On the Pisot substitution conjecture

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Contents

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1. Introduction

Substitutive dynamics. Substitutions are replacement rules that can be of either a symbolic or a geometric nature: by iteration, they produce hierarchically ordered structures (infinite words, point sets, tilings) that display strong self-similarity properties. A symbolic substitution is a morphism on a free monoid defined by replacing letters by finite words, while a geometric substitution inflates each tile in some finite collection of tiles and subdivides the inflated tile into translates of the original tiles (the tiles are like letters and the inflated and subdivided tiles are like higher-dimensional words). Iterating a substitution on a letter (or tile) produces longer and longer words (or larger and larger 'patches'), any subword (or subpatch) of such a word (or patch) is said to be *allowed* for the substitution. The space of all bi-infinite words (or tilings), all of whose finite subwords (or patches) are allowed for a substitution is called a *substitutive system* (or *substitution tiling space*).

The idea of a substitution as a replacement rule is central in symbolic dynamics and tiling theory (see, for instance, [96, 95, 99, 113, 16]). Indeed, substitutions are closely related to the process of induction. For a measure-preserving dynamical system, the Poincaré recurrence theorem guarantees that almost every point in any given set eventually returns to that set. An *induced* system is then defined by first return to the set and a system is *self-induced* if there is a subset with positive measure for which the induced system is isomorphic to the original system. Self-induced dynamics underlies periodic expansions with respect to various algorithms. For example, the continued fraction expansion of a quadratic irrational is derived from successive induction on irrational rotation of the circle. In this case the maximal eigenvalue of the matrix associated with the inverse application for the self-inducing structure is a quadratic Pisot unit and this fact is used to effectively find the fundamental unit of a real quadratic number field. Similar phenomena occur for piecewise isomorphisms and outer billiards.

The field of symbolic dynamics has its origins in the coding of concrete (geometrical) dynamical systems (see, *e.g.*, [90, 91]). Given a partition of the phase space of a dynamical system there is an associated space consisting of the collection of all itineraries of the system with respect to the partition. The symbolic dynamical system consisting of the shift map on the space of itineraries is then a model of the original system. We are interested here in the inverse problem: given a symbolic dynamical system or a tiling system, is there a geometrical dynamical system that it codes? More particularly, we are interested in the geometrical interpretation of symbolic systems consisting of infinite words (or tilings) created by a substitution. In order to have a chance to find the nicest sort of geometrical interpretation - translation on a compact abelian group - of a substitutive symbolic system, it is necessary to restrict to substitutions that create a hierarchical structure with a significant amount of long range order. As translation on a compact abelian group is almost periodic, the substitutions we consider must create sequences, or tilings, with a similar structure.

A (primitive) substitution ϕ stretches words, on average, by some factor λ: if w is a long (allowed) word then the word $\phi(w)$ has roughly λ times the number of letters as has w. The number λ is the *expansion factor* of ϕ. For the infinite words in the substitutive system associated with ϕ to be shift-periodic, λ must be an integer (*e.g.*, $\phi(a) = ab$, $\phi(b) = ab$, $\lambda = 2$). It turns out that for the infinite words making up the substitutive system associated with ϕ to be nearly enough periodic in order that the substitutive system be a coding of a translation on a compact abelian group, it is necessary that higher and higher powers of λ are more and more nearly integers. Pisot characterized such λ ([92]): if $\lambda > 1$ is an algebraic integer, then the distance from λ^n to the nearest integer goes to zero as n goes to infinity if and only if all of the algebraic conjugates of λ (other than λ) lie strictly inside the unit circle. Such λ are called *Pisot numbers* (or sometimes *Pisot-Vijayaraghavan numbers*).

Pisot substitutions. For the above reasons, we restrict ourselves here to substitutions whose expansion factor is a Pisot number. Tiling substitutions also have an associated linear expansion: for one-dimensional tiling substitutions we will assume that this expansion is a Pisot number and in higher dimensions that it has the *Pisot property* (see Section 7.1 for a definition).

By a geometrical interpretation of a symbolic (or tiling) system we mean a factor map from the system onto some system of a geometrical nature. If the geometrical system is translation on a compact abelian group, the existence of such a (non-trivial) factorization is equivalent to the symbolic system having a non-trivial dynamical spectrum. A dynamical system has *pure discrete spectrum* if it factors almost everywhere one-to-one onto a translation on a compact abelian group. The connections between Pisot substitutions and discrete spectrum first appear in [42, 43, 112, 97, 117] and it is shown in [113] that for a one-dimensional substitutive system to have pure discrete spectrum it is necessary that the expansion λ be a Pisot number. Our main focus in this chapter is on the question: what conditions must be placed on a Pisot substitution in order to guarantee that the associated substitutive system (or tiling system) has pure discrete spectrum? The Pisot Substitution Conjecture, and its variants, are proposed answers.

Organization of the chapter. Section 2 introduces various types of substitutions and their associated dynamical systems. These include substitutions acting on words, tilings and point sets. We also consider the related framework of beta-numeration.

In Section 3 we discuss the notions of discreteness of the dynamical and diffraction spectra of the substitutive systems. In particular, the Meyer property is introduced in Section 3.2, and in Section 3.3 we give a general definition of cut-and-project schemes and model sets.

The various (one-dimensional) Pisot Substitution Conjectures are then discussed in Section 4. Techniques for detecting pure discrete spectrum are reviewed in Section 5 and partial results related to the Pisot Substitution

Conjectures are listed in Section 6. In the final Section 7 we propose extensions of the conjectures to higher-dimensional substitutions.

2. Substitutions

In this section we review basic material concerning the dynamical systems generated by substitutions. For more detail on symbolic systems see [96, 95]; standard references for tiling systems are [113, 13, 99, 100].

2.1. Symbolic substitutions

Let $\mathcal{A}$ be a finite set, called the *alphabet*, usually $\mathcal{A} = \{1, \dots, m\}$; its elements are called *symbols* or *letters*. Endowed with the concatenation of words as product operation, the set $\mathcal{A}^*$ of all finite words over $\mathcal{A}$ is the *free monoid* generated by $\mathcal{A}$. If a finite or infinite word u can be factored as $u = vzw$ (v or w may be empty) we say that z is a *subword* of u.

A morphism of the free monoid $\mathcal{A}^*$ is called a *substitution* on $\mathcal{A}$. Such a substitution ϕ is said to be *primitive* if there exists a positive integer k such that, for all $i, j \in \mathcal{A}$, the word $\phi^k(i)$ contains at least one occurrence of the letter j. For any word $w \in \mathcal{A}^*$, we denote by $|w|_j$ the number of occurrences of the letter j in w and by $|w| := \sum_{j \in \mathcal{A}} |w|_j$ the *length* of w. The map $f : \mathcal{A}^* \to \mathbb{N}^d$, $w \mapsto (|w|_1, |w|_2, \cdots, |w|_m)$ is called the *abelianization map*, or *Parikh mapping*. The *substitution* (or *incidence* or *abelianization*) *matrix*, denoted by M_ϕ, is the matrix whose j-th column is $f(\phi(j))$: its ij-th entry is $|\phi(j)|_i$, the number of occurrences of the letter i in $\phi(j)$. Note that the substitution ϕ is primitive if and only if M_ϕ is primitive (that is, some power of M_ϕ is strictly positive). If the characteristic polynomial of M_ϕ is irreducible over $\mathbb{Q}$ then ϕ is said to be *irreducible*, and if $\det M_\phi = \pm 1$, then ϕ is *unimodular*. If ϕ is primitive, the Perron-Frobenius Theorem asserts that M_ϕ has a simple positive eigenvalue λ, which we call the *PF-eigenvalue* of ϕ, that is larger than the absolute value of all other eigenvalues. The *language* of ϕ is the subset $\mathcal{L} \subset \mathcal{A}^*$ consisting of all subwords of words of the form $\phi^k(i)$, $i \in \mathcal{A}$, $k \in \mathbb{N}$; the elements of $\mathcal{L}$ are called *admissible words* for ϕ.

To associate a symbolic dynamical system with a substitution ϕ, we give $\mathcal{A}$ the discrete topology, endow $\mathcal{A}^{\mathbb{Z}}$ with the corresponding product topology and 'extend' ϕ to $\mathcal{A}^{\mathbb{Z}}$ by $\phi((\cdots u_{-1} \cdot u_0 u_1 \cdots)) = (\cdots \phi(u_{-1}) \cdot \phi(u_0)\phi(u_1) \cdots)$, where $\cdot$ indicates the location of the 0-th letter. A *substitution-periodic point* for ϕ, or *ϕ-periodic point*, is a point $u = (u_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ for which $\phi^k(u) = u$ for some $k > 0$, and has the property that $u_{-1}u_0$ belongs to $\mathcal{L}$. Let s stand for the shift on $\mathcal{A}^{\mathbb{Z}}$, *i.e.*, $s((u_n)_{n \in \mathbb{Z}}) = (u_{n+1})_{n \in \mathbb{Z}}$. For a ϕ-periodic point u, let $\overline{\mathcal{O}(u)}$ be the orbit closure of u under the action of the shift s, *i.e.*, the closure in $\mathcal{A}^{\mathbb{Z}}$ of the set $\mathcal{O}(u) = \{s^j(u) : j \in \mathbb{Z}\}$.

It is not hard to see that every primitive substitution has at least one substitution-periodic point. Furthermore, the collection of all finite subwords of any substitution-periodic point of a primitive substitution equals $\mathcal{L}$ ([96]). Hence, if ϕ is a primitive substitution, then $X_\phi := \overline{\mathcal{O}(u)}$ does not depend on the choice of the ϕ-periodic point u: X_ϕ is the space of all bi-infinite words all of whose finite subwords belong to $\mathcal{L}$. The elements of X_ϕ are called *allowed words* for ϕ. The set X_ϕ is a closed, shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ and the subshift obtained by restricting the shift s to X_ϕ is denoted by (X_ϕ, s) and called the *symbolic dynamical system*, or *substitutive system, associated with* ϕ. The system (X_ϕ, s) is *minimal* (every non-empty closed shift-invariant subset equals the whole set) and *uniquely ergodic* (there is a unique, ergodic, shift-invariant Borel probability measure on X_ϕ - see [96] for more detail).

2.2. Tile substitutions

We now consider substitutions acting on tiles, rather than letters. General references on this subject are [81, 99, 100, 114, 120].

We begin with a set of types (or labels, or colors) $\{1, \dots, m\}$. A *tile* in $\mathbb{R}^n$ is a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$, the *support of* T, is a compact set in $\mathbb{R}^n$ which is the closure of its interior, and $i = l(T) \in \{1, \dots, m\}$ is the *type of* T. For $x \in \mathbb{R}^n$, define $T - x := (A - x, i)$. We say that a set $\mathcal{P}$ of tiles is a *patch* if the number of tiles in $\mathcal{P}$ is finite and the tiles of $\mathcal{P}$ have mutually disjoint interiors. A tiling of $\mathbb{R}^n$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^n = \cup_{T \in \mathcal{T}} \mathrm{supp}(T)$ and distinct tiles have disjoint interiors. We assume that any two $\mathcal{T}$-tiles of the same type are translationally equivalent, hence there are finitely many $\mathcal{T}$-tiles up to translation.

A tiling $\mathcal{T}$ is said to have *finite local complexity (FLC)* if for each $R > 0$ there are, up to translation, only finitely many distinct patches in $\mathcal{T}$ with support of diameter less than $2R$ and $\mathcal{T}$ is said to be *repetitive* if for each patch $\mathcal{P} \subset \mathcal{T}$, translates of $\mathcal{P}$ occur with bounded gap in $\mathcal{T}$, *i.e.*, the set $\{x : \mathcal{P} - x \subset \mathcal{T}\}$ is relatively dense in $\mathbb{R}^n$.

Let Λ be an expanding linear map of $\mathbb{R}^n$ (meaning all eigenvalues of Λ have absolute value greater than 1) and let $\mathcal{A} = \{T_1, \dots, T_m\}$, $T_i = (A_i, i)$, be a finite collection of tiles in $\mathbb{R}^n$. The T_i will be called *prototiles*. Let $\mathcal{A}^+$ denote the collection of all patches made of translates of prototiles. We say that $\Phi : \mathcal{A} \to \mathcal{A}^+$ is a *tile substitution* (or simply a *substitution*) with *expansion map* Λ if there are finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^n$, $1 \le i, j \le m$, called *digit sets*, such that

$$\Phi(T_j) = \{T_i + v : \ v \in \mathcal{D}_{ij}, \ i = 1, \dots, m\}, \quad \text{for } 1 \le j \le m, \tag{2.1}$$

with

$$\Lambda A_j = \bigcup_{i=1}^{m} (A_i + \mathcal{D}_{ij}) = \bigcup_{i=1}^{m} \bigcup_{d \in \mathcal{D}_{ij}} (A_i + d), \tag{2.2}$$

where the sets in the last unions of (2.2) have disjoint interiors (some of the $\mathcal{D}_{ij}$ may be empty).

When distinct tiles have translationally inequivalent supports, labeling is not necessary to distinguish tiles and we may simply identify a tile with its support. In this case, Equation (2.2) alone suffices to define a tile substitution.

The substitution (2.1) is extended to translates of prototiles by $\Phi(T_j - x) = \Phi(T_j) - \Lambda x$, and to patches and tilings by $\Phi(\mathcal{P}) = \cup_{T \in \mathcal{P}} \Phi(T)$ and

$\Phi(\mathcal{T}) = \cup_{T\in\mathcal{T}}\Phi(T)$. If $\Phi(\mathcal{T}) = \mathcal{T}$, $\mathcal{T}$ is said to be *a fixed point* of the substitution Φ and $\mathcal{T}$ is Φ*-periodic* if $\Phi^k(\mathcal{T}) = \mathcal{T}$ for some $k \geq 0$. The substitution Φ may be iterated, producing larger and larger patches $\Phi^k(T_j - x)$, and by taking appropriate limits, tilings of $\mathbb{R}^n$.

As for symbolic substitutions, we associate with Φ its $m\times m$ *substitution matrix* M_Φ with ij-th entry $\sharp\mathcal{D}_{ij}$. The substitution Φ is said to be *primitive* if M_Φ is primitive. A patch $\mathcal{P}$ is *admissible for* Φ if there are k, j and x so that $\mathcal{P} \subset \Phi^k(T_j - x)$. The *substitution tiling space associated with* Φ is the collection Ω_Φ of all tilings of $\mathbb{R}^n$ each of whose patches is admissible for Φ: such tilings are said to be *allowed for* Φ. There is a natural metric topology (generated by the *tiling metric*) on Ω_Φ in which two tilings are close if one agrees exactly with a small translate of the other in a large neighborhood of the origin in $\mathbb{R}^n$. Clearly, $\mathbb{R}^n$ acts on Ω_Φ by translation and if Φ is primitive, and any $\mathcal{T} \in \Omega_\Phi$ has FLC, then the dynamical system $(\Omega_\Phi, \mathbb{R}^n)$ is compact, connected, minimal, and uniquely ergodic. In this latter case, let μ be the unique invariant Borel probability measure for the $\mathbb{R}^n$-action; then we get a measure-preserving system $(\Omega_\mathcal{T}, \mathbb{R}^n, \mu)$. Moreover, when $\mathcal{T}$ is non-periodic (meaning $\mathcal{T} - x = \mathcal{T} \implies x = 0$), $\Phi : \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism that interacts with the $\mathbb{R}^n$-action via $\Phi(\mathcal{T} - x) = \mathcal{T} - \Lambda x$ ([114]). For more on substitution tilings and associated dynamical systems, see [99, 113].

