation produced by the fact that a matrix is the abelianized linear version of a substitution. In particular when this matrix turns out to be primitive, that is, when it admits a power with positive entries, this yields for substitutive subshifts strong ergodic and combinatorial properties, as well as particularly convenient tools for decision problems. Indeed, substitutions are particular cases of free group morphisms (they are free monoid morphisms), with the main simplification being that we have no problem of cancellations. In terms of associated matrices, this gives matrices with nonnegative entries and Perron–Frobenius theory enters into play. However, for the study of S-adic expansions, we do not consider powers of a single matrix but infinite products of nonnegative matrices, and there is no clear analogue of Perron-Frobenius theorem for infinite products of matrices. One relevant ergodic strategy is to go through the theory of Lyapunov exponents at the cost that results are given in a metric way, almost everywhere (see as an illustration Theorem 1). A crucial property in this setting is the existence of occurrences of positive blocks of matrices.

If decision problems for substitutive shifts are well investigated (see Section 3.1), in particular thanks to the notion of primitivity, the situation is by nature less effective in the S-adic setting. This survey suggests ways to extend in terms of decidability the substitutive case to the S-adic case (see Section 3.2) and asks several natural decision questions (see Section 3.3). We also allude to connections with continued fractions, as a source of related questions in Section 3.4.

From symbolic to arithmetic dynamics Let us end this introduction by presenting some elements of motivation for the study of S-adic systems. Symbolic dynamical systems come as codings of trajectories of points in a dynamical system according to a given partition and they also occur in a natural way in arithmetics for instance for the representation of numbers. Symbolic dynamics originates in the work of J. Hadamard in 1898, through the study of geodesics on surfaces of negative curvature. Since its inception, symbolic dynamics has gone hand in hand with substitutions which appeared for the first time in papers of A. Thue from 1906 and 1912, in particular with the study of the Thue-Morse word. Symbolic dynamics and Sturmian words were developed by Morse and Hedlund in the 40's [32]. Substitutions then turned out to yield unexpected prominent outcomes in the study of quasicrystals and tilings in the framework of aperiodic order. Aperiodic order refers to the mathematical formalization of quasicrystals, initiated by Y. Meyer. Since their discovery in 1982 (for which Nobel prize was awarded to D. Shechtman in 2011), substitutions and aperiodic tilings have proved to be at the heart of their study [5].

One motivation for developing the S-adic formalism comes from the study of algorithms of an arithmetic nature related to specific numeration systems and their applications, running from number theory to cryptography or computer arithmetic. Indeed, in many examples of digital representations, the digits of expansions are produced step by step by the iteration of a transformation and their ergodic study yields information on their digit distribution. This is for instance the case of decimal expansions, beta-numeration, or continued fractions (see e.g. [9, Chapter 2]). As an illustration, consider the dynamical system producing the q-ary expansions of positive real numbers defined as $([0,1], T_q)$, with $T_q\colon [0,1] \to [0,1],\ x \mapsto qx - \lfloor qx \rfloor = qx \pmod 1$. Indeed, if $x = \sum_{i \geq 1} a_i q^{-i}$, then $a_i = \lfloor q T_q^{i-1}(x) \rfloor$, for $i \geq 1$. When the base q is replaced by some more complicated base β (consider e.g. an algebraic base like the golden ratio $\beta = \frac{1+\sqrt{5}}{2}$), this allows one to expand real numbers in $[0,1]$ under the form of (possibly infinite) sums of negative powers of β of the form $\sum_{i \geq 1} \varepsilon_i \beta^{-i}$. This is called the beta-numeration.

Algorithms of an arithmetic nature can often be decomposed as a succession of algorithmic steps, which in turn can be modeled by dynamical systems. Kronecker translations $x \mapsto x + \alpha$ modulo 1 provide analogues of the addition, whereas positive entropy systems such as beta-numeration $x \mapsto \beta x$ modulo 1 mirror multiplication. This thus requires a specific nonstationary modeling which consists in working with iterated sequences of transformations that are drawn according to a further dynamical system. More precisely, we do not iterate always the same map T, but the map T can be changed with respect to time: one thus iterates a sequence T_{i_n} of maps acting on phase spaces X_{i_n}. One important feature is the order composition $T_{i_1} \circ \cdots \circ T_{i_n}$. Such a time-inhomogeneous formalism is inspired by the well-studied setting of random dynamics (including random Markov chains and random products of matrices). Symbolic codings of such transformations yields the S-adic formalism.

This survey is organized as follows. Definitions and terminology are gathered in Section 2. Specific questions raised by decision issues are then discussed in Section 3.

2 General definitions for substitutive dynamical systems

Ergodic theory Ergodic theory studies the long-term average behavior of dynamical systems. One of its main tools is Birkhoff's ergodic theorem which asserts the existence of a time average along each trajectory, provided that (X, T) is endowed with a T-invariant and ergodic measure μ. More precisely, a *measure-theoretic dynamical system* is a dynamical system endowed with an invariant measure $(X, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the σ-algebra $\mathcal{B}$, and $T : X \to X$ is a measurable map which preserves the measure μ, that is, $\mu(T^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. The measure μ is said to be T*-invariant*. It is said *ergodic* if for every set $B \in B$, $T^{-1}(B) = B$ has either zero or full measure.

Birkhoff's ergodic theorem states that the time average, that is, $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x)$ for some observable function f, is the same for almost all initial points x, and is equal to the space average, i.e., $\int_X f\, d\mu$, almost everywhere. The terms almost all and almost everywhere refer to a set of points of X of full measure μ. The ergodic theorem can thus be stated as

$$\forall f \in L^1(X, \mathbb{R}) \ , \quad \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) \xrightarrow[n\to\infty]{} \int_X f\, d\mu \quad \mu - a.e.$$

If one takes for f the characteristic function of some subspace Y of X, one deduces that, for almost all trajectories, the proportion of the time spent in Y is equal to the size of Y divided by the size of X (that is, $\mu(Y)/\mu(X)$). As a special case of the ergodic theorem, consider equidistribution theory for sequences defined on the unit interval.

