THE CONJUGACY PROBLEM IN SMALL GAUSSIAN GROUPS

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Abstract. Small Gaussian groups are a natural generalization of spherical Artin groups in which the existence of least common multiples is kept as an hypothesis, but the relations between the generators are not supposed to necessarily be of Coxeter type. We show here how to extend the Elrifai-Morton solution for the conjugacy problem in braid groups to every small Gaussian group.

Key words: conjugacy; word problem; Artin groups.

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Introduction

Define a Gaussian monoid to be a cancellative monoid where least common multiples exist and divisibility has no infinite descending chain. Then, in every Gaussian monoid, there exists a well defined (right) residue operation $\backslash$ such that $a(a\backslash b)$ is the right lcm of a and b. A Gaussian group is the group of fractions of a Gaussian monoid. Using Dehornoy's technique of word reversing [6], one can solve the word problem in Gaussian groups. We say that a Gaussian group is small if it is the group of fractions of a small Gaussian monoid, the latter being defined as a Gaussian monoid admitting a finite generating set that is closed under the residue operation. Gaussian groups and small Gaussian groups have been introduced in [9] and [7] as generalizations of finite Coxeter type Artin groups. In this paper, we prove that the conjugacy problem in small Gaussian groups is solvable.

There exists in every small Gaussian group an element Δ whose properties are reminiscent of those of Garside's fundamental braid Δ_n in Artin's braid group B_n. We start from Garside's approach [13], and we show how to extend the algorithm defined by Elrifai & Morton [11] in the case of the braid groups, a special case of small Gaussian groups. On the one hand, our result just says that Garside's and Elrifai-Morton's methods can be extended without change. But, on the other hand, a new proof has to be constructed, for the original argument of [13] and [11] does not work in the general case. The new proof relies on the properties of the residue operation \, and, hopefully, it can give a new insight even in the classical case of braids and of Artin groups.

This paper is organized as follows. In Section 1, we recall earlier results about small Gaussian groups. In Section 2, we establish an elementary lemma which turns out to be the key of the problem. The conjugacy problem in small Gaussian groups is solved in Section 3, and, in addition, we describe, as in [11], an improved algorithm using a special conjugacy operation called cycling. In Section 4, we deal with an example of a small Gaussian group which is not of Artin type, and which is rather different from such groups, in the sense that there exists no morphism from the associated monoid into the integers, and its element Δ is not the lcm of the minimal generators. Finally, in Section 5, we observe that, as regards cycling, we can equally use the Δ-normal form and the fractional normal form.

1. Small Gaussian groups

Assume that M is a monoid. We say that M is *atomic* if, for every set X that generates M and for every a in M, the lengths of the decompositions a as product of elements in X have a finite upper bound. For a, b in M, we say that b is a left divisor of a—or that a is a right multiple of b—if there exists some d in M satisfying $a = bd$. An element c is a right lower common multiple—or a right lcm—of a and b if it is a right multiple of both a and b, and every common right multiple of a and b is a right multiple of c. Right divisor, left multiple, and left lcm are defined symmetrically.

Definition. [9] A monoid M is *Gaussian* if it is atomic, cancellative, and every pair of elements in M admits a right and a left lcm.

If c, c' are two right lcm's of a and b, necessarily c is a left divisor of c', and c' is a left divisor of c. If we assume M to be Gaussian, this implies that c and c' coincide. In this case, the unique right lcm of a and b is denoted by $a \vee b$. If $a \vee b$ exists, and M is left cancellative, there exists a unique element c such that $a \vee b$ is equal to ac. This element is called the *right residue* of a in b, and it is denoted by $a \backslash b$. We define the *left lcm* $\widetilde{\vee}$ and the *left residue* / symmetrically. In particular, we have

$$a \vee b = a(a\backslash b) = b(b\backslash a), \quad \text{and,} \quad a \mathbin{\widetilde{\vee}} b = (b/a)a = (a/b)b.$$

In a Gaussian monoid, every element has only finitely many left divisors, then, for every pair of elements (a, b), the common left divisors of a and b admit a right lcm, which is therefore the *left gcd* of a and b. This left gcd will be denoted by $a \wedge b$. We define the *right gcd* $\widetilde{\wedge}$ symmetrically.

Definition. A Gaussian monoid is *small* if it admits a finite generating subset that is closed under $\backslash$,$/$, $\vee$, $\wedge$, $\widetilde{\vee}$, and $\widetilde{\wedge}$.