We will occasionally consider a space associated with a single tiling $\mathcal{T}$ of $\mathbb{R}^n$. The *hull* of $\mathcal{T}$ is the set $\Omega_\mathcal{T} := \overline{\{\mathcal{T} - v : v \in \mathbb{R}^n\}}$, the closure being taken with respect to the tiling metric. If Φ is a primitive tile substitution and $\mathcal{T} \in \Omega_\Phi$ has FLC, then $\Omega_\mathcal{T} = \Omega_\Phi$ by minimality of the $\mathbb{R}^n$-action. When we say that an FLC tiling $\mathcal{T}$ is a *primitive substitution tiling*, we will mean that $\mathcal{T} \in \Omega_\Phi$ for some primitive tile substitution Φ. In particular, such $\mathcal{T}$ are repetitive.

From symbolic substitutions to tile substitutions. A symbolic substitution naturally gives rise to a one-dimensional tile substitution as follows. Given a primitive substitution ϕ on the alphabet $\mathcal{A} = \{1, \ldots, m\}$, let $\ell = (\ell_1, \ldots, \ell_m)$ be a positive left eigenvector of M_ϕ for the PF-eigenvalue λ. The prototiles for the associated tile substitution Φ are the labeled intervals $T_i := ([0, \ell_i], i)$, $i = 1, \ldots, m$, and Φ is given by $\Phi(T_j) := \{T_{j_k} + \sum_{i=1}^{k-1} \ell_{j_i} : k = 1, \ldots, |\phi(j)|\}$, where j_i denotes the i-th letter of $\phi(j)$. Effectively, the j-th prototile is stretched by a factor of λ and subdivided into translates of prototiles following the pattern $\phi(j)$. (The set $\mathcal{D}_{ij}$ of (2.1) consists of all $v = \sum_{s=1}^{k-1} \ell_{j_s}$ for which $j_k = i$, so the matrix M_Φ for the tile substitution equals the matrix M_ϕ for the symbolic substitution ϕ.) The elements of the substitution tiling space $\Omega_\phi := \Omega_\Phi$ are tilings of the real line whose tiles follow the pattern of some allowed bi-infinite word in X_ϕ. Tilings $\mathcal{T} \in \Omega_\phi$ can be thought of as interpolates of elements of X_ϕ: formally, the system $(\Omega_\phi, \mathbb{R})$ is topologically conjugate with the suspension system $(\hat{X}_\phi, \mathbb{R})$ of (X_ϕ, s) with roof function given by ℓ (see, *e.g.*, [51, Chap.2] for a discussion of suspension). As ϕ is primitive, so is Φ, and $(\Omega_\phi, \mathbb{R})$ is compact, connected, minimal and uniquely ergodic. Moreover, Φ is a homeomorphism of Ω_ϕ and $\Phi(\mathcal{T}-t)=\mathcal{T}-\lambda t$ for $\mathcal{T}\in\Omega_\phi$ and $t\in\mathbb{R}$.

2.3. Point set substitutions

Point sets provide the most flexible context for studying substitutive dynamics. A *Delone set* is a relatively dense and uniformly discrete subset of $\mathbb{R}^n$. We say that $\underline{\Gamma}=\bigcup_{i=1}^m \Gamma_i\times\{i\}$ is a *Delone multi-color set* in $\mathbb{R}^n$ if each Γ_i is Delone and $\mathrm{supp}(\underline{\Gamma}) := \bigcup_{i=1}^m \Gamma_i \subset \mathbb{R}^n$ is Delone. We call such $\underline{\Gamma}$ a *substitution Delone multi-color set* if $\underline{\Gamma}$ is a Delone multi-color set and there exist an expanding map $\Lambda:\mathbb{R}^n\to\mathbb{R}^n$ and finite sets $\mathcal{D}_{ij}$ (the *digit sets*), for $1\le i,j\le m$, such that

$$\Gamma_i=\bigcup_{j=1}^m(\Lambda\Gamma_j+\mathcal{D}_{ij}),\quad 1\le i\le m, \tag{2.3}$$

where the union on the right-hand side is disjoint. The substitution Delone multi-color set $\underline{\Gamma}$ is said to be *primitive* if the matrix $(\sharp\mathcal{D}_{ij})_{m\times m}$ is primitive. For more on substitution Delone sets, see, *e.g.*, [74, 81].

For any given substitution Delone multi-color set $\underline{\Gamma}=\bigcup_{i=1}^m \Gamma_i\times\{i\}$, we define Φ_{ij} to be the collection of affine functions from $\mathbb{R}^n\times\{j\}$ to $\mathbb{R}^n\times\{i\}$

$$\Phi_{ij} := \{f:(x,j)\mapsto(\Lambda x+a,i)\,:\,a\in\mathcal{D}_{ij}\}. \tag{2.4}$$

Then $\Phi_{ij}(\Gamma_j\times\{j\}) := \bigcup_{f\in\Phi_{ij}} f(\Gamma_j\times\{j\}) = (\Lambda\Gamma_j+\mathcal{D}_{ij})\times\{i\}$. We denote by Φ the $m\times m$ array with ij-th entry Φ_{ij} and call Φ a *matrix function system* (MFS). For any $\underline{S}=\bigcup_{i=1}^m S_i\times\{i\}$, $S_i\subset\mathbb{R}^n$, we define $\Phi(\underline{S})$ to be the collection $\bigcup_{i=1}^m(\bigcup_{j=1}^m \Phi_{ij}(S_j\times\{j\}))\times\{i\}$. We may then iterate Φ, obtaining, for any $k\in\mathbb{N}$, $\Phi^k(\underline{\Gamma})=\underline{\Gamma}$ and $\Phi^k(\Gamma_j\times\{j\})=\bigcup_{1\le i\le m}(\Lambda^k\Gamma_j+(\mathcal{D}^k)_{ij})\times\{i\}$ where

$$(\mathcal{D}^k)_{ij}=\bigcup_{1\le n_1,n_2,\dots,n_{(k-1)}\le m}(\mathcal{D}_{in_1}+\Lambda\mathcal{D}_{n_1n_2}+\cdots+\Lambda^{k-1}\mathcal{D}_{n_{(k-1)}j}).$$

A *cluster* of $\underline{\Gamma}$ is a collection $\underline{P}=\bigcup_{i=1,\dots,m}P_i\times\{i\}$ where $P_i\subset\Gamma_i$ is finite for all $1\le i\le m$. We say that a cluster $\underline{P}$ is *legal* if it is a translate of a subcluster of a cluster generated from one point of $\underline{\Gamma}$, *i.e.*, $a+\underline{P}\subset\Phi^k((x,i))$ for some $k\in\mathbb{N}$, $a\in\mathbb{R}^n$ and $x\in\Gamma_i$.

As for tilings, we may consider the hull $\Omega_{\underline{\Gamma}}$ of a Delone multi-color set and the associated dynamical system $(\Omega_{\underline{\Gamma}},\mathbb{R}^n)$.

The equations (2.3) and (2.2) are formally the same. Indeed, one may pass back and forth between substitution tilings and substitution Delone multi-color sets with a fair amount of freedom as we now describe.

From a substitution tiling to a substitution Delone multi-color set. From a given substitution tiling which is a fixed point of a substitution, one easily constructs a substitution Delone multi-color set by taking a representative point from each tile, choosing points in the same relative position in tiles of the same color. In fact, if $\mathcal{T}$ is a repetitive tiling that is a fixed point of a substitution for which $\mathcal{T}=\bigcup_{i=1}^m\{T_i+x : x\in\Gamma_i\}$, then $\underline{\Gamma}=\bigcup_{i=1}^m\Gamma_i\times\{i\}$

is a primitive substitution Delone multi-color set. We call $\underline{\Gamma}_{\mathcal{T}}$ an associated substitution Delone multi-color set of $\mathcal{T}$.

A particularly nice selection of associated Delone multi-color set is as follows. For each $T \in \mathcal{T}$, let $\tau(T) \in \Phi(T)$ be chosen in such a way that if $S = T + x$ then $\tau(S) = \tau(T) + \Lambda x$. Since we are assuming that $\mathcal{T}$ is a fixed point of the substitution, $\tau : \mathcal{T} \to \mathcal{T}$. Let

$$\Gamma_i := \bigcup_{T\in\mathcal{T},\, T \text{ of type } i} \bigcap_{k=0}^{\infty} \Lambda^{-k}\mathrm{supp}(\tau^k(T)).$$

For Γ_i so defined, $\mathcal{C} := \bigcup_{i=1}^m \Gamma_i$ is called a set of *control points* for $\mathcal{T}$.

From a substitution Delone multi-color set to a substitution tiling. On the other hand, it is not so obvious how to reconstruct a substitution tiling from a substitution Delone multi-color set. Lagarias and Wang give a canonical way to carry out such a construction in a restricted setting (see [74, 81] for more detail). We briefly describe the context in which this construction applies. We say that a Delone multi-color set $\underline{\Gamma} = \bigcup_{i=1}^m \Gamma_i \times \{i\}$ is *representable* (by tiles) if there exist tiles $T_i = (A_i, i), 1 \le i \le m$, so that

$$\{x + T_i : \ x \in \Gamma_i, 1 \le \ i \le m\} \quad \text{is a tiling of } \ \mathbb{R}^n, \tag{2.5}$$

that is, $\mathbb{R}^n = \bigcup_{1\le i\le m} \bigcup_{x\in\Gamma_i}(x + A_i)$, and the sets in this union have disjoint interiors. In the case that $\underline{\Gamma}$ is a primitive substitution Delone multi-color set, we will understand the term representable to mean relative to tiles $T_i = (A_i, i), 1 \le i \le m$, that satisfy the adjoint equations (2.2) formed by the digit sets $\mathcal{D}_{ij}$. In [74, Lemma 3.2] it is shown that if $\underline{\Gamma}$ is a substitution Delone multi-color set, then there is a finite multi-color set (cluster) $\underline{P} \subset \underline{\Gamma}$ for which $\Phi^{n-1}(\underline{P}) \subset \Phi^n(\underline{P})$ for $n \ge 1$ and $\underline{\Gamma} = \lim_{n\to\infty} \Phi^n(\underline{P})$. We call such a multi-color set $\underline{P}$ a *generating set* for $\underline{\Gamma}$. It is shown in [81] that if a generating multi-color set of a primitive substitution Delone multi-color set $\underline{\Gamma}$ is legal, then $\underline{\Gamma}$ is representable.

2.4. Beta-shifts

The final context for substitution dynamics that we consider arises from transformations of the interval associated with beta-numeration. For more detail, the reader is referred to [85].

Let $\beta > 1$ be a real number. The *beta transformation*, $T_\beta : [0,1) \to [0,1)$, is defined by $T_\beta(x) = \beta x - d(x)$ with $d(x) = \lfloor \beta x \rfloor \in [0,\beta) \cap \mathbb{Z}$. Putting $d_n(x) = d(T^{n-1}(x))$ for $n = 0, 1, \ldots$, we obtain the *β-expansion of x*:

$$x = \frac{d_1(x)}{\beta} + \frac{d_2(x)}{\beta^2} + \frac{d_3(x)}{\beta^3} + \ldots$$

Define a function $\mathbf{d}_\beta : [0,1] \to \mathcal{A}^{\mathbb{N}}$ by $\mathbf{d}_\beta(x) = d_1(x)d_2(x)d_3(x)\ldots$ with $\mathcal{A} = [0,\beta) \cap \mathbb{Z}$. For $x > 1$, we can find the minimum non-negative integer m with $\beta^{-m}x \in [0,1)$. Putting $d_{-m+k}(x) = d_k(\beta^{-m}x)$, we obtain:

$$x = d_{-m}(x)\beta^m + d_{-m+1}(x)\beta^{m-1} + \cdots + d_0(x) + \frac{d_1(x)}{\beta} + \frac{d_2(x)}{\beta^2} + \frac{d_3(x)}{\beta^3} + \ldots$$

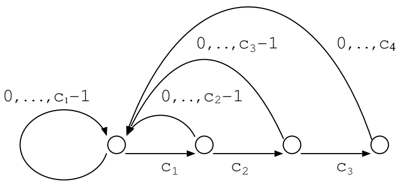
which we denote by $d_{-m}(x)d_{-m+1}(x)\dots d_0(x)d_1(x)d_2(x)d_3(x)\dots$. It is sometimes convenient to introduce a special symbol $\bullet$, the 'decimal point', to indicate the initial position: $d_{-m}(x)d_{-m+1}(x)\dots d_0(x)\bullet d_1(x)d_2(x)d_3(x)\dots$ Here $\bullet$ appears only once in the expansion and we ignore $\bullet$ when we treat this bi-infinite word as an element of $\mathcal{A}^{\mathbb{Z}}$. Note that the mark here appears to the right of the 0-th element.

Not every word in $\mathcal{A}^{\mathbb{N}}$ is realized as a beta expansion, as, for example, 9^∞ is not allowed in the tail of a decimal expansion. A word in $\mathcal{A}^{\mathbb{N}}$ is called *admissible* if it is a beta expansion of some $x \geq 0$. We also say a finite word $\omega \in \mathcal{A}^*$ is admissible if $\omega 0^\infty = \omega 00\dots$ is admissible.