Symbolic dynamics Symbolic dynamics offers the advantage of working with coded trajectories, together with combinatorial and topological methods. Consider a finite set $\mathcal{A}$, and let shift map S act on the set $\mathcal{A}^{\mathbb{N}}$ of infinite words with values in the alphabet $\mathcal{A}$ as $S((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}$. (Note that all the notions and results here hold also for biinfinite words in $\mathcal{A}^{\mathbb{Z}}$.) Here, the set $\mathcal{A}^{\mathbb{N}}$ is equipped with the product topology of the discrete topology on each copy of $\mathcal{A}$. Thus, this set is a compact space. This topology is the topology defined by the following distance: for $u \neq v \in \mathcal{A}^{\mathbb{N}},\ d(u, v) = 2^{-\min\{n\in\mathbb{N};\ u_n\neq v_n\}}$. Thus, two infinite words are close to each other if their first terms coincide.

A *subshift* is a closed shift invariant subset of some $\mathcal{A}^{\mathbb{N}}$ for $\mathcal{A}$ finite alphabet. A *factor* of an infinite word u is a string of consecutive letters that occurs in u. The *factor complexity* of an infinite word is the function that counts the number of factors of a given length that occur in it. The set of factors $\mathcal{L}_u$ of an infinite word u is called its *language*. This definition extends to the language of a subshift (X, S): this is the set of factors of infinite words in X. One important feature of a subshift is that it is defined by its language, since it is closed and shift-invariant. Subshifts of *finite type* are then defined as the subshifts whose set of words (in their language) is finite. *Sofic subshifts* are images of subshifts of finite type under a factor map, where a factor map $\pi : X \to Y$ between two subshifts X and Y is a continuous, surjective map such that $\pi \circ S = S \circ \pi$. A subshift (X, S) is *minimal* if X has no nontrivial closed shift-invariant subset; then, all the infinite words in X have the same language.

The *frequency* of a letter i in an infinite word u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0 u_1 \cdots u_{n-1}$ divided by n. Frequencies of factors are defined analogously. Let (X, S) be a minimal subshift. For a given (finite) word w of the language of X, the cylinder $[w]$ is the set of infinite words in X that have w as a prefix, i.e., $[w] = \{v \in X; v_0 \dots v_{n-1} = w\}$. If μ is an ergodic measure on u, then we deduce from Birkhoff's ergodic theorem that, for μ-almost every infinite word in X, any w has frequency $\mu([w])$. If the shift (X, S) is uniquely ergodic (i.e., there exists a unique shift-invariant probability measure on X), then the unique invariant measure on X is ergodic and the convergence in Birkhoff's ergodic theorem holds uniformly for every infinite word in X. For more details on invariant measures and ergodicity, we refer to [36] and [9, Chap. 7].

Primitive substitutions A *substitution* σ is an application from an alphabet $\mathcal{A}$ into the set of nonempty finite words on $\mathcal{A}$; it extends to a morphism of the free monoid $\mathcal{A}^*$ by concatenation, that is, $\sigma(ww') = \sigma(w)\sigma(w')$. It also extends in a natural way to a map defined over $\mathcal{A}^{\mathbb{N}}$. The substitutive symbolic dynamical system (X_σ, S) generated by σ is then defined as the set of infinite words w such that there exists a letter a and a nonnegative integer n such that w is a factor of $\sigma^n(a)$.

Substitutions are very efficient tools for producing infinite words. As an example, consider the substitution σ on $\mathcal{A} = \{a, b\}$ defined by $\sigma(a) = ab$ and $\sigma(b) = a$. Then, the sequence of finite words $(\sigma^n(a))_n$ starts with $\sigma^0(a) = a$, $\sigma^1(a) = ab$, $\sigma^2(a) = aba$, $\sigma^3(a) = abaababa, \dots$ Each $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$, and the limit word in $\mathcal{A}^{\mathbb{N}}$ is

$$abaababaabaababaababaabaababaabaababaababaabaababaababaabaab\dots$$

The above limit word is called the *Fibonacci word* (for more on the Fibonacci word, see e.g. [31,35]).

For $i \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, let $|w|_i$ stand for the number of occurrences of the letter i in the word w. Let d stand for the cardinality of $\mathcal{A}$. The *incidence matrix* M_σ of the substitution σ is the square matrix with entries $|\sigma(j)|_i$. This is a commutative linear version of the substitution σ and it offers the advantage to bring all the strength of Perron-Frobenius theorem for nonnegative matrices (see e.g. [39]). A substitution is said *primitive* if there exists a power of its incidence matrix whose entries are all positive. According to Perron–Frobenius theorem, if a substitution is primitive, then its incidence matrix admits a dominant eigenvalue (it dominates strictly in modulus the other eigenvalues) that is (strictly) positive. It is called its *Perron–Frobenius eigenvalue*.

The dynamical system (X_σ, S) associated with a primitive substitution σ can be endowed with a natural shift-invariant Borel probability measure μ defined by its values on the cylinders. The measure of the cylinder $[w]$ is defined as the frequency of the finite word w in any infinite word of X_σ, which is proved to exist, with a proof based again on Perron–Frobenius theorem [36].

Primitive substitutions have numerous interesting properties: the subshift (X_σ, S) is minimal, linearly recurrent, uniquely ergodic and any of its elements has at most linear factor complexity. (An infinite word u is said to be *linearly recurrent* if there exists a constant C such that $R(n) \leq Cn$, for all n.) For more details, see [36].