It is proved in [8] that for a Gaussian monoid to be small actually follows from a much weaker hypothesis, namely from the existence of a finite generating set closed under $\backslash$ solely.

Assume that M is a monoid. We say that an element a in M is an *atom* if a is not 1 and $a = bc$ implies $b = 1$ or $c = 1$. A small Gaussian monoid admits a finite set of atoms, and this set is the minimal generating set [9]. We define the norm $\|.\|$ of a small Gaussian monoid M such that, for every a in M, $\|a\|$ is the upper bound of the decompositions of a as products of atoms. The hypothesis that there exists a finite generating subset that is closed under $\backslash$ implies that the closure of the atoms under $\backslash$ is finite. In particular, the closure of the atoms under $\backslash$ and $\vee$ is finite—its elements are called *simple elements*, and their right lcm is denoted by Δ. Let us observe that the set of the simple elements is also the closure of atoms under $/$ and $\widetilde{\vee}$. In that way, Δ is both the right and the left lcm of the simple elements: Δ is called the *Garside element*.

We summarize here those results of [8] we use in the sequel.

Proposition 1.1. *Assume that M is a small Gaussian monoid, and S is a finite generating subset of M that is closed under $\backslash$ and $\vee$. Let Δ be the right lcm of S.*

(i) Let k be a nonnegative integer. Then, S^k is both the set of all left divisors of Δ^k and the set of all right divisors of Δ^k.

(ii) The functions $\partial : a \mapsto a\backslash\Delta$ and $\widetilde{\partial} : a \mapsto \Delta/a$ map M into S. Their restrictions to S are mutually inverse permutations of S.

(iii) The functions $a \mapsto (a\backslash\Delta)\backslash\Delta$ and $a \mapsto \Delta/(\Delta/a)$ from S into itself extend into automorphisms ∂^2 and $\widetilde{\partial}^2$ of M that map S^k into itself for every k, and the equalities

$$a\Delta = \Delta\partial^2(a), \quad \text{and} \quad \Delta a = \widetilde{\partial}^2(a)\Delta$$

hold for every a in M.

If M is a Gaussian monoid, then M satisfies Ore's conditions [5], and it embeds in a group of right fractions, and, symmetrically, in a group of left fractions. In this case, by construction, every right fraction ab^{-1} with a, b in M can be expressed as a left fraction $c^{-1}d$, and conversely. Therefore, the two groups coincide, and there is no ambiguity in speaking of the group of fractions of a Gaussian monoid.

Definition. A group G is a *(small) Gaussian group* if there exists a (small) Gaussian monoid of which G is the group of fractions.

By [3], all spherical Artin monoids are small Gaussian monoids. The braid monoids of the complex reflection groups $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}$ and G_{22} given in [4], the braid monoids of Birman-Ko-Lee [2], several of monoids of torus knots or links [16], and several of braid monoids of Sergiescu [17] are also small Gaussian monoids.

Remark. A given group may admit several Gaussian structures of different type. For instance, the 3-strand braids group $\langle\ \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\ \rangle$ also admits the presentations $\langle\ x, y : xyx = yy\ \rangle$ and $\langle\ x, y : xy^2x = yxy\ \rangle$. The associated monoids admit Garside elements, but the monoid $\langle\ x, y : xyx = yy\ \rangle$ is small, while $\langle\ x, y : xy^2x = yxy\ \rangle$ is not.

Example 1.2. Let us consider the monoid M_χ with presentation

$$\langle\ x, y, z : xzxy = yzx^2\ ,\ yzx^2z = zxyzx\ ,\ zxyzx = xzxyz\ \rangle.$$

The monoid M_χ is a typical example of a small Gaussian monoid, and, in addition, M_χ has the distinguishing feature to not be antiautomorphic, contrary to all spherical Artin monoids. The lattice of simple elements in M_χ is represented in Figure 1. In Section 5, we shall study another example of a small Gaussian monoid, which, as for it, admits no additive norm.