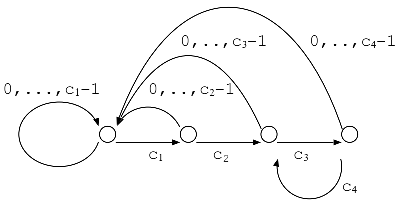
The *beta shift* X_β is the subshift of the full shift on $\mathcal{A}^{\mathbb{Z}}$ consisting of the collection of bi-infinite words over $\mathcal{A}$, all of whose finite subwords are admissible. The function T_β is a discontinuous piecewise linear transform on $[0,1)$ and its discontinuities are essential in describing admissibility. A nice feature of beta-expansion is that admissibility, as we now describe, is easily computed through consideration of the *expansion of one*:

$$\mathbf{d}_\beta(1-) = \lim_{\varepsilon\downarrow 0} \mathbf{d}_\beta(1-\varepsilon)$$

where the limit is in the topology of $\mathcal{A}^{\mathbb{N}}$. The expansion $\mathbf{d}_\beta(1-) = c_1c_2\dots$ is made concrete in the following way. We have $c_1 = \lfloor\beta\rfloor$ and if $T^n(\beta - \lfloor\beta\rfloor) \neq 0$ for all $n \geq 0$, then $c_{n+2} = \lfloor \beta T^n(\beta - \lfloor\beta\rfloor)\rfloor$. If there is n with $T^n(\beta - \lfloor\beta\rfloor) = 0$, then take the smallest n with this property, in which case we have $\mathbf{d}_\beta(1-) = (c_1c_2\dots c_n(c_{n+1}-1))^\infty$. The admissibility condition, called the *Parry condition* is then: a word $x \in \mathcal{A}^{\mathbb{N}}$ is admissible if and only if $s^n(x) \ll \mathbf{d}_\beta(1-)$, where $\ll$ is the lexicographic order (see [98, 66]) and s denotes the shift on $\mathcal{A}^{\mathbb{N}}$. By this characterization, we see that X_β is the set of all bi-infinite sequences of labels of bi-infinite walks on a finite labeled directed graph. More precisely, X_β is a subshift of finite type if and only if $\mathbf{d}_\beta(1-)$ is purely periodic, and it is sofic if and only if $\mathbf{d}_\beta(1-)$ is eventually periodic. If X_β is sofic, then β is a Perron number, and if β is a Pisot number, then X_β must be sofic. Assume that X_β is sofic and put $\mathbf{d}_\beta(1-) = c_1c_2\dots c_m(c_{m+1}c_{m+2}\dots c_{m+\ell})^\infty$. Note that we always have $\ell > 0$, and if X_β is a subshift of finite type, then $m = 0$. Figure 1 shows examples of such graphs for X_β with $\mathbf{d}_\beta(1-)$ as above.



(A) $(c_1c_2c_3c_4)^\infty$



(B) $c_1c_2(c_3c_4)^\infty$

FIGURE 1. Graph of X_β

We write an element $x = (a_n)_{n\in\mathbb{Z}} \in X_\beta$ as $x = \dots a_{-2}a_{-1}a_0 \bullet a_1a_2\dots$ and say that $x_I := \dots a_{-2}a_{-1}a_0$ (resp. $x_F := a_1a_2\dots$) is the *integer part* (resp. the *fractional part*) of x. The *future* of the integer part $x_I = \dots a_{-2}a_{-1}a_0$ is defined by:

$$\mathcal{F}(x_I) = \{b_1b_2\dots \in \mathcal{A}^{\mathbb{N}} \mid \dots a_{-2}a_{-1}a_0 \bullet b_1b_2\dots \in X_\beta\}$$

and the *past* of the fractional part $x_F = a_1a_2\dots$ is defined by:

$$\mathcal{P}(x_F) = \{\dots b_{-2}b_{-1}b_0 \in \mathcal{A}^{\mathbb{N}} | \dots b_{-2}b_{-1}b_0 \bullet a_1a_2\dots \in X_\beta\}.$$

By the above graph characterization, it is clear that X_β is sofic if and only if there are only finitely many distinct future sets $\mathcal{F}(x_I)$ and past sets $\mathcal{P}(x_F)$. If X_β is sofic, we may associate with β a tile substitution with expansion β and (unlabeled) prototiles $\mathcal{F}(x_I)$ by means of the natural map

$$\pi : a_1a_2\dots \mapsto \sum_{i=1}^{\infty} a_i\beta^{-i} \tag{2.6}$$

that satisfies

$$\beta\pi(\mathcal{F}(x_I)) = \bigcup_{x_Ia \text{ is admissible}} \pi(\mathcal{F}(x_Ia)) + \pi(a\bullet) \tag{2.7}$$

where x_Ia denotes the concatenation of x_I and $a \in \mathcal{A}$ and $\pi(a\bullet)$ is just the integer a.

From a beta-shift to a symbolic substitution. A symbolic substitution, denoted ϕ_β, and called the *β-substitution*, may be associated with a sofic beta-shift. This β-substitution is defined by:

$$\begin{aligned} \phi_\beta(1) &= 1^{c_1}2 \\ \phi_\beta(2) &= 1^{c_2}3 \\ &\vdots \\ \phi_\beta(m) &= 1^{c_m} \end{aligned}$$

for the shift of finite type case with $\mathbf{d}_\beta(1-) = (c_1\dots c_m)^\infty$,

$$\begin{aligned} \phi_\beta(1) &= 1^{c_1}2 \\ \phi_\beta(2) &= 1^{c_2}3 \\ &\vdots \\ \phi_\beta(m+\ell-1) &= 1^{c_{m+\ell-1}}(m+\ell) \\ \phi_\beta(m+\ell) &= 1^{c_{m+\ell}}(m+1) \end{aligned}$$

for the general sofic case.

It is easy to see that ϕ_β is primitive, although it may not be irreducible. By primitivity, we get minimal and uniquely ergodic substitutive symbolic and tiling dynamical systems, (X_{ϕ_β}, s) and $(\Omega_{\phi_\beta}, \mathbb{R})$. Up to a rescaling of the $\mathbb{R}$-action and the labeling of tiles, the system $(\Omega_{\phi_\beta}, \mathbb{R})$ is just the system derived from the tiling substitution (2.7).

3. Discreteness of the dynamical and diffraction spectra

3.1. Eigenvalues and coincidence rank

Recall that an eigenfunction for an $\mathbb{R}^n$-action on a space Ω with invariant measure μ is an $L^2(\Omega,\mu)$ function[1] f for which there is an associated eigenvalue $\alpha \in \mathbb{R}^n$ so that $f(T-v) = e^{2\pi i\langle\alpha,v\rangle} f(T)$ for all $v \in \mathbb{R}^n$ and $T \in \Omega$. The $\mathbb{R}^n$-action is said to have *pure discrete spectrum*[2] or *pure point spectrum* if the linear span of the eigenfunctions is dense in $L^2(\Omega,\mu)$. When the $\mathbb{R}^n$-action is ergodic, eigenfunctions must have constant absolute value and eigenvalues are simple: if f and g are eigenfunctions with eigenvalue α, then $f = cg$ for some $c \in \mathbb{C}$. In this situation we may choose eigenfunctions to have values in the unit circle $\mathbb{T}^1$. Furthermore, if $\Omega = \Omega_\Phi$ is a substitution tiling space, eigenfunctions can be chosen to be continuous ([115]). We may view each continuous eigenfunction as a semi-conjugacy, or factor map, between $(\Omega, \mathbb{R}^n)$ and an action of $\mathbb{R}^n$ by translation on the compact abelian group $\mathbb{T}^1$.

The definitions for a $\mathbb{Z}$-action on a space are analogous: for example, an eigenfunction f for the shift s on a symbolic space X with associated eigenvalue $\alpha \in \mathbb{R}$ satisfies $f(s^n(x)) = e^{2\pi i\alpha n} f(x)$ for all $n \in \mathbb{Z}$ and $x \in X$.

It is a consequence of the Halmos-von Neumann theory that the $\mathbb{R}^n$-action on a tiling space (likewise, the shift on a symbolic space) has pure discrete spectrum if and only if the action is measurably conjugate to translation on a compact abelian group (see [121, 96], and see Theorems 5.2 and 7.8 for illustrations).

Every primitive FLC substitution tiling dynamical system $(\Omega_\Phi, \mathbb{R}^n)$ has a maximal equicontinuous factor $(\hat{\mathbb{T}}, \mathbb{R}^n)$ with factor map $g : \Omega \to \hat{\mathbb{T}}$ and g is a.e. m-to-1 for some $m \in \mathbb{N} \cup \{\infty\}$. (Here $\hat{\mathbb{T}}$ is a torus or solenoid, the $\mathbb{R}^n$-action is a Kronecker action, and the map g is obtained by considering all eigenfunctions.) The number m is called the *coincidence rank* of Φ and is denoted by $cr(\Phi)$: by the Halmos-von Neumann theory, $(\Omega_\Phi, \mathbb{R}^n)$ has pure discrete spectrum if and only if $cr(\Phi) = 1$. It follows from [113, 29, 21] that, for one-dimensional primitive tile substitutions Φ, $cr(\Phi) < \infty$ if and only if the expansion λ for Φ is a Pisot number.

If the dynamical spectrum of a tiling $\mathcal{T}$ is pure discrete, the eigenvalues for the dynamical system $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ span $\mathbb{R}^n$. Since every additive combination of eigenvalues for the dynamical system is also an eigenvalue, the eigenvalues are relatively dense. In the next section we translate this necessary condition for pure discrete spectrum (that is, relative denseness of the eigenvalues) into a condition expressed in terms of substitution Delone multi-color sets, namely, the Meyer property.

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[1] In this measure-theoretical framework, statements about eigenfunctions should be interpreted to hold a.e..

[2] Note that here 'discrete' refers to a property of the span of the eigenfunctions and not to discreteness of the eigenvalues as a set.

3.2. The Meyer property

For the study of the discreteness of the dynamical spectrum of a tiling space, the so-called Meyer property plays an important role. A Delone set Y is called a *Meyer set* if $Y-Y$ is uniformly discrete. Necessarily, if Y is a Meyer set, then Y has FLC. Suppose that $\mathcal{T}$ is a primitive FLC substitution tiling in $\mathbb{R}^n$ and that $\underline{\Gamma}_{\mathcal{T}} = \bigcup_{i=1}^m \Gamma_i \times \{i\}$ is a substitution Delone multi-color set associated with $\mathcal{T}$, as described in Section 2.3. When $\Gamma_{\mathcal{T}} = \cup_{1\le i\le m}\Gamma_i$ is a Meyer set, we say that the tiling $\mathcal{T}$ has the *Meyer property.* For characterizations of the Meyer property, see [73] for $\mathbb{R}^n$, and [20] for the general case of a σ-compact, locally compact abelian group.

The connection between Meyer sets and the spectrum of an FLC substitution tiling can be viewed through the mathematical concept of diffraction measure developed by Hof [61, 62]. For a Delone set Γ, consider $\delta_\Gamma = \sum_{x\in\Gamma}\delta_x$. Let $\gamma(\delta_\Gamma)$ denote its autocorrelation (assuming it is unique), that is, the vague limit[3]

$$\gamma(\delta_\Gamma) = \lim_{r\to\infty}\frac{1}{\mathrm{Vol}(B_r)}\left(\delta_\Gamma|_{B_r} * \widetilde{\delta_\Gamma|_{B_r}}\right), \tag{3.1}$$

where B_r is a ball of radius r. The Fourier transform $\widehat{\gamma(\delta_\Gamma)}$ is called the *diffraction measure* for δ_Γ. We say that the measure δ_Γ, or the Delone set Γ has *pure point diffraction spectrum*, if $\widehat{\gamma(\delta_\Gamma)}$ is a pure point or discrete measure. The point masses of the diffraction measure are called *Bragg peaks.*

There is a close correspondence between pure pointedness of the diffraction measure and pure discrete spectrum of the associated dynamical system. For example, if $\mathcal{T}$ is a primitive FLC substitution tiling in $\mathbb{R}^n$ and $\underline{\Gamma}_{\mathcal{T}} = \bigcup_{i=1}^m \Gamma_i \times \{i\}$ is a substitution Delone multi-color set associated with $\mathcal{T}$, then each Γ_i has pure point diffraction spectrum if and only if $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ has pure discrete spectrum ([17, 18, 48, 60, 80, 81, 82, 102]).

The following theorem is due to Strungaru [116] and Lee-Solomyak [77].

Theorem 3.1. [116, 77] *Suppose that $\mathcal{T}$ is a primitive FLC substitution tiling in $\mathbb{R}^n$ and that $\underline{\Gamma}_{\mathcal{T}} = \bigcup_{i=1}^m \Gamma_i \times \{i\}$ is a substitution Delone multi-color set associated with $\mathcal{T}$. The following are equivalent:*
(i) The set of locations of Bragg peaks for each Γ_i is relatively dense in $\mathbb{R}^n$.
(ii) The set of eigenvalues for $(\Omega_{\mathcal{T}}, \mathbb{R}^n, \mu)$ is relatively dense in $\mathbb{R}^n$.
(iii) $\Gamma = \cup_{1\le i\le m}\Gamma_i$ is a Meyer set.

In particular, if $\mathcal{T}$ is a primitive FLC substitution tiling for which the dynamical system $(\Omega_{\mathcal{T}}, \mathbb{R}^n, \mu)$ has pure discrete spectrum, then $\Gamma_{\mathcal{T}} = \cup_{1\le i\le m}\Gamma_i$ is a Meyer set. In other words, the Meyer property is necessary for pure discrete spectrum of an FLC substitution tiling space. This is not generally true for the hull of a tiling that does not arise from a substitution: it fails, for example, for modulated lattices (see [16, 79, 110, 118]).

[3] Recall that if f is a function in $\mathbb{R}^n$, then $\tilde{f}$ is defined by $\tilde{f}(x) = \overline{f(-x)}$. If μ is a measure, $\tilde{\mu}$ is defined by $\tilde{\mu}(f) = \overline{\mu(\tilde{f})}$ for all $f \in \mathcal{C}_0(\mathbb{R}^n)$.

3.3. Cut-and-project schemes and model sets

We now endeavor to fill the gap between relative density for eigenvalues and pure discreteness of the dynamical spectrum. This is the aim of the present section where the notions of regular model set and inter-model set are introduced in the setting of cut-and-project schemes. For more about model sets see, *e.g.*, the surveys [88, 89] and [16, Chap. 7] where an explicit discussion with the silver mean as an example can be found.

A *cut-and-project scheme* (or *CPS* for short) consists of a triple $(G, H, \widetilde{L})$ where G and H are σ-compact, locally compact abelian groups, $\widetilde{L}$ is a lattice, *i.e.*, a discrete subgroup for which the quotient group $(G \times H)/\widetilde{L}$ is compact, such that the restriction of the canonical projection $\pi_1 : G \times H \to G$ to $\widetilde{L}$ is injective and the image $L^\star = \pi_2(\widetilde{L})$ of $\widetilde{L}$ under the canonical projection $\pi_2 : G \times H \to H$ is dense in H. Schematically, we summarize the situation as follows by setting $L = \pi_1(\widetilde{L})$:

$$\begin{array}{ccccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ \cup & & \cup & & \cup \text{ dense} \\ L & \xleftarrow{1-1} & \widetilde{L} & \longrightarrow & L^\star \end{array} \tag{3.2}$$

For a subset $W \subset H$, called the *acceptance window*, let

$$\varGamma(W) = \{\pi_1(x) \in G \,:\, x \in \widetilde{L}, \pi_2(x) \in W\}\,.$$

A subset of G of the form $\varGamma = \varGamma(W) + g$, where $g \in G$ and $W \subset H$ has non-empty interior and compact closure, is called a *model set*. The non-empty interior property yields that any model set is relatively dense in G ([109]). Intuitively, being a model set means that all the points in $\varGamma$ are obtained through the selection process by the window W.