***S*-adic words** An S-adic dynamical system is defined in terms of a sequence of substitutions. Let $\mathcal{S}$ be a set of substitutions. Let $s = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, with $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, be a sequence of substitutions, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters with $a_n \in \mathcal{A}_n$ for all n. We say that the infinite word $u \in \mathcal{A}^{\mathbb{N}}$ admits $((\sigma_n, a_n))_n$ as an *S-adic representation* if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n).$$

The sequence s is called the *directive sequence* and the sequences of letters $(a_n)_n$ will only play a minor role compared to the directive sequence. If the set $\mathcal{S}$ is finite, it makes no difference to consider a constant alphabet (i.e., $\mathcal{A}_n^* = \mathcal{A}^*$ for all n and for all substitution σ in $\mathcal{S}$). Note that the S-adic dynamical system of a periodic directive sequence $(\sigma_0, \cdots, \sigma_{n-1})^\infty$ is equal to the substitutive dynamical system generated by the substitution $\sigma_0 \circ \cdots \circ \sigma_{n-1}$.

To be "S-adic" is not an intrinsic property of an infinite word, but a way to construct it and an infinite word admits many possible S-adic representations [8]. But some S-adic representations might be useful to get information about the word. One thus adds the following requirement. An S-adic representation defined by the directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ is *everywhere growing* if for any sequence of letters $(a_n)_n$, one has

$$\lim_{n\to+\infty} |\sigma_{[0,n)}(a_n)| = +\infty,$$

with $\sigma_{[0,n)} := \sigma_0 \circ \sigma_{k+1} \ldots \circ \sigma_{n-1}$.

Given an everywhere growing directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ of substitutions that are all defined over the same finite alphabet $\mathcal{A}$, the *subshift* associated with $(\sigma_n)_n$ is defined as the set of infinite words whose set of factors is included in some $\sigma_{[0,n)}(i)$, for some $i \in \mathcal{A}$.

As a prominent example, consider Sturmian words for which the Fibonacci word is a particular case. The substitutions τ_a and τ_b are defined over the alphabet $\mathcal{A} = \{a, b\}$ by $\tau_a \colon a \mapsto a, b \mapsto ab$ and $\tau_b \colon a \mapsto ba, b \mapsto b$. Let $(i_n) \in \{a, b\}^{\mathbb{N}}$. The following limits

$$u = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1} \cdots \tau_{i_{n-1}}(a) = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1} \cdots \tau_{i_{n-1}}(b) \tag{1}$$

exist and coincide whenever the directive sequence $(i_n)_n$ is not ultimately constant (it is easily shown that the shortest of the two images by $\tau_{i_0}\tau_{i_1}\dots\tau_{i_{n-1}}$ is a prefix of the other). The infinite words thus produced belong to the class of Sturmian words, and a *Sturmian word* is an infinite word whose set of factors coincides with the set of factors of a sequence u of the form (1), with the sequence $(i_n)_{n\geq 0}$ being not ultimately constant.

We have seen that the notion of primitivity plays an important role for substitutions. There are two ways of extending this notion in the S-adic setting. An S-adic expansion with directive sequence $(\sigma_n)_n$ is said *weakly primitive* if, for each n, there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive. It is said *strongly primitive* if the set of substitutions $\{\sigma_n\}$ is finite, and if there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive, for each n.

The following statement from [16] illustrates the role of primitivity in the S-adic context: if a directive sequence is weakly primitive, then the associated shift is minimal. If it is strongly primitive, the associated shift is minimal, uniquely ergodic, and it has at most linear factor complexity. With an extra condition of properness one even obtains the following characterization of linear recurrence [16]: a subshift (X, S) is linearly recurrent if and only if it is a strongly primitive and proper S-adic subshift. (A substitution over $\mathcal{A}$ is said *proper* if there exist two letters $b, e \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $\sigma(a)$ begins with b and ends with e. An S-adic system is said to be *proper* if the substitutions in $\mathcal{S}$ are proper.) An essential ingredient in the proofs of these results is the uniform growth of the matrices $M_{(0,n)}$ as for substitutive systems.

Discrepancy and balancedness properties Ergodic deviations control the convergence of ergodic sums and measure the difference between $1/n\sum_{k=0}^{n-1} f \circ T^k(x)$ and the expected value $\int f d\mu$. Of course, ergodic deviations depend on the nature of the dynamical system and on the regularity of f. If f is the indicator function of some given subset of X, this corresponds to the classical notion of discrepancy in equidistribution theory. Discrepancy measures the difference between the actual number of points in a subset and the expected number of points. This makes particularly sense in a metric number-theoretic framework for Kronecker sequences $(n\alpha)_n \mod 1$. Low discrepancy sequences are widely used in the Monte Carlo method and dynamical systems may be applied to generate higher-dimensional low-discrepancy sequences.

In symbolic dynamics, ergodic deviations measure the convergence toward frequencies, via symbolic discrepancy. Counting frequencies of words in a given symbolic dynamical system (or in a tiling space) is among the most fundamental questions of the field. Recurrence (repetitiveness for tilings and point-sets) is a closely related fundamental notion of order which describes how often a finite given configuration occurs (see e.g. [9, Chap. 7]).

The symbolic version of discrepancy is defined as follows [1]. Let $u \in \mathcal{A}^{\mathbb{N}}$ be an infinite word and assume that each letter i has frequency f_i in u. The *letter discrepancy* of u is $\Delta_n(u) = \sup_{i\in\mathcal{A}} ||u_0u_1\dots u_{n-1}|_i - nf_i|$. It can also be defined by general factors (not only letters).