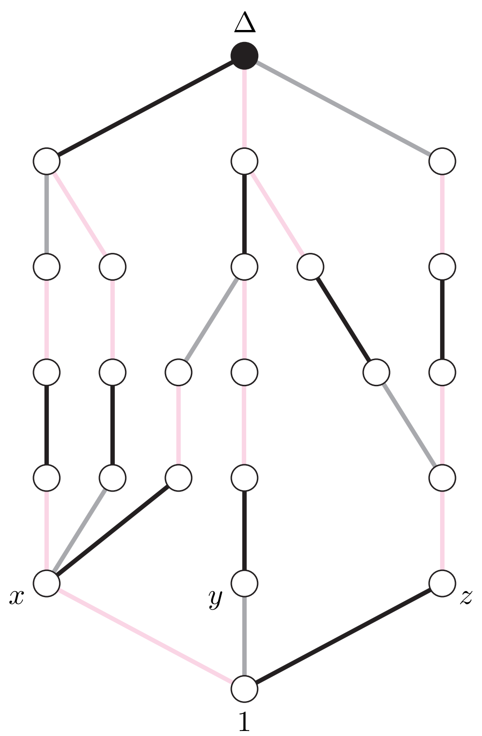

Figure 1. The lattice of simples in M_χ.

2. The Δ-normal form

If M is a small Gaussian monoid, and G is its group of fractions, then, for a, b in G, we write $a \le b$ if there exist elements a', a'' in M satisfying $b = a'aa''$; we say that a is a *left divisor* (*resp. right divisor*) of b if there exists an element c in M satisfying $b = ac$ (*resp.* $b = ca$).

Lemma 2.1. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then, for all a, b in G and every integer k, the following are equivalent:*

(i) $a \le \Delta^k \le b$ holds;

(ii) a is a left divisor of Δ^k, and Δ^k is a left divisor of b;

(iii) a is a right divisor of Δ^k, and Δ^k is a right divisor of b.

Proof. Assume $a \le \Delta^k \le b$. There exist elements a', a'', b', b'' in M satisfying $\Delta^k = a'aa''$ and $b = b'\Delta^k b''$. We obtain $\Delta^k = aa''\partial^{2k}(a') = \widetilde{\partial}^{2k}(a'')a'a$ and $b = \Delta^k\partial^{2k}(b')b'' = b'\widetilde{\partial}^{2k}(b'')\Delta^k$, which implies (ii) and (iii). The implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) are obvious. $\square$

Lemma 2.2. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then, for a, b in G, $\Delta^{r_1} \le a \le \Delta^{s_1}$ and $\Delta^{r_2} \le b \le \Delta^{s_2}$ imply $\Delta^{r_1+r_2} \le ab \le \Delta^{s_1+s_2}$.*

Proof. By Lemma 2.1, there exist elements a', a'', b', b'' in M satisfying $a = a'\Delta^{r_1}$, $\Delta^{s_1} = a''a$, $b = \Delta^{r_2}b'$, and $\Delta^{s_2} = bb''$. We deduce $ab = a'\Delta^{r_1+r_2}b'$ and $\Delta^{s_1+s_2} = a''abb''$. Therefore, we have $\Delta^{r_1+r_2} \le ab \le \Delta^{s_1+s_2}$. $\square$

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. For a in G, the *power* and the *copower* of a are respectively

$$v_\Delta^-(a) = \max\{r \in \mathbf{Z} : \Delta^r \le a\}, \quad \text{and} \quad v_\Delta^+(a) = \min\{s \in \mathbf{Z} : a \le \Delta^s\}.$$

The *gap* of a is defined to be the difference $v_\Delta^+(a) - v_\Delta^-(a)$. The idea is that $v_\Delta^\pm$ is a sort of Δ-valuation. A similar notion of power is introduced in [13], and similar notions of copower and of gap are introduced in [11], where the power of a is denoted by $\inf(a)$, its copower by $\sup(a)$, and the gap is called canonical length.

The following lemma is crucial. It already appears in [11] and [13] in the case of the braid monoids. The proof there is significantly different.

Lemma 2.3. *Assume that M is a small Gaussian monoid. Then, for a, b in M, $\Delta \le ba$ implies $\Delta \le b(\Delta \wedge a)$.*

Proof. Assume $\Delta \le ba$. Then, by Lemma 2.1, Δ is a left divisor of ba. As b is a left divisor of ba, the element $\Delta \vee b$ is a left divisor of ba. So, having $\Delta \vee b = b(b\backslash\Delta)$ by definition, by left cancellation, $b\backslash\Delta$ is a left divisor of a. By Proposition 1.1(ii), the element $b\backslash\Delta$ is simple, hence divides Δ, and, therefore, $b\backslash\Delta$ is a left divisor of $\Delta \wedge a$. Finally, $\Delta \vee b$ is a left divisor of $b(\Delta \wedge a)$, and Δ is a left divisor of $b(\Delta \wedge a)$. $\square$

Remark. A similar argument shows that $\Delta \le ba$ implies $\Delta \le (b \,\widetilde{\wedge}\, \Delta)(\Delta \wedge a)$.