Models sets are deeply connected to Meyer sets. In fact all model sets are Meyer sets ([78]). The converse reads as follows: a relatively dense set in G is a Meyer set if and only if it is a subset of a model set of G ([88]). See [19, 20, 76, 89, 101] for more on model sets.

Applying Theorem 3.1, we obtain the following.

Corollary 3.2. *Suppose that $\mathcal{T}$ is a primitive substitution tiling in $\mathbb{R}^n$ and that $\underline{\varGamma}_{\mathcal{T}} = \bigcup_{i=1}^m \varGamma_i \times \{i\}$ is the substitution Delone multi-color set associated with $\mathcal{T}$. If $\varGamma = \cup_{1 \le i \le m} \varGamma_i$ is a model set, then the set of eigenvalues for $(\Omega_{\mathcal{T}}, \mathbb{R}^n, \mu)$ is relatively dense in $\mathbb{R}^n$.*

Note that additional conditions are needed to guarantee pure discrete spectrum. We say that a model set $\varGamma$ is *regular* if the boundary $\partial W = \overline{W} \setminus W^\circ$ of W is of (Haar) measure 0.

Theorem 3.3. [102] *If $\varGamma \subset R^n$ is a regular model set, then $(\Omega_\varGamma, \mathbb{R}^n)$ has pure discrete spectrum.*

Theorem 3.4. [81] *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^n$ with associated Delone multi-color set $\underline{\varGamma}_{\mathcal{T}} = \bigcup_{i=1}^m (\varGamma_i \times \{i\})$ and suppose that $L = \bigcup_{1 \le i \le m} \varGamma_i$ is a lattice. Then each $\varGamma_i$ is a regular model set for $1 \le i \le m$ if and only if $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ has pure discrete spectrum.*

In the general case, we define an *inter-model set* as a subset Γ of G for which $s+\Gamma(W^\circ)\subset\Gamma\subset s+\Gamma(W)$ for some $s\in\mathbb{R}^n$, where W is compact in H and $W=\overline{W^\circ}\neq\emptyset$. Inter-model sets are Delone sets, and all regular models also are inter-model sets (see [109] for more detail).

Theorem 3.5. [75] *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^n$ with FLC and $\underline{\Gamma}_{\mathcal{T}}=\bigcup_{i=1}^m(\Gamma_i\times\{i\})$ be an associated substitution Delone multi-color set. Then Γ_i is an inter-model set if and only if $(\Omega_{\mathcal{T}},\mathbb{R}^n)$ has pure discrete spectrum.*

Dynamical systems generated by regular or inter-model sets coming from primitive substitutions thus have pure discrete spectrum. Section 3.4 and 3.5 exhibit natural candidates for acceptance windows and cut-and-project schemes to be associated with symbolic substitutions and beta-numeration. Their particular structure allows a convenient formulation of sufficient conditions ensuring that these constructions indeed yield regular model sets. As will be discussed in Section 5, pure discrete substitution tiling systems can be viewed as inter-model sets. This will require the introduction of intermediary conditions based on notions of coincidences (see in particular Theorems 5.4 and 5.5). Checking whether these additional conditions are satisfied is at the core of the Pisot Substitution Conjecture.

3.4. Rauzy fractals and symbolic substitutions

In the case of a symbolic substitution ϕ, there is a natural way to exhibit a cut-and-project scheme for the associated substitution tiling Φ. This approach was initiated by Rauzy in the seminal paper [97] and is based on the notion of Rauzy fractal, an object that will be our candidate both for being an acceptance window for a cut-and-project scheme such as defined in (3.2), and equivalently, for being a fundamental domain for the Kronecker action in case of pure discrete spectrum.

Suppose that ϕ is a symbolic Pisot substitution, and, for ease of exposition, that ϕ is also unimodular and irreducible. Thus, if d is the degree of the PF-eigenvalue, λ, of M_ϕ (d is also equal to the cardinality of the alphabet), then M_ϕ is $d\times d$ and $\mathbb{R}^d$ splits as the direct sum $\mathbb{R}^d=E^s\oplus E^u$ of $d-1$ and 1-dimensional M_ϕ-invariant subspaces. Let $\pi_s:\mathbb{R}^d\to E^s$ and $\pi_u:\mathbb{R}^d\to E^u$ denote the corresponding projections and let r be a positive right PF-eigenvector for M_ϕ (so $E^u=\{t\,r:t\in\mathbb{R}\}$). Let u be a ϕ-periodic point. The *i-th Rauzy piece* for ϕ is the subset of E^s

$$\mathcal{R}_i:=\overline{\{\pi_s(f(u_0\cdots u_j)):j\geq 0,u_{j+1}=i\}}$$

and the *Rauzy fractal* for ϕ is the union of the pieces

$$\mathcal{R}:=\cup_{i=1}^d\mathcal{R}_i.$$

The $\mathcal{R}_i$ do not depend on u, each is the closure of its interior and has zero measure boundary. For general properties of Rauzy fractals, see, *e.g.*, [97, 14, 95, 35, 39].

As constructed above, the Rauzy fractal is obtained as the closure of the stable projection of a certain subset of points from the integer lattice $\mathbb{Z}^d$. We will see in Section 3.5 below, in the context of beta-numeration, that the Rauzy can also be obtained using the Minkowski embedding of $\mathbb{Q}(\lambda)$ in $\mathbb{R}^d$. This is the approach of [117] and has been extended through the use of non-Euclidean representation spaces to allow for a definition of the Rauzy fractal when the Pisot number λ (the PF-eigenvalue) is not a unit (see [107, 109]).

Rauzy fractals and cut-and-project schemes. Given an irreducible unimodular Pisot substitution ϕ on d letters, the d entries of left and right PF-eigenvectors of the substitution matrix M_ϕ are linearly independent over $\mathbb{Q}$. It follows that the projection $\pi_u : \mathbb{R}^d = E^s \oplus E^u \to E^u \simeq \mathbb{R}$ is one-to-one on the lattice $\mathbb{Z}^d$ and that $\pi_s(\mathbb{Z}^d)$ is dense in $E^s \simeq \mathbb{R}^{d-1}$. Using the Rauzy fractal $\mathcal{R} \subset E^s$ as an acceptance window, we may define a Delone set $\Gamma(\mathcal{R}) \subset E^u$ by projecting into E^u, via π_u, those elements of $\mathbb{Z}^d$ that lie in $\pi_s^{-1}(\mathcal{R})$:

$$\Gamma(\mathcal{R}) := \{\pi_u(x) \in E^u : x \in \mathbb{Z}^d,\ \pi_s(x) \in \mathcal{R}\}.$$

Let $V := \{f(u_0 \cdots u_{k-1}) : k \in \mathbb{N}\} \cup \{-f(u_{-k} \cdots u_{-1}) : k \in \mathbb{N}\}$. (The set V is the set of vertices of the *strand* S_u to be defined in Section 5.2 below.) One sees easily from the definition of $\mathcal{R}$ that $\pi_u(V) \subset \Gamma(\mathcal{R})$.

Delone sets that arise from such a cut-and-project procedure have particularly nice spectral properties and it thus becomes relevant for the study of the dynamical systems (X_ϕ, s) and $(\Omega_\phi, \mathbb{R})$ to ask the question: is $\pi_u(V)$ essentially the same as $\Gamma(\mathcal{R})$? By observing that the boundary of $\mathcal{R}$ has zero measure, a positive answer yields that (X_ϕ, s) has pure discrete spectrum. Hence, most conditions ensuring pure discrete spectrum expressed in terms of Rauzy fractals can be reformulated as sufficient conditions for the associated substitution Delone multi-color set to be a regular model set with acceptance window given by the Rauzy fractal $\mathcal{R}$. As an illustration, the equality between $\pi_u(V)$ and $\Gamma(\mathcal{R})$ can be proved in particular cases by studying the way that particular polygonal approximations of the boundary of $\mathcal{R}$ converge to this boundary ([52, 65, 84]).

Periodic multi-tilings in $\mathbb{R}^d$. In order to make explicit the fact that the Rauzy fractal is a natural candidate for a fundamental domain for the Kronecker action for the maximal equicontinuous factor of Ω_ϕ, we endeavor to construct tilings of E^s by translates of the $\mathcal{R}_i$. For each $i = 1, \ldots, d$, let e_i denote the i-th standard unit vector. Let us fix a vector $y \in E^u$. Let $t_i \in \mathbb{R}^+$ be such that $t_i\, y = \pi_u(e_i)$, and let $\mathcal{C}_i := \{x + t\, y : x \in \mathcal{R}_i \text{ and } 0 \le t \le t_i\}$. The collection $\mathcal{M} := \{\mathcal{C}_i + v : 0 \le i \le d \text{ and } v \in \mathbb{Z}^d\}$ is a periodic *multi-tiling* of $\mathbb{R}^d$ by translations of the $\mathcal{C}_i$: there is $m \in \mathbb{N}$, called the *degree*, or *multiplicity* of the multi-tiling so that almost every $x \in \mathbb{R}^d$ lies in exactly m tiles of $\mathcal{M}$. In fact, the degree m is equal to the coincidence rank $cr(\phi)$ of ϕ introduced in Section 3.1 (see [30]).

Theorem 3.6. [65, 30] *Let ϕ be a primitive Pisot substitution that is also unimodular and irreducible. The spaces (X_ϕ, s) and $(\Omega_\phi, \mathbb{R})$ have pure discrete spectrum if and only if the degree m of the multi-tiling associated with ϕ is 1.*

(For ϕ primitive, Pisot, and irreducible, pure discreteness of (X_ϕ, s) and $(\Omega_\phi, \mathbb{R})$ are equivalent ([45]) - see Section 4.)

Several approaches have been developed in order to make the tiling condition of Theorem 3.6 algorithmic. Indeed, the intersections between the sets $\mathcal{C}_i + v$ have a self-similar structure that can be described in terms of finite graphs (see the monograph [108] and the bibliography therein for more detail). Another approach consists in studying the *dual multi-tiling* associated with ϕ, that is, the multi-tiling of E^s given by $\{T \cap E^s : T \in \mathcal{M}\}$. This dual multi-tiling of the $(d-1)$-dimensional space E^s can be described via a discrete combinatorial action, namely a generalized (also called dual) substitution ([14, 30]). Each prototile of this dual multi-tiling can be characterized by a pair made of a Rauzy piece and a face of the discrete plane associated with the contracting space E^s. This discrete plane is stable under the action of the associated dual substitution. As a consequence, further combinatorial formulations of the pure discrete spectrum property can be stated in effective terms (see [14, 39] for more detail). These conditions are dual versions of the notions of geometric coincidence developed in Section 5.2 (where the tilings live in the unstable space E^u). Observe also that the terminology 'dual' is here consistent with the notion of star-duality for self-similar cut-and-project tilings such as developed in [117, 55]; see also [37] for an illustration of these connections in the two-letter case.

3.5. Cut-and-project schemes and beta-numeration

We present here an arithmetical version of the previous construction for the beta shift due originally to Thurston [117], which was itself inspired by Rauzy's construction [97]. For more on these constructions, see [1, 3, 5, 58].

Let β be a Pisot unit of degree d with real algebraic conjugates $\beta = \beta_1, \ldots, \beta_r$, and complex algebraic conjugates $\beta_{r+1}, \bar{\beta}_{r+1}, \ldots, \beta_{r+s}, \bar{\beta}_{r+s}$. The *conjugate map* $\Psi : \mathbb{Q}(\beta) \to \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}$ is defined by $x \mapsto (x^{(2)}, \ldots, x^{(r)}, x^{(r+1)}, \ldots x^{(r+s)})$ where $x^{(i)}$ denotes the image of x under the embedding of $\mathbb{Q}(\beta)$ into $\mathbb{C}$ that takes β to β_i. Hereafter, we identify $\mathbb{R}^{r-1} \times \mathbb{C}^s$ with $\mathbb{R}^{d-1}$.

Beta-tiles. Given a fractional part x_F of an element $x \in X_\beta$, and given $\ldots b_{-2}b_{-1}b_0$ in the past set $\mathcal{P}(x_F)$ (see Section 2.4), we define $\tilde{\Psi}(\ldots b_{-2}b_{-1}b_0) \in \mathbb{R}^{d-1}$ by $\tilde{\Psi}(\ldots b_{-2}b_{-1}b_0) := \sum_{i=0}^{\infty} b_i \Psi(\beta^i)$. The series converges since the conjugates of β have absolute value less than 1 and the *prototiles* $\tilde{\Psi}(\mathcal{P}(x_F))$ correspond to the Rauzy pieces. More generally, versions of the Rauzy fractal and arithmetical coding when the Pisot number β is not a unit require either the inclusion of a non-archimedean component in the conjugate map (in the arithmetical setting) or the use of inverse limits (in the topological setting). See [109, 35, 21, 10] for more detail.

Non-periodic multi-tilings. Beta-tiles are analogues of Rauzy pieces. As in the symbolic substitution case, there is a natural multi-tiling that can be formed with the beta-tiles. The difference here is that this multi-tiling is not periodic. In fact, it is strongly connected with the dual multi-tiling of E^s discussed above for symbolic substitutions and as highlighted in [35, 36].

For $x_F = a_1a_2\dots$ finite (that is, $a_i = 0$ for all i bigger than some k), the tile

$$\mathcal{T}(x_F) := \tilde{\Psi}(\mathcal{P}(x_F)) + \Psi(\pi(x_F))$$

is a translate of a prototile. (We recall that π is defined in (2.6).) Let points $\ell_i \in [0,1]$ be determined by $\{0 = \ell_0 < \ell_1 < \cdots < \ell_{m+\ell=1}\} = \{0\} \cup \{\pi(s^n(\mathbf{d}_\beta(\mathbf{1}-)))\}$. Then, if $x, y \in X_\beta$ have finite fractional parts and if $\pi(x_F), \pi(y_F) \in [\ell_{i-1}, \ell_i)$, we have $\mathcal{P}(x_F) = \mathcal{P}(y_F)$ so that $\mathcal{T}(y_F) = \mathcal{T}(x_F) + \Psi(\pi(y_F) - \pi(x_F))$ and thus there are (at most) $m + \ell$ different tiles up to translation.

The linear isomorphism of $\mathbb{Q}(\beta)$ given by multiplication by $1/\beta$ extends, after embedding $\mathbb{Q}(\beta)$ in $\mathbb{R}^{d-1}$ via Ψ, to an expanding linear isomorphism G of $\mathbb{R}^{d-1}$ and the tiles $\mathcal{T}(x_F)$ satisfy the substitution rule:

$$G(\mathcal{T}(x_F))) = \bigcup_{ax_F \text{ is admissible}} \mathcal{T}(ax_F).$$

The collection $\{\mathcal{T}(x_F) : x_F \text{ finite}\}$ is a multi-tiling of $\mathbb{R}^{d-1}$ invariant under this substitution rule ([3]). We will see in Section 6 equivalent arithmetic formulations of the following result.