Symbolic letter discrepancy is closely related to balancedness, which is a measure of disorder which counts the difference between the numbers of occurrences of a given word in factors of the same length. An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is said to be *C-balanced* if for any pair v, w of factors of the same length of u, and for any letter $i \in \mathcal{A}$, one has $||v|_i - |w|_i| \leq C$. It is said *balanced* if there exists $C > 0$ such that it is C-balanced. Then, if the letters have frequencies in u, u is balanced if and only if its letter discrepancy is finite (uniformly in n).

As examples of balanced words, Sturmian words are known to be 1-balanced; they even are exactly the 1-balanced infinite words that are not eventually periodic [31]. Balance for substitutions can also be studied via the incidence matrix and a characterization of balanced words generated by primitive substitutions is given in [1]. Indeed, let σ be a primitive substitution and λ be its Perron-Frobenius eigenvalue; if its second eigenvalue is smaller than 1, then the letter discrepancy is finite. These results can be extended from letters to words.

Generalized Perron-Frobenius eigenvectors Given a directive sequence $(\sigma_n)_n$ that is everywhere growing, the cone $\bigcap_n M_{[0,n)}\mathbb{R}^d_+$ determined by the incidence matrices of the substitutions σ_n is intimately related to letter frequencies in the corresponding S-adic shift: it is the convex hull of the set of half lines $\mathbb{R}_+\mathbf{f}$ generated by the vector $\mathbf{f}$ whose components are the frequencies of letters of infinite words in the associated S-adic shift.

In the primitive substitutive case, letter frequencies are given by the Perron-Frobenius eigenvector [36]. For a primitive matrix M, the cones $M^n\mathbb{R}^d_+$ nest down to a single line directed by this eigenvector at an exponential convergence speed, according to Perron–Frobenius theorem (see e.g. [39]).

Concerning primitivity, the situation is more contrasted for S-adic systems. For instance, weak primitivity is known not to imply unique ergodicity [28]. Let $M_{[0,n)} := M_0M_1 \cdots M_{n-1}$. The following convenient sufficient condition from [23] for the sequence of cones $M_{[0,n)}\mathbb{R}^d_+$ to nest down to a single strictly positive direction as n tends to infinity (provided that the square matrices M_n have all non-negative entries) is a (weak) analogue of Perron-Frobenius theorem; in other words, the columns of the product $M_{[0,n)}$ tend to be proportional. It is stated in terms of infinite occurrences of a positive block of matrices. Let $(M_n)_n$ be a sequence of non-negative integer matrices of size d. Assume that there exist a strictly positive matrix B and indices $j_1 < k_1 \leq j_2 < k_2 \leq \cdots$ such that $B = M_{j_1} \cdots M_{k_1-1} = M_{j_2} \cdots M_{k_2-1} = \cdots$. Then, there exists a vector $\mathbf{f} \in \mathbb{R}^d_+$ with positive entries such that the sequence of cones $M_{[0,n)}\mathbb{R}^d_+$ nests down to the direction carried by $\mathbf{f}$ as n tends to infinity, i.e., $\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{R}^d_+ = \mathbb{R}_+\mathbf{f}$. It is widely used in symbolic dynamics [12,44].

The proof of this theorem relies on classical methods for non-negative matrices, namely Birkhoff contraction coefficient estimates and projective Hilbert metric. The vector $\mathbf{f}$ is a generalized right eigenvector of $\boldsymbol{\sigma}$. In particular, the letter frequency vector is a generalized right eigenvector for an infinite S-adic word generated by a sequence of substitutions whose incidence matrices satisfy the above conditions. Note that there is no way to define a generalized left eigenvector as the sequence of rows vary significantly in the sequence of matrices $(M_{[0,n)})_n$, in contrast to the columns. In the case of a single matrix, one simply takes the transpose, with the cones nesting down for rows as well as for columns.

Lyapunov exponents We can go beyond the previous sufficient condition (yielding the existence of a generalized right eigenvector), at the cost of working in average. Lyapunov exponents then replace (logarithms) of Perron–Frobenius eigenvalues. They describe the asymptotic behaviour of the singular values of large products of random matrices, under the ergodic hypothesis.

Let $\mathcal{S}$ be a finite set of substitutions with invertible incidence matrices, and let (D, S, μ) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic subshift equipped with a probability measure μ. It can be for instance a shift of finite type or a sofic shift. Here S stands also for the shift acting on D. We thus have a second dynamical system governing the substitutions to be iterated. Given an infinite sequence of substitutions $\sigma = (\sigma_n)_n \in D$, we define $M_{[0,n]}(\sigma) := M_0 M_1 \cdots M_{n-1}$, where M_i is the incidence matrix of the substitution σ_i. The Lyapunov exponents of the products $M_{[0,n]}(\sigma)$ with respect to the ergodic probability measure μ provide the exponential growth of eigenvalues of the matrices $M_{[0,n]}(\sigma)$ along a μ-generic sequence σ. The existence of Lyapunov exponents generalizes Birkhoff's ergodic theorem in a non-commutative setting. The fact that $\mathcal{S}$ is a finite set of substitutions with invertible incidence matrices ensures log-integrability, that is, $\int_D \log \max(\|M_1(\sigma)\|, \|M_1(\gamma)^{-1}\|) d\mu(\sigma) < \infty$. By ergodicity of μ, the first Lyapunov exponent of (D, S, ν) is the μ-almost everywhere[1] limit

$$\theta_1^\mu = \lim_{n\to\infty} \frac{\log \|M_{[0,n)}(\sigma)\|}{n}.$$

The other Lyapunov exponents $\theta_2^\mu \geq \theta_3^\mu \ldots \geq \theta_d^\mu$ are defined recursively by the following μ-almost everywhere limits, for $k = 1, \ldots, d$:

$$\theta_1^\mu + \theta_2^\mu + \cdots + \theta_k^\mu = \lim_{n\to\infty} \frac{1}{n} \int_D \log \| \wedge^k M_{[0,n)} \| \, d\nu$$

where $\wedge^k$ stands for the k-th exterior product (the k-fold wedge product). We are mostly interested by the two first Lyapunov exponents θ_1^μ and θ_2^μ that govern the convergence of column vectors for the products $M_{[0,n)}$.