Equipped with Lemma 2.3, we can now establish easily the following result, which already appears in [11] in the case of braids.

Lemma 2.4. *Assume that M is a small Gaussian monoid. For a in M, let $(a_i)_{i\geq 1}$ be the sequence of simple elements defined by $a_1 = \Delta \wedge a$ and $a_{i+1} = \Delta \wedge ((a_1 \dots a_i)\backslash a)$ for $i \geq 1$. Then $a_j \neq 1$ holds if and only if $j \leq v_\Delta^+(a)$ holds.*

Proof. By atomicity of M, there exists an integer k satisfying $a_k \neq 1$ and $a_j = 1$ for $j > k$. Then, a is $a_1 \dots a_k$ and, by Lemma 2.2, a satisfies $v_\Delta^+(a) \leq k$. We show $k \leq v_\Delta^+(a)$ by using induction on k. For $k = 0$, we have $a = 1$ and $v_\Delta^+(a) = 0$. Assume $k \geq 1$, *i.e.*, $a \neq 1$ and $v_\Delta^+(a) \geq 1$. Let $q = v_\Delta^+(a)$. There exists an element b in M satisfying $\Delta^q = ba$. By Lemma 2.3, we have $\Delta \leq ba_1$, and there exists an element b' in M satisfying $ba_1 = \Delta b'$. We deduce $\Delta^q = ba = ba_1a_2 \dots a_k = \Delta b' a_2 \dots a_k$, hence $\Delta^{q-1} = b'a_2 \dots a_k$ and $a_2 \dots a_k \leq \Delta^{q-1}$. Now, by induction hypothesis, we have $v_\Delta^+(a_2 \dots a_k) = k - 1$, hence $k - 1 \leq v_\Delta^+(a) - 1$. □

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. For a in G, the (left) Δ-*normal form* of a is the unique decomposition $\Delta^p a_1 \dots a_\ell$ with $p = v_\Delta^-(a)$, $a_1 = \Delta \wedge (\Delta^p\backslash a)$, $a_{i+1} = \Delta \wedge ((\Delta^p a_1 \dots a_i)\backslash a)$ for $i \geq 1$, and $\ell = v_\Delta^+(a) - v_\Delta^-(a)$.

Definition. [11] Assume that M is a small Gaussian monoid, and G is its group of fractions. The interval $[r, s]$ is the subset of G consisting of the elements a satisfying $\Delta^r \leq a \leq \Delta^s$. So we have

$$[r, s] = \{a \in G \ : \ r \leq v_\Delta^-(a) \quad \text{and} \quad v_\Delta^+(a) \leq s\}.$$

If S denotes the set of the simple elements, it comes $[r, s] \subseteq \{a \in G : a = \Delta^r t_{r+1} \dots t_s,\ t_i \in S\}$. In particular, we have the bound $\mathrm{card}[r, s] \leq (\mathrm{card}\ S)^{s-r}$.

Lemma 2.5. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then we have $[r_1, s_1][r_2, s_2] = [r_1 + r_2, s_1 + s_2]$ in G.*

Proof. By Lemma 2.2, we have $[r_1, s_1][r_2, s_2] \subseteq [r_1 + r_2, s_1 + s_2]$. Let us show $[r_1 + r_2, s_1 + s_2] \subseteq [r_1, s_1][r_2, s_2]$. As $[r, s] = \Delta^r[0, s - r]$ holds, it suffices to show $[0, s_1 + s_2] \subseteq [0, s_1][0, s_2]$. Assume $a \in [0, s_1 + s_2]$. By Lemma 2.4, we have $v_\Delta^+(a) \leq s_1 + s_2$. Let $(a_i)_{i\geq 1}$ be the infinite sequence of simple elements defined by $a_1 = \Delta \wedge a$ and $a_{i+1} = \Delta \wedge ((a_1 \dots a_i)\backslash a)$ for $i \geq 1$. Then, by construction, $a' = a_1 \dots a_{s_1}$ lies in $[0, s_1]$, and $a'' = a_{s_1+1} \dots a_{s_1+s_2}$ lies in $[0, s_2]$. □

3. The conjugacy problem

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. For a, b in G, we say that a, b are *positively conjugate* if there exists an element c in M satisfying $a = c^{-1}bc$, and that they are *simply conjugate* if there exists a simple element s satisfying $a = s^{-1}bs$.