Theorem 3.7. [3, 21, 6] *Let β be a Pisot unit of degree d. If the collection $\{\mathcal{T}(x_F) : x_F \text{ finite}\}$ is a tiling of $\mathbb{R}^{d-1}$, then $(\Omega_{\phi_\beta}, \mathbb{R})$ has pure discrete spectrum.*

Markov partitions. A Markov partition for a hyperbolic toral automorphism provides an explicit measurable conjugacy with a subshift of finite type by recording the itinerary of a point under the action of the automorphism with respect to the partition. According to [44], all hyperbolic toral automorphisms admit Markov partitions, but they can be rather difficult to find.

Sidorov describes in [105] the following procedure for constructing Markov partitions for certain total automorphisms (see also [93] and [103, 104]). In [70] and [106] this construction arises in the general framework of arithmetic dynamics and arithmetic codings. Given a Pisot unit β of degree d, let M be the companion matrix of its minimal polynomial. There is then an associated hyperbolic automorphism, F_M, of the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ defined by $F_M(x + \mathbb{Z}^d) := Mx + \mathbb{Z}^d$.

With E^s and E^u denoting the stable and unstable spaces of M, the stable and unstable manifolds of $0 \in \mathbb{T}^d$ under F_M are $W^s(0) := E^s + \mathbb{Z}^d$ and $W^u(0) := E^u + \mathbb{Z}^d$, and the *homoclinic group* of 0 is the intersection $\mathcal{H} = W^s(0) \cap W^u(0)$. A point $\bar{y} \in \mathcal{H}$ is a *fundamental homoclinic point* if its orbit $\{F_M^n(\bar{y}) : n \in \mathbb{Z}\}$ generates the group $\mathcal{H}$. Given any $\bar{y} \in \mathcal{H}$, let $h_{\bar{y}} : X_\beta \to \mathbb{T}^d$ be defined by

$$h_{\bar{y}}(\cdots x_{-1}x_0 \bullet x_1 \cdots) := \sum_{n\in\mathbb{Z}} x_n F_M^{-n}(\bar{y}).$$

The map $h_{\bar{y}}$, called an *arithmetical coding* of F_M, is continuous, a.e. m-to-1 for some $m \in \mathbb{N}$, and semi-conjugates the shift s on X_β with F_M. Moreover, if $\underline{x}, \underline{y}$, and $\underline{x+y}$ denote the (greedy) β-expansions of the non-negative real numbers x, y and $x+y$, then $h_{\bar{y}}(\underline{x+y}) = h_{\bar{y}}(\underline{x}) + h_{\bar{y}}(\underline{y})$.

For each past set $\mathcal{P}$, let $[\mathcal{P}] := h_{\bar{y}}(\{x \in X_\beta : x_P = \mathcal{P}\})$. The intersections of $E^s \simeq \mathbb{R}^{d-1}$ with the sets $[\mathcal{P}]$ under the immersion which maps $x \in E^s$ to $x + \mathbb{Z}^d \in W^s(0)$ determine a degree m multi-tiling. If $m = 1$, the sets $[\mathcal{P}]$, where $\mathcal{P}$ are past sets constructed from a fundamental homoclinic point $\bar{y}$, form a Markov partition for the hyperbolic automorphism F_M associated with M ([105]). When the associated beta-substitution ϕ_β is irreducible, the sets $[\mathcal{P}]$ actually match with the sets $\mathcal{C}_i$ of the periodic multi-tiling of $\mathbb{R}^d$ built from the Rauzy pieces $\mathcal{R}_i$ introduced in Section 3.4. One similarly determines Markov partitions for toral automorphisms provided by the substitution matrix of unimodular irreducible Pisot substitutions when the tiling system has pure discrete spectrum; see, *e.g.*, [95, 65, 15].

It is proved in [105] that $m = 1$ (for a fundamental homoclinic point $\bar{y}$) if and only if $(\Omega_{\phi_\beta}, \mathbb{R})$ has pure discrete spectrum ([3, 21]). This property is itself equivalent to β satisfying the Property (W) discussed in Section 6 (see also [93] in the same vein).

4. The Pisot Substitution Conjecture

In this section, we focus on the one-dimensional case $n = 1$, that is, we consider either symbolic substitutions, beta-numeration, or tile substitutions defined in $\mathbb{R}$.

Recall that an algebraic integer $\lambda > 1$ is a *Pisot number* if all its algebraic conjugates α other than λ itself satisfy $|\alpha| < 1$. In terms of its influence on spectral properties of associated dynamical systems, the key property of a Pisot number λ is that the distance from λ^n to the nearest integer tends to zero as n tends to infinity. (Conversely, if λ is any algebraic number bigger than 1 with this property, then λ must be a Pisot number ([92]); it is a conjecture of Pisot that no transcendental number has this property.) A primitive substitution ϕ is said to be *Pisot* if its Perron–Frobenius eigenvalue λ is a Pisot number. By *Pisot dynamics* we mean, loosely, the shift dynamics on the symbolic space associated with a Pisot substitution, or the translation dynamics on a substitution tiling space with expansion $\Lambda = (\lambda)$, λ a Pisot number (or, more generally, Λ with the Pisot property, see Section 7), etc.

For the shift, or translation, dynamics associated with a substitution to have pure discrete spectrum a Pisot condition is necessary. The Pisot Substitution Conjectures have grown out of attempts to answer the question: what additional conditions guarantee that Pisot dynamics have pure discrete spectrum? In previous sections we have considered several types of Pisot dynamical systems, all arising from substitutions in a symbolic or geometrical context. The most easily recognized feature shared by all known examples that fail to have pure discrete spectrum is reducibility of the characteristic polynomial of the substitution matrix. Correspondingly, the most basic of the

Pisot Substitution Conjectures, formulated separately for the symbolic and tiling contexts, are the following.

Conjecture 4.1 (Pisot Substitution Conjecture: symbolic substitutive case). If ϕ is an irreducible Pisot substitution then the substitutive system (X_ϕ, s) has pure discrete spectrum.

Conjecture 4.2 (Pisot Substitution Conjecture: tiling of the line case). If ϕ is an irreducible Pisot substitution then the tiling dynamical system $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum.

That these two conjectures are equivalent is a consequence of a theorem of Clark and Sadun ([45]). Indeed, (X_ϕ, s) has pure discrete spectrum if and only if the $\mathbb{R}$-action on the suspension $\hat{X}_\phi$, with constant roof function, has pure discrete spectrum. If the constant value c of the roof function is chosen so that $c(1, \ldots, 1) - \ell$, ℓ the positive left eigenvector of the substitution matrix M_ϕ giving the tile lengths, is in the left-contracting space of the substitution matrix M_ϕ (so that $(c(1, \ldots, 1) - \ell)M_\phi^m \to 0$ as $m \to \infty$) then, according to Theorem 3.1 of [45], the $\mathbb{R}$-actions on $\hat{X}_\phi$ and Ω_ϕ are topologically conjugate. Such a choice of c is always possible in the irreducible case as then the left contracting space has codimension one.

The Thue-Morse substitution ($1 \mapsto 12$, $2 \mapsto 21$) provides a simple example showing that the above conjectures are false if the hypothesis of irreducibility is dropped. No such examples have been found for β-substitutions (see Section 2.4) and in this setting there is a stronger conjecture.

Conjecture 4.3 (Pisot Substitution Conjecture: β-substitution case). If β is a Pisot number with associated β-substitution ϕ_β then the tiling dynamical system $(\Omega_{\phi_\beta}, \mathbb{R})$ has pure discrete spectrum.

For counterexamples to the above (with reducible substitution matrix M_{ϕ_β}) for the substitutive system (X_{ϕ_β}, s), rather than the tiling dynamics, see [49]. If the tiling dynamical system $(\Omega_{\phi_\beta}, \mathbb{R})$ has pure discrete spectrum, these examples show that the symbolic substitutive system (X_{ϕ_β}, s) may not. Nonetheless, in this situation (X_{ϕ_β}, s) is at least measurably isomorphic with an induced system of a group translation (see [21]).

The assumption of 'irreducibility' in Conjectures 4.1 and 4.2 is unnatural in the following sense. It is easy to take an irreducible Pisot substitution and 'rewrite' it to obtain another substitution that is not irreducible but has topologically conjugate dynamics ([24]). One substitution will satisfy the hypotheses while the other won't, yet their spectral properties are identical. In fact, if two one-dimensional substitution tiling spaces are homeomorphic, then the corresponding $\mathbb{R}$-actions are (up to rescaling) conjugate ([31], and [71] for higher dimensions). It is thus desirable to find a topological condition to replace the assumption of irreducibility of the substitution matrix.

The substitution homeomorphism $\Phi : \Omega_\phi \to \Omega_\phi$ induces a linear isomorphism $\Phi^* : H^1(\Omega_\phi) \to H^1(\Omega_\phi)$ on the (Čech, with rational coefficients) cohomology of Ω_ϕ. The expansion λ is an eigenvalue of this isomorphism so the dimension of $H^1(\Omega)$ is at least the algebraic degree d of λ ([26]). Thus Φ^* is irreducible (that is, its characteristic polynomial is irreducible over $\mathbb{Q}$) if and only if the dimension of $H^1(\Omega_\phi)$ equals d. If Φ^* is irreducible, we will say that ϕ is *homologically irreducible* and that ϕ is a *homological Pisot substitution* if ϕ is also a Pisot substitution.

Conjecture 4.4 (Homological Pisot Substitution Conjecture). If ϕ is a homological Pisot substitution whose expansion is an algebraic unit then the tiling dynamical system $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum.

By [46], and as in the comments above following Conjecture 4.2, if ϕ is homological Pisot, then (X_ϕ, s) has pure discrete spectrum if and only if $(\Omega_\phi, \mathbb{R})$ does: a similar remark applies to the Coincidence Rank Conjecture below.

Conjecture 4.4 is neither weaker nor stronger than Conjectures 4.1 and 4.2 (with an additional assumption of unimodularity). However, the only way an irreducible substitution ϕ can fail to be homologically irreducible is for there to be an 'asymptotic cycle' of arc components in Ω_ϕ which is associated with a root of unity eigenvalue for Φ^* ([25]).

There are counterexamples to Conjecture 4.4 if the expansion is not assumed to be a unit ([33]) but the conjecture can be extended in the following way. The *norm* of an algebraic integer λ is the product of λ with all its algebraic conjugates. So λ is a unit if and only if its norm is ± 1 and if λ is the PF-eigenvalue of a substitution matrix with irreducible characteristic polynomial M_ϕ, then the norm of λ is $\pm \det(M_\phi)$.

See Section 3.1 for the definition of the 'coincidence rank' of a substitution and recall that the coincidence rank of a Pisot substitution equals one if and only if the system $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum. Thus the following conjecture extends the Homological Pisot Substitution Conjecture to the non-unit case.

Conjecture 4.5 (Coincidence Rank Conjecture). If ϕ is a homological Pisot substitution with expansion λ then the coincidence rank of ϕ divides the norm of λ.

5. Coincidences

We present here several variations of the notion of coincidence. Strong coincidences (see Section 5.1), geometric coincidences (Section 5.2) and balanced pairs (Section 5.3) are defined in the one-dimensional case $n = 1$, whereas coincidences in higher dimensions are handled in Section 5.4 with the notions of modular coincidence, overlap coincidence and algebraic coincidence. We then revisit the notion of algebraic coincidence in the particular case of beta-numeration in Section 5.5.

5.1. Strong coincidences

A primitive substitution ϕ is said to satisfy the *strong coincidence condition* if each pair of ϕ-periodic points $x = (\dots x_{-1}x_0x_1 \dots), y = (\dots y_{-1}y_0y_1 \dots) \in X_\phi$ are *strongly coincident*, *i.e.*, there is $n \in \mathbb{N}$ so that $x_n = y_n$ and the abelianizations $f(x_0 \dots x_n)$ and $f(y_0 \dots y_n)$ are equal. This combinatorial condition, originally due to Arnoux and Ito [14], is an extension of a similar condition considered by Kamae [67] and Dekking [47] in the case of *constant length substitutions* (that is, when $|\phi(a)| = |\phi(b)|$ for all $a, b \in \mathcal{A}$). In [47], Dekking proves that a constant length substitution having trivial height satisfies the strong coincidence condition if and only if (X_ϕ, s) has pure discrete spectrum.

Conjecture 5.1 (Strong coincidence Conjecture). Every irreducible Pisot substitution satisfies the strong coincidence condition.

It is not known whether or not the Coincidence Conjecture is equivalent to the Pisot Substitution Conjecture (that is, if strong coincidence implies pure discrete spectrum). We formulate now a geometrical version of coincidence that is equivalent to pure discrete spectrum.

5.2. Geometric coincidences

The idea of associating a geometrical strand with an element of a symbolic substitution space originates with Arnoux and Ito in [14]. There, the authors present a geometrical version of a substitution: letters become segments in $\mathbb{R}^d$ with integer vertices, and their images under substitution become 'broken lines'. This allows for a dualization of the substitution: dual to a segment is a $d-1$ cell which, under the dual substitution becomes a piece of a broken hyperplane. With these tools, the authors prove that if an irreducible unimodular Pisot substitution satisfies the strong coincidence condition, then its substitutive system is measurably conjugate to a domain exchange in $\mathbb{R}^{d-1}$ and semi-conjugate to a rotation on the $(d-1)$-torus. The idea is further developed in [30, 21], where the segments are no longer required to have vertices on the integer lattice. With this innovation, the tiling space itself, in the form of strand space, emerges as the global attractor of a geometrical substitution. Each element of the tiling space now corresponds uniquely to a strand in $\mathbb{R}^d$ and the map taking the tiling to a vertex of its strand, modulo $\mathbb{Z}^d$, semi-conjugates the tiling flow with a Kronecker action on $\mathbb{T}^d$ and the substitution homeomorphism with a hyperbolic automorphism of $\mathbb{T}^d$.

For simplicity in what follows, we assume that ϕ is an irreducible and unimodular Pisot substitution on d letters; also, we construct strand space as an orbit closure, rather than as a global attractor.

To begin with, for $x \in \mathbb{R}^d$ and e_i a standard unit vector, we call the set

$$[x, i] := \{x + te_i : 0 \le t \le 1\}$$

a *segment of type* i with *initial vertex* x and *terminal vertex* $x + e_i$. Suppose that $u \in X_\phi$ is ϕ-periodic. Employing the abelianization map f, the *strand* *associated with* u is the union of segments:

$$S_u = \bigcup_{N\in\mathbb{N}} [f(u_0\cdots u_{N-1}), u_N] \cup \bigcup_{N\in\mathbb{N}} [-f(u_{-N-1}\cdots u_{-1}), u_{-N-1}] \, .$$

The tiling $\mathcal{T}_u$ is recovered from the strand S_u by projection onto the unstable space $E^u = \{tr : t \in \mathbb{R}\}$ spanned by the right PF-eigenvector r of the abelianization M_ϕ of ϕ [34].