Lyapunov exponents enable to state convergence results (see Theorem 1 below). By following the vocabulary of Markov chains [39], or of continued fractions [38], it is natural to consider the following definitions for convergence for the columns of the products of nonnegative matrices $M_{[0,n)}$. Assume the existence of a generalized right eigenvector $\mathbf{f}$. We work on the alphabet $\mathcal{A} = \{1, \ldots, d\}$. Let $\mathbf{f}_i^{(n)}$ stand for the column vectors of $M_{[0,n)}$. The column vectors $\mathbf{f}_i^{(n)}$, $1 \leq i \leq d$, produce d sequences of *rational convergents* $(\mathbf{f}_i^{(n)}/\|\mathbf{f}_i^{(n)}\|_1)_{n\in\mathbb{N}}$ that converge to $\mathbf{f}$.

More precisely,

[1] Here μ-almost everywhere refers to directive sequences of substitutions chosen in D with respect to the measure μ.

- the convergence is said to be *weak* if $\lim_{n\to\infty} \mathbf{f}_i^{(n)}/\|\mathbf{f}_i^{(n)}\|_1 = \mathbf{f}$ holds for all $i \in \{1,\dots,d\}$;
- it is said to be *strong* if $\lim_{n\to\infty} \|\mathbf{f}_i^{(n)} - \|\mathbf{f}_i^{(n)}\|_1 \mathbf{f}\| = 0$ holds for all $i \in \{1,\dots,d\}$;
- it is said *exponential* if there are positive constants $\kappa, \delta \in \mathbb{R}$ such that $\|\mathbf{f}_i^{(n)} - \|\mathbf{f}_i^{(n)}\|_1 \mathbf{x}\| < \kappa e^{-\delta n}$ holds for all $i \in \{1,\dots,d\}$ and all $n \in \mathbb{N}$.

Weak convergence means that the angle between the column vectors and $\mathbf{f}$ tends to 0 whereas strong convergence means that the distance between the column vectors and $\mathbf{f}$ tends to 0. In the case of a single primitive nonnegative matrix, Perron–Frobenius theorem yields exponential convergence.

Working with Lyapunov exponents provides the following statement [8]. Note that one of its limits, in terms of effectiveness, relies in the fact that it holds almost every where, that is, for a set of full measure with respect to the measure μ. This is often a very delicate task to be able to describe in effective terms such a full measure set.

Theorem 1. *Let $\mathcal{S}$ be a finite set of substitutions with invertible incidence matrices, and let $(D, \mathcal{S}, \mu)$, with $D \subset \mathcal{S}^{\mathbb{N}}$, be an ergodic shift. Assume that there exists a product of substitutions $\sigma_0 \dots \sigma_{k-1}$ with positive incidence matrix $M_{\sigma_0} \cdots M_{\sigma_{k-1}}$ whose associated cylinder in D has positive measure for μ. For μ-almost every directive sequence of substitutions $\sigma \in D$, the corresponding S-adic system X_σ is minimal and uniquely ergodic and one has weak convergence. Furthermore, $\theta_1^\mu > 0$ and $\theta_1^\mu > \theta_2^\mu$. If $\theta_2^\mu < 0$, then, for μ-almost every S-adic sequence in D, (X_σ, S) has bounded letter discrepancy and the convergence is strong.*

The quantity $1-\frac{\theta_2^\mu}{\theta_1^\mu}$ can also be found in the context of continued fractions [29] as the uniform approximation exponent for multidimensional continued fractions algorithms such as the Jacobi-Perron algorithm. We will come back to it in Section 3.4. The existence of a product of substitutions $\sigma_0 \dots \sigma_{k-1}$ with positive incidence matrix is crucial here and plays the role of primitivity.

3 From substitutions to S-adic systems

Since substitutions are finite in nature, many of their properties are decidable. In particular, we can decide whether a matrix is primitive and we have seen with Section 2 the interest of being primitive for substitutions. We first recall decidability properties for primitive substitutions in Section 3.1. The next question is to have a suitable decision framework for S-adic shifts. Elements in this direction are given in Section 3.2. Specific decision questions are listed in Section 3.3. We conclude by developing an analogy with continued fractions in Section 3.4.

3.1 Some decisions problems for substitutions

Numerous decidability results exist for fixed points of substitutions and their images by general morphisms (see e.g. [20, Section 10]). For primitive substitutions, various decision problems have been proved recently using the notion of return words and derived sequences. A return word to a factor u of an infinite word is a factor w of this infinite word such that uw admits exactly two occurrences of u, with the second occurrence of u being a suffix of uw. One then can recode infinite words generated by a primitive substitution via return words, obtaining derived sequences (see e.g. [16]). This has allowed the resolution of several long-standing decision problems. Let us quote the decidability of the equality between two substitutive infinite words [17]; see also [9, Chapter 10]. The decidability of the ultimate periodicity of substitutive infinite words is also decidable (see [17] for the primitive case, and [18] for the general case) as well as the uniform recurrence of substitutive words [19]. This problem is closely related to the decidability of the ultimate periodicity of recognizable sets of integers in some abstract numeration systems [6]. One can even decide whether two minimal substitution subshifts are topologically isomorphic and even whether one is a factor of the other [20]. In the particular case of constant-length substitutions (automatic sequences), the connections between first-order logic and automata produce efficient decision procedures (see e.g. [15,41,10]) relying on the equivalence between being p-recognizable and p-definability [14].

More generally, decidability in symbolic dynamics has already a long history since the undecidability of the emptiness problem (the domino problem) for multi-dimensional subshifts of finite type [7,37]; see also [9, Chapter 8]. Since the beginning, substitutions were used to input computation in tilings and they produced the first examples of aperiodic tilings, such as Robinson or Penrose tilings, proving the undecidability of the domino problem. Moreover, computability is a notion that has appeared as a major understanding tool in the study of multi-dimensional subshifts of finite type with the breakthrough characterization of the entropies of multi-dimensional subshifts of finite type as the non-negative right recursively enumerable numbers [25]. Let us mention also the realization of effective subshifts (with factor and projective subaction operation) from higher-dimensional subshifts of finite type [24,3,21]; see also [9, Chapter 9] and [26].

3.2 On effectiveness in the S-adic framework

Decision problems for substitutions make particular sense since the data that describe substitutions are finite. However, the description of an S-adic system is not finite, it is based on the infinite directive sequence of substitutions. The first issue which arises in this context is thus to give a meaning to effectiveness. There are several notions that can be considered and that are intimately related [11]: effectiveness of the directive sequence, computability of frequencies/invariant measures, and decidability of the language. We describe them below.

We recall that a subshift X can be defined by providing its language, that is, the set of finite words that occur in infinite words in X. It can be defined equivalently by providing the set of forbidden factors. This leads to the following definitions. A subshift is said to be Π_1-computable or effective if its language is co-recursively enumerable; it is said Σ_1-computable if its language is recursively enumerable; it is said Δ_1-computable or decidable if its language is recursive. A subshift (X, S) is said to have computable frequencies if the frequencies of factors exist and are uniformly computable. A shift-invariant measure is said to be computable if the measure of any cylinder is uniformly computable. A closed subset $D \subset S^{\mathbb{N}}$ is *effectively closed* if the set of (finite) words which do not appear as prefixes of elements of S is recursively enumerable (one enumerates forbidden prefixes). An effectively closed set is not necessarily a subshift.

The following relations between these concepts offers a convenient framework for decision problems for S-adic shifts (see [11]). Let X be a subshift. If X is effective and uniquely ergodic, then its invariant measure is computable and X is decidable. If X is minimal and its frequencies are computable, then its language is recursively enumerable. If X is minimal and effective, then it is decidable. If it is minimal, uniquely ergodic ergodic, and defined with respect to a directive sequence $\sigma \in \mathcal{S}^{\mathbb{N}}$, then the following conditions are equivalent: there exists a computable directive sequence σ' such that $X_\sigma = X_{\sigma'}$; the unique invariant measure of X_σ is computable; the subshift X_σ is decidable.

3.3 Around decision problems for *S*-adic systems

The concepts described in Section 2 lead to numerous decision questions. For instance, can we decide rational independence for the coordinates of a generalized right eigenvector? Note that sufficient conditions can be given in terms of strong convergence [12]. Concerning classification issues, can the isomorphism between two (effective) S-adic systems be decided (in the spirit of [20] which handles the substitutive case)? Can we decide whether balancedness on letters holds? and on factors? Can we decide recurrence properties such as having linear recurrence? Due to the fact that we lose the self-similarity properties present for substitutive systems, such results require new ideas and do not run along the lines of the substitutive setting.

We have seen the importance of Lyapunov exponents. Their computation also raises specific questions. Hardness is considered in [42] where the largest Lyapunov exponent is proved not to be algorithmically approximable. Practically, ergodic theorems provide efficient ways of estimating numerically Lyapunov exponents by following trajectories and then taking averages over truncated trajectories. Moreover, with ergodic theory and probability come methods issued from thermodynamic formalism, and more particularly transfer operators. Indeed, the theory of transfer operators can be considered as the analogue for invariant densities of dynamical systems of Perron–Frobenius theory for nonnegative matrices. They govern the evolution in time of the mass distribution of points and their action on densities models the action a dynamical system on input distributions. Since they are bounded linear operators, this makes them suitable for computer approximations (via truncations with respect to functional spaces) [33,30]. Note that the study of extremal matrix products and of the joint spectral radius (the largest asymptotic rate of growth that can be obtained by forming long products of matrices) is also particularly relevant in symbolic dynamics (see e.g. [9, Chapter 11]).

Pisot dynamics is a further specific setting that leads to numerous decision problems. A Pisot number is an algebraic integer whose algebraic conjugates (other than itself) are smaller than 1. In the eighties, Pisot substitutions (i.e., primitive substitutions whose Perron–Frobenius factor is a Pisot number) attracted much attention in the context of mathematical quasicrystals. We recall that quasicrystals are solids that are ordered but not periodic, and since their discovery, fruitful mathematical formulations have been proposed for the understanding of how atoms must be arranged in a material for it to diffract like a quasicrystal [5]. Pisot substitutions play a crucial role here since they create a hierarchical structure with a significant amount of long range order. They are conjectured to have pure discrete spectrum, that is, to be isomorphic (in the measure-theoretic sense) to a translation on a compact abelian group [9,2]. In a nutshell, algebraicity plus the Pisot arithmetic condition equals order. Order is expressed here in spectral and dynamical terms as being isomorphic to the simplest dynamical systems, namely group translations. This conjecture remains open since the 80's. For Pisot substitutions, several tools based on graphs are proposed in order to decide pure discrete spectrum [40]. In [12], the Pisot conjecture is extended to the S-adic setting going beyond algebraicity. The Pisot condition is then replaced by the requirement that the second Lyapunov exponent of the dynamical system is negative. Deciding pure discrete spectrum has to be formulated in this context.