The *strand space* associated with ϕ is the translation-orbit closure

$$\Sigma_\phi := \overline{\{S_u - tr : t \in \mathbb{R}\}},$$

the closure being taken in the topology in which two strands are close if their intersections with a large closed ball about the origin are Hausdorff-close. Given a segment $[x, i]$, let

$$\Phi([x, i]) := \cup_{j=1,\ldots,k}[M_\phi x + f(i_1\cdots i_{j-1}), i_j],$$

where $\phi(i) = i_1\cdots i_k$. Then, for $S = \cup_j[x_j, l_j] \in \Sigma_\phi$, $\Phi(S) := \cup_j\Phi([x_j, l_j])$ defines a self-homeomorphism of Σ_ϕ. Strand space is simply another presentation of tiling space: the $\mathbb{R}$-action $S \mapsto S - tr$ and the homeomorphism Φ are conjugated with the corresponding dynamics on Ω_ϕ through projection of strands to tilings of E^u. Strand space has two advantages: there is an easily defined *geometric realization*, $g : \Sigma_\phi \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by $g(S) = x + \mathbb{Z}^d$, where x is a vertex of a segment in S; and there is a *geometric coincidence condition*, by means of which pure discrete spectrum can be checked. To formulate the former, we say that segments $[x, i]$ and $[y, j]$ are *coincident* if there is $n \in \mathbb{N}$ so that $\Phi^n([x, i]) \cap \Phi^n([y, j])$ contains a segment. The substitution ϕ *satisfies the geometric coincidence condition* if for all $i \in \mathcal{A}$ and for all $x \in \mathbb{Z}^d$ and $j \in \mathcal{A}$ so that the interior of $[x, j]$ meets the stable space, E^s, of M_ϕ, it happens that $[0, i]$ and $[x, j]$ are coincident. This condition is called *super coincidence* in [65]. The equivalence of pure discrete spectrum and geometric/super coincidence for irreducible unimodular Pisot substitutions appears in [30] and [65] and is generalized to the reducible and non-unimodular setting in [21] and [50].

Theorem 5.2 (Geometric coincidence condition and geometric realization). *If ϕ is an irreducible unimodular Pisot substitution then the geometric realization $g : \Sigma_\phi \to \mathbb{T}^d$ semi-conjugates Φ with the homeomorphism of $\mathbb{T}^d$ induced by M_ϕ and the $\mathbb{R}$-action on Σ_ϕ with a Kronecker action on $\mathbb{T}^d$. The following are equivalent:*
(i) $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum,
(ii) g is almost everywhere one-to-one,
(iii) ϕ satisfies the geometric coincidence condition.

The Kronecker action in Theorem 5.2 is the maximal equicontinuous factor of the $\mathbb{R}$-action on Ω_ϕ mentioned in Section 3.1; more will be said about this in Section 7.1. Equivalence of the geometric coincidence condition with measurable conjugacy of the tiling and Kronecker dynamical systems can be found in [30] and [21]. The map g in Theorem 5.2 is, in general, almost everywhere m-to-one with m equal to the maximal cardinality of a collection of segments $\{[x_i, j_i]\}$ having the properties: each of these segments meets E^s in its interior; $x_i - x_k \in \mathbb{Z}^d$ for all i, k; and $[x_i, j_i]$ and $[x_k, j_k]$ are not coincident for $i \neq k$. This m is the coincidence rank of ϕ (see Section 3.1 for the definition and Conjecture 4.5). The geometric and super coincidence conditions are versions of the more general *overlap coincidence* condition introduced in [113] for tilings of the plane and specialized to one-dimensional tilings in [111]. Akiyama and Lee have automated an overlap coincidence algorithm in [6]. According to the formalism of dual substitutions developed in [14], dual versions of these conditions have been expressed in [14, 65, 108, 39].

5.3. The balanced pair algorithm

The following *balanced pair algorithm* is a purely combinatorial adaptation of overlap coincidence presented in [111] for application to Pisot substitutive systems.

The balanced pair algorithm for establishing pure discrete spectrum originated with [87, 83, 96]. A pair of finite words $(x, y) \in \mathcal{A}^* \times \mathcal{A}^*$ is said to be a *balanced pair*, if the abelianizations $f(x)$ and $f(y)$ are equal. A *coincidence* is a one-letter balanced pair (a, a). A balanced pair is *irreducible* if it cannot be properly factored as a product (with respect to the multiplication $(x, y)(u, v) := (xu, yv)$) of balanced pairs. Clearly, each balanced pair can be factored uniquely as a product of irreducible balanced pairs. Suppose that $u = vwv \cdots$ is a fixed (or periodic) word for the substitution ϕ. One says that *the balanced pair algorithm for ϕ terminates with coincidence* if only finitely many distinct irreducible factors occur in factorizations of all balanced pairs of the form $(\phi^n(vw), \phi^n(wv))$, $n \in \mathbb{N}$, and for each such occurring irreducible factor (x, y), $(\phi^k(x), \phi^k(y))$ has a coincidence in its irreducible factorization for some $k \in \mathbb{N}$. For more on the balanced pair algorithm, and the theorem below, see [111].

Theorem 5.3 (Balanced pair algorithm). *Given an irreducible Pisot substitution ϕ, the substitutive dynamical system (X_ϕ, s) has pure discrete spectrum if and only if the balanced pair algorithm for ϕ terminates with coincidence.*

In fact, it suffices to check termination with coincidence of the balanced pair algorithm starting from any particular seed of the form (ij, ji), $i \neq j \in \mathcal{A}$. For this, and extensions to the reducible setting, as well as geometric versions for tilings in arbitrary dimension, see [86] and [32].

5.4. Coincidences in higher dimensions

There are various notions of coincidence for n-dimensional substitution tilings which characterize pure discrete spectrum of the tiling dynamical system. We mention a few of these here.

Modular coincidence. In the case that the underlying structure of a substitution tiling is a lattice (this is the analog of a constant length substitution tillng in one-dimension), it is easy to check whether or not the tiling system has pure discrete spectrum by checking for 'modular coincidence', as we explain now.

Suppose that $\underline{\Gamma} = \bigcup_{i=1}^m \Gamma_i \times \{i\}$ is a primitive substitution Delone multicolor set with expansion map Λ and that $L = \bigcup_{1\le i\le m} \Gamma_i$ is a lattice (that is, a co-compact discrete subgroup of $\mathbb{R}^n$). Let $L_i := \langle \Gamma_i - \Gamma_i \rangle$ be the Abelian group generated by $\Gamma_i - \Gamma_i$ and let $L' := L_1 + L_2 + \cdots + L_m$. For $a \in L$, Φ_{ij} as in (2.4), and $f \in \Phi_{ij}$, let $t(f) \in \mathcal{D}_{ij}$ be so that $f(x,j) = (\Lambda x + t(f), i)$. Set

$$\begin{aligned} \Phi_{ij}[a] \quad &:= \quad \{f \in \Phi_{ij} :\ \Lambda y + t(f) \equiv a \bmod \Lambda L', \\ &\qquad\quad \text{where } \Gamma_j \subset y + L'\} \\ &= \quad \{f \in \Phi_{ij} : \mathrm{supp}(\Gamma_j \times \{j\}) \subset a + \Lambda L')\} \end{aligned} \tag{5.1}$$

Then

$$\bigcup_{i,j\le m} \ \bigcup_{f\in\Phi_{ij}[a]} \mathrm{supp}(\Gamma_j \times \{j\}) = a + \Lambda L'.$$

Let $\Phi[a] := \cup_{1\le i,j\le m} \Phi_{ij}[a]$.

We say that $\underline{\Gamma}$ admits *a modular coincidence relative to* $\Lambda L'$ if $\Phi[a]$ is contained entirely in one row of Φ for some $a \in L$. It is easy to see that $\underline{\Gamma}$ admits a modular coincidence relative to $\Lambda L'$ if and only if $(a + \Lambda L') \subset \Gamma_i$ for some $1 \le i \le m$.

Theorem 5.4. [76] *Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}^n$ with associated Delone multi-color set $\underline{\Gamma}_{\mathcal{T}} = \bigcup_{i=1}^m (\Gamma_i \times \{i\})$ and suppose that $L = \bigcup_{1\le i\le m} \Gamma_i$ is a lattice. Let $L' = L_1 + \cdots + L_m$, where $L_i = \langle \Gamma_i - \Gamma_i \rangle$. The following are equivalent:*
(i) *$(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ has pure discrete spectrum.*
(ii) *A modular coincidence relative to $\Lambda^M L'$ occurs in Φ^M for some M.*
(iii) *Each Γ_i is a regular model set for $1 \le i \le m$.*

The upper bound for the number of iterations to check modular coincidence is given in [56] where the underlying structure is on a lattice and in [6] for more general substitution tilings.

Algebraic coincidences. For substitution tilings in $\mathbb{R}^n$ whose associated Delone sets may not be on lattices, the notion of algebraic coincidence (defined below) generalizes the notion of modular coincidence and provides an opportunity for a concise expression of the connection between model sets and pure discrete spectrum for substitution tiling spaces.

Let $\underline{\Gamma}$ be a primitive substitution Delone multi-color set with an expansive map Λ. Let $\Xi(\underline{\Gamma}) = \bigcup_{j\le m} (\Gamma_j - \Gamma_j)$. We say that $\underline{\Gamma}$ admits an *algebraic coincidence* if there exist $M \in \mathbb{Z}_+$ and $\xi \in \Gamma_i$ for some $i \le m$ such that $\xi + \Lambda^M \Xi(\underline{\Gamma}) \subset \Gamma_i$.

Theorem 5.5. [75] *Let $\mathcal{T}$ be a primitive substitution tiling with FLC and let $\underline{\Gamma}_{\mathcal{T}} = \bigcup_{i=1}^m (\Gamma_i \times \{i\})$ be an associated substitution Delone multi-color set. Then the following are equivalent:*

(1) $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ *has pure discrete spectrum.*
(2) $\underline{\mathit{\Gamma}}_{\mathcal{T}}$ *admits an algebraic coincidence.*
(3) *Each* $\mathit{\Gamma}_i$ *is an inter-model set.*

Overlap coincidences. Unlike modular coincidence, algebraic coincidence (for substitution tilings whose associated Delone sets are not on lattices) is not so easily checked. For the computation of pure discrete spectrum, the condition of 'overlap coincidence', which we define now, proves to be more convenient.

Let $\mathcal{T}$ be a tiling and let

$$\Xi(\mathcal{T}) := \{v \in \mathbb{R}^n : \text{ there is } T \in \mathcal{T} \text{ with } T + v \in \mathcal{T}\} \tag{5.2}$$

be the set of *return vectors* for $\mathcal{T}$. A triple (T, y, S), with $T, S \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if $\mathrm{supp}(y + T) \cap \mathrm{supp}(S)$ has non-empty interior.

An overlap (T, y, S) is a *coincidence* if $y + T = S$. The *support* of an overlap (T, y, S) is $\mathrm{supp}(T, y, S) = \mathrm{supp}(y+T) \cap \mathrm{supp}(S)$. Let $\mathcal{O} = (T, y, S)$ be an overlap. Recall that for a tile-substitution Φ, $\Phi(y+T) = \Lambda y + \Phi(T)$ is a patch of $\Lambda y + \mathcal{T}$, and $\Phi(S)$ is a $\mathcal{T}$-patch; moreover,

$$\mathrm{supp}(\Lambda y + \Phi(T)) \cap \mathrm{supp}(\Phi(S)) = \Lambda(\mathrm{supp}(T, y, S)).$$

For each $l \in \mathbb{Z}_+$,

$$\begin{aligned} \Lambda^l(\mathcal{O}) = \{(T', \Lambda^l y, S') : \quad & T' \in \Phi^l(T), S' \in \Phi^l(S), \\ & \mathrm{supp}(\Lambda^l y + T') \cap \mathrm{supp}(S') \neq \emptyset\}. \end{aligned}$$

We say that a substitution tiling $\mathcal{T}$ admits an *overlap coincidence* if there exists $l \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $\Lambda^l(\mathcal{O})$ contains a coincidence. We recall that the Meyer property was introduced in Section 3.2.

Theorem 5.6. [81, Th. 4.7 and Lemma A.9] *Let $\mathcal{T}$ be a primitive substitution tiling which has the Meyer property. Then $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ has pure discrete spectrum if and only if $\mathcal{T}$ admits an overlap coincidence.*

When a tiling has the Meyer property, the number of equivalence classes of overlaps is finite. Thus, once all equivalences of overlaps are found, by applying the substitution to each overlap it can be determined if the overlap coincidence condition holds. When the dimension of the tiling is more than 2, however, it is not easy to check if a given triple $\mathcal{O} = (T, y, S)$ is an overlap. Instead, one can consider potential overlaps (T', x, S') for which $T' + x$ and S' are within certain distance. Then, computing how many potential overlaps come from each potential overlap after substitution, one can tell which potential overlaps are actually overlaps by simple computation of spectral radii. This easy procedure is justified by proving that the (slightly modified) Hausdorff dimension of the tile boundary of the self-affine tiling is strictly less than the dimension of the space. This leads to the algorithm of Akiyama and Lee ([6]) for determining overlap coincidence. We recall that $\Xi(\mathcal{T})$ is the set of return vectors for the tiling $\mathcal{T}$ such as defined in (5.2).

Theorem 5.7. *Let $\mathcal{T}$ be a primitive substitution tiling for which $\Xi(\mathcal{T})$ is a Meyer set and the digit sets of the tile substitution are provided. Then there is a terminating algorithm determining overlap coincidence.*

5.5. Beta-numeration: Property (W) and algebraic coincidences

We now return to the one-dimensional case in the particular framework of beta-numeration with the introduction of the condition called Property (W), which, while not directly stated in terms of coincidences, is nonetheless closely related to the notion of algebraic coincidence introduced in the previous section.

The β-expansion $\mathbf{d}_\beta(x)$ is *finite*, if there is n_0 such that $d_n(x) = 0$ for $n > n_0$. The image by π (defined in (2.6)) of such an element is written as $d_{-m}(x)d_{-m+1}(x)\dots d_{n_0}(x)$ for simplicity. Let $\mathrm{Fin}(\beta)$ denote the set of all $x \geq 0$ whose β-expansion is finite. We consider several properties concerning $\mathrm{Fin}(\beta)$.

- (W) For each $x \in \mathbb{Z}[1/\beta] \cap [0, \infty)$ and each $\varepsilon > 0$, there are $y, z \in \mathrm{Fin}(\beta)$ with $z < \varepsilon$ and $x = y - z$.
- (H) $\mathbb{Z}[1/\beta] \cap [0, 1) \subset (\mathrm{Fin}(\beta) \cap [0, \beta)) - (\mathrm{Fin}(\beta) \cap [0, 1))$.