Note that the study of Sturmian words and their various extensions is a setting that has confirmed the crucial role played by primitivity. Indeed, the S-adic expansion of Sturmian words is governed by continued fractions that play a renormalization role via the geodesic flow [35]. This has been very successfully extended with the study of interval exchanges in relation with the Teichmüller flow. Finding occurrences of positive matrices in the associated infinite products of matrices is at the heart of their study (see e.g. [4,43,44]).

3.4 Continued fractions and S-adic expansions

Decision questions in the S-adic setting are nourished by the dictionary that exists between continued fractions and S-adic expansions. Indeed, one important principle governing the S-adic approach relies in the the translation of a continued fraction expansion in symbolic terms via the matrix/substitution correspondence between a substitution and its incidence matrix. A continued fraction algorithm in dimension d generates products of nonnegative square matrices $(M_n)_n$ of size d, with the expanded vector belonging to the cone $\cap_n M_1 \cdots M_n \mathbb{R}_+^d$. A continued fraction algorithm then provides an S-adic subshift.

More generally, continued fraction expansions provide increasingly good rational approximations of real numbers. A continued fraction is expected to yield simultaneous better and better rational approximations with the same denominator for d-uples $(\alpha_1, \cdots, \alpha_d)$ in $[0,1]^d$, in an effective way and with a good approximation quality. It has to produce a sequence of positive integers $(q_n)_n$ such that the distance to the nearest integer $|||q_n(\alpha_1, \cdots, \alpha_d)|||$ converges exponentially fast to 0 with respect to q_n, and ideally in $q_n^{-\frac{1}{d}}$, with respect to Dirichlet's theorem.

There is no canonical extension of regular continued fractions to higher dimensions (the monoid $SL(3,\mathbb{N})$ is not free, contrarily to $SL(2,\mathbb{N})$), and the zoology of existing types of algorithms is particularly rich. Indeed, regular continued fractions rely on Euclid algorithm: starting with two numbers, one subtracts the smallest from the largest. If we start with at least three numbers, it is not clear to decide which operation has to be performed on these numbers in order to get something analogous to Euclid algorithm, hence the diversity and multiplicity of existing generalizations. Famous examples are the Jacobi-Perron, Brun or Selmer algorithms [38].

Note that in this correspondence, substitutions are associated with matrices in a noncanonical way. Indeed, a matrix can be the incidence matrix of several substitutions. A substitution offers in fact more information than a matrix. Given a continued fraction algorithm, a first step is thus to choose correctly the substitutions associated with matrices. Once a suitable choice of substitutions will be provided, a continued fraction algorithm will provide an S-adic system, it remains to investigate its combinatorial properties (factor complexity, recurrence and frequencies, symbolic discrepancy).

The main advantage of most classical unimodular continued fractions is that they can be expressed as dynamical systems whose ergodic study has already been well understood [38]. Convergence issues are then to be discussed and statements such as Theorem 1 have to be made effective. However, in higher dimension, continued fraction algorithms seem to present a major drawback concerning the quality of approximation. The convergence is governed by the quantity $1-\frac{\theta_2}{\theta_1}$ [29], where θ_1 and θ_2 are the two largest Lyapunov exponents of the associated dynamical system) (cf. Theorem 1). It has to be compared with Dirichlet's exponent $1+1/d$. However, recent striking numerical experimentations [13] indicate that the second Lyapunov exponent is not even negative for the most classical continued fraction algorithms, such as the Brun, Jacobi-Perron or Selmer algorithms in dimension d with $d \geq 10$, contrarily to what was expected. In other words, strong convergence is lost when increasing the dimension. A first challenge is to confirm these experimental results theoretically. This also raises the need for designing efficient strongly convergent continued fraction algorithms in dimension larger than 2, conducting to S-adic systems, thus providing further decision questions to explore.

References

1. Adamczewski, B.: Symbolic discrepancy and self-similar dynamics. Ann. Inst. Fourier (Grenoble) **54**, 2201–2234 (2004)
2. Akiyama, S., Barge, M., Berthé, V., Lee, J.-Y., Siegel, A.: On the Pisot substitution conjecture. In: Mathematics of aperiodic order, Progr. Math., vol. 309, p. 33–72, Birkhäuser/Springer (2015)

3. Aubrun, N., Sablik, M.: Simulation of effective subshifts by two-dimensional subshifts of finite type. Acta Applicandae Mathematicae **126**, 35–63 (2013)
4. Avila, A., Forni, G.:Weak mixing for interval exchange transformations and translation flows. Ann. of Math. **165**, 637–664 (2007)
5. Baake, M., Grimm, U.: Aperiodic order. Volume 1: A Mathematical invitation. Cambridge University Press, Cambridge (2013)
6. Bell, J.P., Charlier, E., Fraenkel, A.S., Rigo, M.:. A decision problem for ultimately periodic sets in non-standard numeration systems. Internat. J. Algebra Comput **9**, 809– 839 (2009)
7. Berger, R: The undecidability of the domino problem. Mem. Amer. Math. Soc. **66** (1966)
8. Berthé, V., Delecroix, V.: Beyond substitutive dynamical systems: S-adic expansions. RIMS Lecture note 'Kokyuroku Bessatu' **B46**, 81–123 (2004)
9. Berthé, V., Rigo, M. (Eds.): Combinatorics, Automata and Number Theory. Encyclopedia Math. Appl., vol. 135, Cambridge University Press, Cambridge (2010)
10. Berthé, V., Rigo, M. (Eds.): Combinatorics, Words and Symbolic dynamics. Encyclopedia of Math. Appl., vol. 159, Cambridge University Press (2016)
11. Berthé, V., Fernique, T., Sablik, M: Effective S-adic symbolic dynamical systems. In: CiE 2016, LNCS, Vol. 9709, p. 13–23, Springer, Heidelberg (2016)
12. Berthé, V., Steiner, W. Thuswaldner, J.: Geometry, dynamics, and arithmetic of s-adic shifts. Annales de l'Institut Fourier **69**, 1347–1409 (2019)
13. Berthé, V., Steiner, W., Thuswaldner, J.: On the strong convergence of higher-dimensional continuous fractions. Mathematics of computation, to appear
14. Bruyère, V., Hansel, G., Michaux, C., Villemaire, R.: Logic and p-recognizable sets of integers. Bull. Belg. Math. Soc. Simon Stevin 1 **2**, 191–238 (1994)
15. Charlier, E., Rampersad, N., Shallit, J.: Enumeration and decidable properties of automatic sequences. Internat. J. Found. Comput. Sci. **23** 1035–1066 (2012)
16. Durand, F.: Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergodic Theory Dynam. Systems*, **20** 1061–1078 (2000). Corrigendum and addendum. *Ergodic Theory Dynam. Systems*, **23** 663–669 (2003)
17. Durand, F. : HD0L ω-equivalence and periodicity problems in the primitive case. Unif. Distrib. Theory **7** 199–215 (2012)
18. Durand, F.: Decidability of the HD0L ultimate periodicity problem. RAIRO - Theoretical Informatics and Applications **47** 201–214 (2013)
19. Durand, F.: Decidability of uniform recurrence of morphic sequences. Internat. J. Found. Comput. Sci. **24** 123–146 (2013)
20. Durand, F., Leroy. J.: Decidability of the isomorphism and the factorization between minimal substitution subshifts. https://arxiv.org/abs/1806.04891
21. Durand, B., Romashchenko, A. E., Shen, A.: Effective closed subshifts in $1d$ can be implemented in $2d$, Fields of Logic and Computation. LNCS, Vol. 6300, p. 208–226, Springer, Heidelberg (2010)
22. Fijalkow, N., Ohlmann, P., Ouaknine, J., Pouly, A., Worrell, J.: Complete Semialgebraic Invariant Synthesis for the Kannan-Lipton Orbit Problem. Theoretical Computer Science, **63** 1027–1048 (2019)
23. Furstenberg, H.: Stationary processes and prediction theory. Annals of Mathematics Studies, vol. 44, Princeton University Press, Princeton, N.J. (1960)
24. Hochman, M.: On the dynamics and recursive properties of multidimensional symbolic systems. Inventiones Mathematicae **176**,131–167 (2009)
25. Hochman, M., Meyerovitch, T.: A Characterization of the Entropies of Multidimensional Shifts of Finite Type. Annals of Mathematics **171**, 2011–2038 (2010)

26. Jeandel, E., Vanier, P.: A characterization of subshifts with computable language. In: STACS Art. No. 40, LIPIcs. Leibniz Int. Proc. Inform., 126, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern (2019)
27. Kannan, R.; Lipton, R. J. Polynomial-time algorithm for the orbit problem. J. Assoc. Comput. Mach. **33**, no. 4, 808–821 (1986)
28. Keane, M.: Non-ergodic interval exchange transformations. Israel J. Math. **26**, 188–196 (1977).
29. Lagarias, J.C.: The quality of the Diophantine approximations found by the Jacobi-Perron algorithm and related algorithms. Monatsh. Math. **115**, 299–328 (1993)
30. Lhote, L.: Computation of a class of continued fraction constants. In: Analytic Algorithmic and Combinatorics ANALCO 2014,SIAM, p. 199–210 (2004)
31. Lothaire, M.: Algebraic Combinatorics on Words, vol. 90, Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2002.
32. Morse, M., Hedlund, G. A.: Symbolic dynamics II. Sturmian trajectories. Amer. J. Math. **62**, 1–42 (1940)
33. Pollicott, M.: Maximal Lyapunov exponents for random matrix products. Invent. Math. **181**, 209–226 (2010)
34. Priebe Frank, N., Sadun, L.: Fusion: a general framework for hierarchical tilings of $\mathbb{R}^d$. Geometriae Dedicata **171**, 149–186 (2014)
35. Pytheas Fogg, N.: Substitutions in Dynamics, Arithmetics and Combinatorics. V. Berthé, Ferenczi, S., Mauduit, C., Siegel, A. (eds.) Lecture Notes in Mathematics, vol. 1794, Springer Verlag, (2002)
36. Queffélec, M.: *Substitution dynamical systems—spectral analysis*, vol. 1294, Lecture Notes in Mathematics, Springer-Verlag, Berlin, second ed. (2010)
37. Robinson, R. M.: Undecidability and nonperiodicity for tilings of the plane. Invent. Math. **12**, 177–209 (1971)
38. Schweiger, F.: *Multidimensional continued fractions.* Oxford Science Publications, Oxford University Press, Oxford (2000)
39. Seneta, E.: *Non-negative matrices and Markov chains.* Springer Series in Statistics, Springer-Verlag, New York (1981)
40. Siegel, A. Thuswaldner, J. M.: Topological properties of Rauzy fractals. Mém. Soc. Math. Fr. Vol. 118 (2009)
41. Shallit, J.: Decidability and enumeration for automatic sequences: a survey. In: Computer science—theory and applications, LNCS, Vol. 7913, p. 49–63, Springer, Heidelberg (2013)
42. Tsitsiklis, J. N., Blondel, J. N.: The Lyapunov exponent and joint spectral radius of pairs of matrices are hard—when not impossible—to compute and to approximate. Math. Control Signals Systems **10**, 31–40 (1997)
43. Yoccoz, J.-C.: Continued fraction algorithms for interval exchange maps: an introduction. In: Frontiers in number theory, physics, and geometry. I, p. 401–435, Springer, Berlin, (2006)
44. Zorich, A.:, Deviation for interval exchange transformations. Erg. Th. and Dyn. Syst. **17**, 1477–1499 (1997)