The conditions (W) and (H) are equivalent (see [9]). The notation (H) derives from Hollander [63], who showed that the slightly stronger property $\mathbb{Z}[1/\beta] \cap [0, 1) = (\mathrm{Fin}(\beta) \cap [0, 1)) - (\mathrm{Fin}(\beta) \cap [0, 1))$ implies that (X_{ϕ_β}, s) has pure discrete spectrum when X_β is a subshift of finite type and ϕ_β is irreducible.

Recall from Section 3.5 that if β is a Pisot unit of degree d then $\{\mathcal{T}(x_F) : x_F \text{ finite}\}$ is a multi-tiling of $\mathbb{R}^{d-1}$. The system $(\Omega_{\phi_\beta}, \mathbb{R})$ has pure discrete spectrum if and only if the degree (multiplicity) of this multi-tiling is one. If ϕ_β is irreducible, that is, if $d = n$, this is equivalent to saying that the system (X_{ϕ_β}, s) has pure discrete spectrum.

We now introduce a topological criterion:

- (Ex) There is an exclusive inner point in $\mathcal{T}(x_F)$ for some finite fractional part x_F (*i.e.*, $\pi(x_F) \in \mathbb{Z}[1/\beta] \cap [0, 1)$) of an $x \in X_\beta$.

Here an exclusive inner point means it does not lie in any tile $\mathcal{T}(y_F)$ with $y_F \neq x_F$. It is shown in [3] that the degree of the multi-tiling is one if and only if (Ex) holds, and that (W) and (Ex) are equivalent, that is, (W), (Ex), and pure discreteness of $(\Omega_{\phi_\beta}, \mathbb{R})$ are equivalent.

Indeed, let us call an element of $\mathcal{P}(x_F)$ *finite* if it has the form $0^\infty y$. The map π extends to such finite elements and $\mathit{\Gamma}_{x_F} = \{\pi(y) \mid 0^\infty y \in \mathcal{P}(x_F)\}$ is a set of control points for the tiling of the half-line obtained by iterating the tile substitution (2.7) on the tile $\pi(\mathcal{F}(x_I))$ with $I = \cdots 000$. Let $\mathbf{P} = \{x \in \mathbb{Z}[\beta] \cap [0, 1) \mid (T_\beta^n(x))_{n=0,1,\dots} \text{ is purely periodic}\}$. When β is a Pisot number, we easily see that $\mathbf{P}$ is a finite set and for each $x \in \mathbb{Z}[1/\beta] \cap [0, \infty)$ there is $k \in \mathbb{Z}$ such that $T_\beta^k(x) \in \mathbf{P}$. In fact, there is a uniform k so that every non-negative element z in $\mathit{\Gamma}_{x_F} - \mathit{\Gamma}_{x_F}$ satisfies $T_\beta^k(z) \in \mathbf{P}$. From this we are able to see the connection between (Ex) and the algebraic coincidence formulated by Lee [75] (and discussed in Section 5.4) as follows. By Proposition 1 of [3], (Ex) is equivalent to the existence of an element $x \in \mathbb{Z}[\beta]$ and a constant K_0 so that $\pi(x) \in \mathcal{P}(0^\infty)$ and $\pi(\beta^K u + x) \in \mathcal{P}(0^\infty)$ for any integer $K \geq K_0$ and any $u \in \mathbf{P}$. One readily sees that this condition is a special form of algebraic coincidence of the 1-dimensional tiling. Thus Property (W) is equivalent to pure discreteness of the translation action on the hull of a 1-dimensional tiling generated by β-expansion.

On the other hand, once we have a Pisot dual tiling (see Section 3.5), *i.e.*, the degree of the dual multi-tiling is one, we can immediately show algebraic coincidence for such a $(d-1)$-dimensional tiling. We see that $\pi(\mathcal{F}(0^\infty)) = \mathbb{Z}[\beta] \cap [0,1)$ and $\Phi(\mathbb{Z}[\beta] \cap [0,1))$ is the union of control points of Pisot dual tilings. Taking algebraic conjugates into consideration, to have algebraic coincidence, we only need to find another constant K_1 and an element $x \in \mathbb{Z}[\beta] \cap [0,1)$ such that $x + \beta^{-K}(\mathbb{Z}[\beta] \cap [0,1) - \mathbb{Z}[\beta] \cap [0,1)) \subset \mathbb{Z}[\beta] \cap [0,1)$ for $K \geq K_1$. This turns out to be trivial from the Parry condition introduced in Section 2.4.

Summing up, the Pisot Substitution Conjecture for β-substitutions is equivalent to (W) which is equivalent to the associated dual multi-tiling having degree one. Once the multi-tiling has degree one, the associated $(d-1)$-dimensional tiling dynamical system is pure discrete, as is $(\Omega_{\phi_\beta}, \mathbb{R})$.

6. Partial results toward pure discrete spectrum

Substitution case. Conjectures 4.1 and 4.2 have been established for symbolic substitutions on two letters ([111]) and are known to hold for some families of substitutions, such as, *e.g.*, the Arnoux-Rauzy substitutions ([32, 38, 41, 23]) and the substitutions associated with the Brun and Jacobi-Perron continued fraction algorithms ([40, 64, 41, 23]). They have also been checked, mainly by the methods of Section 5.3, for many more-or-less randomly chosen substitutions and special cases of substitutions on three letters ([7]). Otherwise, they remain wide open.

The Coincidence Rank Conjecture 4.5 is verified for degree one (that is, when the expansion factor λ is an integer) in [33] and for all Pisot λ in the case that the coincidence rank is two in [22]. Thus there are no counterexamples to the Homological Pisot Substitution Conjecture with coincidence rank two and any coincidence rank two counterexample to the Pisot Substitution Conjectures 4.1 and 4.2 must have an asymptotic cycle of arc components.

Beta-numeration case. The context of beta-substitutions, being narrower, has seen more progress. Here we list known sufficient conditions for the validity of the beta-substitution case of the Pisot Substitution Conjecture which make use only of algebraic conditions on the Pisot number β. Let $x^d - \sum_{i=0}^{d-1} k_i x^i$ be the minimal polynomial of β. We recall that Property (W), which is equivalent to pure discrete spectrum of the tiling system, has been introduced in Section 5.5.

- The condition $k_{d-1} > \sum_{i=0}^{d-2} |k_i|$ implies (W) (see [9]).
- Pisot units with $d \leq 3$ satisfy (W) (see [9]).

- The condition $\mathbf{d}_\beta(1-) = (c_1 c_2 \dots c_m)^\infty$ implies (W) (see [23]).

Moreover, recall that $\mathrm{Fin}(\beta)$ denotes the set of all $x \geq 0$ whose β-expansion is finite. In addition to the (W) and (H) properties introduced in Section 5.5, we consider two stronger properties concerning $\mathrm{Fin}(\beta)$:

- (F) $\mathrm{Fin}(\beta) \supset \mathbb{Z}[1/\beta] \cap [0, \infty)$;
- (PF) $\mathrm{Fin}(\beta) \supset \mathbb{Z}_+[1/\beta]$ where $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$.

The conditions (F) and (PF) imply (W) (see [9]). They were introduced by Frougny and Solomyak in [57] who proved the following:

- $c_1 \leq c_2 \leq c_3 \dots$ implies (F) or (PF) ([57]).

Those β with property (PF) but not (F) are characterized in [4].

The finiteness condition (F) means all possible candidates have finite β-expansion and it is equivalent to state $\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty)$. It is useful in many situations with ergodic and number theoretical flavor: Akiyama showed in [1] that every sufficiently small rational number has purely periodic T_β-orbit under (F) and, in [93], Praggastis constructed Markov partitions for toral automorphisms related to β-expansion under condition (F).

The property $\mathbb{Z}_+ \subset \mathrm{Fin}(\beta)$ implies that β is a Pisot number (Proposition 1 in [2]), thus a number β satisfying (F) is a Pisot number, but the converse is false. For example, if β has property (F) then it cannot have another positive conjugate. The characterization problem of Pisot numbers with property (F) is difficult and has been transformed into a problem of shift radix systems (see [12, 11]). The idea of a shift radix system is essentially due to Gilbert [59] and Hollander [63] (see [8]). For a version of the finiteness property (F) in the symbolic substitution case and its relation with topological properties of Rauzy fractals, see [35, 108, 39].

7. The Pisot property and hyperbolicity in higher dimensions

For the $\mathbb{R}$-action on a one-dimensional substitution tiling space to have pure discrete spectrum it is necessary that the expansion λ be a Pisot number ([113]). Generally, for the $\mathbb{R}^n$-action on an n-dimensional substitution tiling space to have pure discrete spectrum it is necessary that the total number (with multiplicity) of algebraic conjugates η of eigenvalues of the linear expansion Λ with $|\eta| > 1$, equals n. The term *Pisot property* is designed to capture this condition.

7.1. Pisot families and the Pisot property

First let us remark that if Φ is an n-dimensional primitive tiling substitution with expansion Λ then the eigenvalues of Λ are all algebraic integers, so speaking of their algebraic conjugates makes sense (see [69] for the diagonalizable case and [72] for the general result). Let $J[\lambda, r]$ denote a real Jordan block with either real eigenvalue λ and size $r \times r$ or complex eigenvalues $\lambda, \bar{\lambda}$ and size $2r \times 2r$. Then Λ is said to have the *Perron property* if, whenever $J[\lambda, r]$ occurs in the real Jordan form of Λ with multiplicity k and λ' is an algebraic conjugate of λ with $|\lambda'| \geq |\lambda|$, then $J[\lambda', r]$ is also a block in the real Jordan form of Λ with multiplicity at least k (we consider $J[\lambda, r]$ and $J[\bar{\lambda}, r]$ to be the same). The terminology is due to Kwapisz who proves in [72], generalizing a result of Kenyon and Solomyak [69], that linear expansions for primitive tile substitutions must have the Perron property. Let us say that Λ has the *Pisot property* if, whenever $J[\lambda, r]$ is a block in the real Jordan form of Λ with multiplicity k and λ' is an algebraic conjugate of λ with $|\lambda'| \geq 1$, then $J[\lambda', r]$ also occurs in the real Jordan form of Λ and with multiplicity k.

If, in the definition of the Pisot property, one drops reference to Jordan blocks and speaks instead only of eigenvalues, the *Pisot family condition* results (we caution the reader that the definition of Pisot family is somewhat variable in the literature). But the Pisot family condition plus the Perron property is equivalent to the Pisot property, so in light of the Kwapisz result cited above, if Λ is the expansion for a primitive FLC tile substitution, then Λ has the Pisot property if and only if Λ satisfies the Pisot family condition.

Every primitive FLC substitution tiling dynamical system $(\Omega_\Phi, \mathbb{R}^n)$ has a maximal equicontinuous factor $(\hat{\mathbb{T}}, \mathbb{R}^n)$ with factor map $g : \Omega_\Phi \to \hat{\mathbb{T}}$ with g a.e. m-to-1 for some $m \in \mathbb{N} \cup \{\infty\}$, where $\hat{\mathbb{T}}$ is a torus or solenoid and the $\mathbb{R}^n$-action is a Kronecker action. The number $m = cr(\Phi)$ is the coincidence rank of Φ ([28]). Also, by [113, 29, 21], for one-dimensional primitive tile substitutions Φ, $cr(\Phi) < \infty$ if and only if the expansion λ for Φ is a Pisot number.

Conjecture 7.1. If Φ is a primitive FLC tile substitution then $cr(\Phi) < \infty$ if and only if Φ has the Pisot property.

7.2. Pisot families and discrete spectrum

We sketch below the sufficiency of a strong form of the Pisot property in Conjecture 7.1; it is reasonable to expect that the full conjecture will follow along the lines of [72].

Let us consider two additional conditions on the expansion Λ:

[A1] The expansion map Λ is diagonalizable over $\mathbb{C}$.

[A2] All eigenvalues of Λ are algebraic conjugates with the same multiplicity.

Let J be the multiplicity of each eigenvalue of Λ. After a linear change of coordinates, we may write

$$\Lambda = \begin{bmatrix} \Psi_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \Psi_J \end{bmatrix}$$

where $\Psi_j = \Psi$ for any $1 \leq j \leq J$, Ψ is an $m \times m$ matrix, and O is the $m \times m$ zero matrix . For each $1 \leq j \leq J$, let

$$H_j = \{0\}^{(j-1)m} \times \mathbb{R}^m \times \{0\}^{n-jm}\,.$$

We define $\boldsymbol{\alpha}_j \in H_j$ such that for each $1 \leq d \leq n$,

$$(\boldsymbol{\alpha}_j)_d = \begin{cases} 1 & \text{if } \ (j-1)m + 1 \leq d \leq jm; \\ 0 & \text{else}\,. \end{cases} \tag{7.1}$$

In the following theorem, which is key for the proof of Theorem 7.3, $\mathcal{C}$ is a set of control points for $\mathcal{T}$ (see Section 2.3).

Theorem 7.2. [78] *Let $\mathcal{T}$ be a primitive FLC substitution tiling of $\mathbb{R}^n$ with expansion Λ satisfying [A1] and [A2]. Then there exists a linear isomorphism $\rho : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\rho\Lambda = \Lambda\rho \quad \text{and} \quad \mathcal{C} \subset \rho(\mathbb{Z}[\Lambda]\boldsymbol{\alpha}_1 + \cdots + \mathbb{Z}[\Lambda]\boldsymbol{\alpha}_J)\,, \tag{7.2}$$

where the $\boldsymbol{\alpha}_j$, $1 \le j \le J$, are as above.

As a consequence of the containment (7.2), one has the following *rigid structure property* of $\mathcal{T}$ (we recall that $\Xi(\mathcal{T})$ stands for the set of return vectors of the tiling $\mathcal{T}$ as defined in (5.2)):

$$\Xi(\mathcal{T}) \subset \rho(\mathbb{Z}[\Lambda]\boldsymbol{\alpha}_1 + \cdots + \mathbb{Z}[\Lambda]\boldsymbol{\alpha}_J). \tag{7.3}$$

Theorem 7.3. [78] *Let $\mathcal{T}$ be a primitive FLC substitution tiling of $\mathbb{R}^n$ with expansion Λ satisfying [A1] and [A2]. Then the following are equivalent.*
(i) $Spec(\Lambda)$ is a Pisot family: if $\lambda \in Spec(\Lambda)$ is of multiplicity k and λ' is a conjugate of λ with $|\lambda'| \ge 1$, then $\lambda' \in Spec(\Lambda)$ and λ' has multiplicity at least k.
(ii) The set of eigenvalues of $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ is relatively dense in $\mathbb{R}^n$.
*(iii) $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ is not weakly mixing (*i.e.*, it has a non-zero eigenvalue).*
(iv) $\Xi(\mathcal{T})$ is a Meyer set.

Under the assumptions of Theorem 7.3, it is shown in [28] that the coincidence rank of the underlying substitution is finite, establishing sufficiency in Conjecture 7.1 in case [A1] and [A2] hold.

Proof. Let us sketch the proof of Theorem 7.3. The proof of $(i) \Rightarrow (ii)$ is based on the rigid structure property of $\mathcal{T}$; we sketch the argument in a simple case. Assume that Λ has only real eigenvalues, λ_i, of multiplicity one and $\mathcal{T}$ has no translational periods. There is then a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ such that $\mathcal{C} \subset \mathbb{Z}[\Lambda]\boldsymbol{\alpha}$. Observe that

$$\Xi(\mathcal{T}) \subset \mathcal{C} - \mathcal{C} \subset \mathbb{Z}[\Lambda]\boldsymbol{\alpha}.$$

The set of control points $\mathcal{C}$ is relatively dense, consequently the vector $\boldsymbol{\alpha} := [a_1, \ldots, a_n]^T$ must have all non-zero coordinates. Consider now the vector $\boldsymbol{\beta} := [a_1^{-1}, \ldots, a_n^{-1}]^T$. We claim that the set $\{\Lambda^j\boldsymbol{\beta}\}_{j=0}^{n-1}$ is contained in the set of eigenvalues of $(\Omega_{\mathcal{T}}, \mathbb{R}^n)$. This set is a basis for $\mathbb{R}^n$ (over $\mathbb{R}$), and, since the set of eigenvalues forms an additive group, we may conclude that the eigenvalues are relatively dense. For $\mathbf{x} = \Lambda^i\boldsymbol{\alpha}$, $\boldsymbol{\gamma} = \Lambda^j\boldsymbol{\beta}$ we have:

$$\langle \Lambda^l\mathbf{x}, \boldsymbol{\gamma}\rangle = \langle \Lambda^{l+i}\boldsymbol{\alpha}, \Lambda^j\boldsymbol{\beta}\rangle = \sum_{k=1}^{n} \lambda_k^{l+i+j} \xrightarrow[l\to\infty]{} 0 \pmod{\mathbb{Z}}. \tag{7.4}$$

The convergence follows from the Pisot family property: the numbers $\lambda_1, \ldots,$ λ_n are all roots of the same integer polynomial, and the 'missing roots' in (7.4) are all less than one in modulus. The sum of $(l+i+j)$-th powers over all roots of an integer polynomial is an integer, yielding (7.4). It follows from (7.4) that

$$\lim_{l\to\infty} e^{2\pi i\langle \Lambda^l \mathbf{x},\gamma\rangle} = 1 \quad \text{for all } \mathbf{x}\in \Xi(\mathcal{T}), \tag{7.5}$$

which means that $\gamma \in \mathbb{R}^d$ is an eigenvalue from [113].

The proof of $(ii) \Rightarrow (iii)$ is trivial. The necessity $(iii) \Rightarrow (i)$ was proved by Robinson [99] in a more general case; it is a consequence of [113]. For the equivalence of (iv) with the rest, see [77]. $\square$

7.3. Examples

The tile substitutions of the following two examples are primitive and 2-dimensional but without FLC. Under the additional assumptions [A1] and [A2] on a substitution, FLC implies the rigid structure property. Example 7.5 shows that the converse is not true, hence substitution tilings with the rigid structure property constitute a strictly larger class than substitution tilings with FLC. The rigid structure property (7.3) is easy to check from the digit sets of the substitution and one can study various spectral properties of tiling systems in this larger class.

Example 7.4 ([68, 78]). Consider the substitution tiling $\mathcal{T}$ in $\mathbb{R}^2$ with a single prototile T and expansion $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ such that

$$\Lambda T = \bigcup_{d\in\mathcal{D}} (T+d)$$

where

$$\mathcal{D} = \{(0,-1),(0,0),(0,1),(-1,-1),(-1,0),(-1,1),(1,-1+a),(1,a),(1,1+a)\}$$

is a digit set and $a \in \mathbb{R}$ is irrational. Note that

$$\Xi(\mathcal{T}) \subset \mathbb{Z}[\Lambda](1,0) + \mathbb{Z}[\Lambda](0,1) + \mathbb{Z}[\Lambda](0,a)$$

and $\mathbb{Z}[\Lambda](1,0)+\mathbb{Z}[\Lambda](0,1)+\mathbb{Z}[\Lambda](0,a)$ is the minimal module over $\mathbb{Z}[\Lambda]$ containing $\Xi(\mathcal{T})$. Thus $\mathcal{T}$ does not have the rigid structure property.

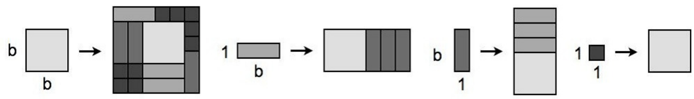

FIGURE 2. The non-FLC Frank-Robinson substitution.

Example 7.5 ([53]). The Frank-Robinson substitution is pictured in Figure 2, in which b, the scalar expansion, is the largest (and non-Pisot) root of $x^2 - x - 3 = 0$. The digit sets $\mathcal{D}_{ij}$ satisfy $\mathcal{D}_{ij} \subset \mathbb{Z}[b](1,0)+\mathbb{Z}[b](0,1)$. Hence

$$\Xi(\mathcal{T}) \subset \mathbb{Z}[b](1,0)+\mathbb{Z}[b](0,1)$$

and the rigid structure property holds. See also [16, Example 5.8] for a discussion of the Frank-Robinson tiling and see [94] for more examples of non-FLC 'fusion' tilings.

7.4. The Pisot property and hyperbolicity

In one dimension, the construction of tile substitutions is purely combinatorial. In higher dimensions, there is geometry to deal with (tile shapes) making it much more difficult to construct examples. And then it is more time-consuming to check their spectral properties. For these reasons, possible extensions of the Pisot Substitution Conjecture to higher-dimensional tile substitutions have not been well vetted. We nevertheless discuss the possibilities. The most straightforward route is:

Conjecture 7.6. Suppose that Φ is an n-dimensional primitive FLC tile substitution whose expansion Λ has the Pisot property. If the characteristic polynomial of the substitution matrix M_Φ is irreducible, then $(\Omega_\Phi, \mathbb{R}^n)$ has pure discrete spectrum.

It seems to be a fairly stringent requirement, in higher dimensions, that the characteristic polynomial of the substitution matrix M_Φ be irreducible and the irreducibility hypothesis suffers the same unnaturality as in one dimension (a different 'presentation' of the tiling space may well change the irreducibility of the characteristic polynomial of the substitution matrix). By considering a homological condition instead, we will at least be led to an interesting connection with hyperbolic dynamics. Suppose that Φ is an n-dimensional primitive FLC tile substitution with expansion Λ. Let us say that Φ is *unimodular* if every eigenvalue of Λ is an algebraic unit, and *hyperbolic* if no eigenvalue of Λ has an algebraic conjugate on the unit circle. Let $\langle \Xi(\Phi) \rangle$, called the *module of generalized return vectors* of Φ, be the additive subgroup of $\mathbb{R}^n$ generated by the return vectors $\Xi(\Phi)$ (that is, $\Xi(\Phi) := \Xi(\mathcal{T})$, any $\mathcal{T} \in \Omega_\Phi$). If $v \in \Xi(\Phi)$, say $T, T - v \in \mathcal{T} \in \Omega_\Phi$, then $\Phi(T), \Phi(T) - \Lambda v \subset \Phi(\mathcal{T}) \in \Omega_\Phi$, so that $\Lambda v \in \Xi(\Phi)$. Thus Λ induces a homomorphism $\Lambda : \langle \Xi(\Phi) \rangle \to \langle \Xi(\Phi) \rangle$. If Φ is unimodular, then $\langle \Xi(\Phi) \rangle$ is a finitely generated free $\mathbb{Z}$-module: let $D = D(\Phi) := \operatorname{rank}(\langle \Xi(\Phi) \rangle)$. The following is proved in [27] using the global shadowing technique in hyperbolic dynamics pioneered by Franks [54].

Theorem 7.7. [27] *If Φ is a primitive n-dimensional FLC unimodular hyperbolic tile substitution with linear expansion Λ, and module of generalized return vectors $\langle \Xi(\Phi) \rangle$, there is a continuous and boundedly finite-to-one map $G : \Omega_\Phi \to \mathbb{T}^D$ so that $G \circ \Phi = F_A \circ G$. Here A is an integral unimodular hyperbolic matrix representing $\Lambda : \langle \Xi(\Phi) \rangle \to \langle \Xi(\Phi) \rangle$, and $F_A : \mathbb{T}^D \to \mathbb{T}^D$ is the hyperbolic toral automorphism associated with A. The map G is topologically essential in that the homomorphism $G^* : H^1(\mathbb{T}^D) \to H^1(\Omega_\phi)$ induced on first cohomology is injective and there is $r \in \mathbb{N}$ so that G is a.e. r-to-1 with respect to the translation invariant measure μ on Ω_Φ.*

If the unstable dimension of A (the sum of the dimensions of all the generalized eigenspaces of A corresponding to eigenvalues of modulus greater than one) is greater than n, then G does not semi-conjugate the translation action on Ω_Φ with a Kronecker action by $\mathbb{R}^n$ on $\mathbb{T}^D$: the map $v \mapsto G(\mathcal{T} - v)$ wiggles around in a very jagged (probably nowhere smooth) manner in the unstable manifold of $G(\mathcal{T})$ in $\mathbb{T}^D$. If the unstable dimension of A equals n, leaving no room for such wiggling, it seems plausible that G would also semi-conjugate $\mathbb{R}^n$-actions. This appears to be what happens if Λ has the Pisot property (note that if Λ has the Pisot property, then Φ is hyperbolic).

Let us partition the eigenvalues of Λ into conjugacy classes $\mathcal{F}_i$ and, for each i, let d_i be the algebraic degree of the elements of $\mathcal{F}_i$ and let J_i be the maximum multiplicity (as eigenvalues of Λ) of the elements of $\mathcal{F}_i$. The *generalized degree* of Λ is

$$d(\Lambda) := \sum J_i d_i.$$

One can show that $D(\Phi) \geq d(\Lambda)$. Thus, if Λ has the Pisot property, $D(\Phi) = d(\Lambda)$ forces the unstable dimension of A to be n, the dimension of the substitution. We know of no Pisot property substitution with $D(\Phi) > d(\Lambda)$.

Theorem 7.8. [27] *Suppose that Φ is a primitive FLC n-dimensional unimodular substitution whose expansion has the Pisot property. If the rank D of the module $\langle \Xi(\Phi) \rangle$ of generalized return vectors for Φ equals the generalized degree $d(\Lambda)$ of Λ, then the map $G : \Omega_\Phi \to \mathbb{T}^D$ of Theorem 7.7 is surjective and also semi-conjugates the translation action on Ω_Φ with a Kronecker action by $\mathbb{R}^n$ on $\mathbb{T}^D$. Furthermore, G is the maximal equicontinuous factor map for $(\Omega_\Phi, \mathbb{R}^n)$. That is, G=g, and hence $r = cr(\Phi) < \infty$.*

In the proof of Theorem 7.7 a *universal abelian cover*, $\tilde{\Omega}_\Phi$, is constructed and the map Φ on Ω_Φ is lifted to $\tilde{\Phi}$ on $\tilde{\Omega}_\Phi$. It is shown that the structure relation for G, denoted $\sim_{gs}$, is the global shadowing relation: $G(\mathcal{T}) = G(\mathcal{T}')$ if and only if $\mathcal{T} \sim_{gs} \mathcal{T}'$ if and only if there are $\tilde{\mathcal{T}}, \tilde{\mathcal{T}}' \in \tilde{\Omega}_\Phi$, lying over $\mathcal{T}, \mathcal{T}'$ so that the distance between $\tilde{\Phi}^k(\tilde{\mathcal{T}})$ and $\tilde{\Phi}^k(\tilde{\mathcal{T}}')$ is uniformly bounded for $k \in \mathbb{Z}$. It is a fundamental theorem of Veech [119] that the structure relation for the maximal equicontinuous factor map g is regional proximality: tilings $\mathcal{T}, \mathcal{T}' \in \Omega_\Phi$ are *regionally proximal*, $\mathcal{T} \sim_{rp} \mathcal{T}'$, if and only if, for each $\epsilon > 0$ there are $\mathcal{S}, \mathcal{S}' \in \Omega_\Phi$ and $v \in \mathbb{R}^n$ so that: (i) $d(\mathcal{T}, \mathcal{S}) < \epsilon$, (ii) $d(\mathcal{T}', \mathcal{S}') < \epsilon$, and (iii) $d(\mathcal{S} - v, \mathcal{S}' - v) < \epsilon$. (For a general discussion of regional proximality in tiling spaces see [28].) In the context of Theorem 7.7 one can show that $\mathcal{T} \sim_{gs} \mathcal{T}' \implies \mathcal{T} \sim_{rp} \mathcal{T}'$. Under the hypotheses of Theorem 7.8, the global shadowing and regional proximal relations are the same.

The rank of $\langle \Xi(\Phi) \rangle$ is bounded above by the dimension of $H^1(\Omega_\Phi)$ so, in light of Theorem 7.8, a natural generalization to higher dimensions of the one-dimensional Homological Pisot Substitution Conjecture is:

Conjecture 7.9. Suppose that Φ is an n-dimensional primitive FLC unimodular tile substitution whose expansion Λ has the Pisot property and has generalized degree $d(\Lambda)$. If $\dim(H^1(\Omega_\Phi)) = d(\Lambda)$, then $(\Omega_\Phi, \mathbb{R}^n)$ has pure discrete spectrum.

An extension of this to the more general hyperbolic setting is:

Conjecture 7.10. Suppose that Φ is a primitive FLC unimodular and hyperbolic tile substitution and that the induced isomorphism $\Phi^*\colon H^1(\Omega_\phi) \to H^1(\Omega_\phi)$ is hyperbolic on $H^1(\Omega_\Phi)$. There is then a continuous, μ-a.e. one-to-one, topologically essential map $G : \Omega_\Phi \to \mathbb{T}^D$, $D = \dim(H^1(\Omega_\Phi))$, and a hyperbolic toral automorphism $F_A : \mathbb{T}^D \to \mathbb{T}^D$, with $G \circ \Phi = F_A \circ G$.

Just as it is rare that characteristic polynomials of substitution matrices are irreducible in higher dimensions, it is unusual for the first cohomology of the tiling space to have the minimal dimension required to accommodate the generalized degree of the expansion. This limits the range of the above conjectures. They can be strengthened, as can Conjectures 4.4 and 4.5, by replacing the cohomology of Ω_Φ by its *essential cohomology*, which does not see, for example, the contributions of asymptotic cycles (see [27]). The resulting conjectures imply the non-homological conjectures in the unimodular case.

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