Lemma 3.1. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Let a, b be conjugate elements in G. Then a, b are positively conjugate.*

Proof. There exists an element c in G satisfying $a = c^{-1}bc$. Let c' be the element of M satisfying $c = \Delta^{v_\Delta^-(c)}c'$. By Proposition 1.1, there exists a positive integer p satisfying $\widetilde{\partial}^{2p}(b) = b$, *i.e.*, $b\Delta^p = \Delta^p b$; indeed, it suffices to take for p the common index of permutations ∂ and $\widetilde{\partial}$. Now, there exist integers q and r satisfying $v_\Delta^-(c) = pq + r$ and $0 \le r < p$. We deduce

$$a = c'^{-1}\Delta^{-pq-r}b\Delta^{pq+r}c' = c'^{-1}\Delta^{-pq-r}\Delta^{pq}b\Delta^r c' = (\Delta^r c')^{-1}b\Delta^r c'.$$

As r is nonnegative, $\Delta^r c'$ lies in M. $\square$

Lemma 3.2. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Let a, b be conjugate elements in G. Assume $a = c^{-1}bc$ with c in M. Let $c_1 = \Delta \wedge c$. Then we have $v_\Delta^-(c_1^{-1}bc_1) \ge \min(v_\Delta^-(a), v_\Delta^-(b))$.*

Proof. Let $m = \min(v_\Delta^-(a), v_\Delta^-(b))$. There exists elements a', b' in M satisfying $a = \Delta^m a'$ and $b = \Delta^m b'$. We find

$$c_1^{-1}bc_1 = c_1^{-1}\Delta\Delta^{m-1}b'c_1 = \partial(c_1)\Delta^{m-1}b'c_1 = \Delta^{m-1}\partial^{2m-1}(c_1)b'c_1. \tag{3.1}$$

There exists an element c' in M satisfying $c = c_1c'$, and we have

$$c^{-1}bc = c'^{-1}\Delta^{m-1}\partial^{2m-1}(c_1)b'c_1c' = \Delta^{m-1}\partial^{2m-2}(c')^{-1}\partial^{2m-1}(c_1)b'c_1c'.$$

We obtain

$$\partial^{2m-2}(c')^{-1}\partial^{2m-1}(c_1)b'c_1c' = \Delta a',$$

hence

$$\partial^{2m-1}(c_1)b'c_1c' = \Delta\partial^{2m}(c')a'.$$

Therefore, Δ divides $\partial^{2m-1}(c_1)b'c_1c'$. Now, by Lemma 2.3, Δ divides $\partial^{2m-1}(c_1)b'c_1$, and we deduce from (3.1) that Δ^m divides $c_1^{-1}bc_1$. $\square$

Proposition 3.3. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Let a, b be conjugate elements belonging to $[r, s]$. Then there exists a sequence $b = b_0, b_1, \ldots, b_k = a$ of elements in $[r, s]$ such that the elements b_{i-1} and b_i are simply conjugate for $1 \le i \le k$.*

Proof. By Lemma 3.1, there exists an element c in M satisfying $a = c^{-1}bc$. Assume that c is $c_1 \dots c_k$ with $c_1 = \Delta \wedge c$ and $c_{i+1} = \Delta \wedge ((c_1 \dots c_i)\backslash c)$. Let $b_0 = b$ and $b_i = c_i^{-1} b_{i-1} c_i$. We have to show that b_1 lies in $[r, s]$. On the one hand, Lemma 3.2 gives $\Delta^r \le b_1$, *i.e.*, $r \le v_\Delta^-(b_1)$. On the other hand, we have $\Delta^{-s} \le a^{-1}$, $\Delta^{-s} \le b^{-1}$, and $a^{-1} = c^{-1}b^{-1}c$. Therefore, always by Lemma 3.2, we obtain $\Delta^{-s} \le c_1^{-1}b^{-1}c_1 = b_1^{-1}$, hence $b_1 \le \Delta^s$, *i.e.*, $v_\Delta^+(b_1) \le s$. An induction on k completes the proof. □

Corollary 3.4. *The conjugacy problem in a small Gaussian group is solvable.*

Proof. Assume that M is a small Gaussian monoid, G is its group of fractions, and a, b belong to G. We have to decide whether a and b are conjugate. First, we find an interval $[r, s]$ both a and b lie in—it suffices to take $r = \min(v_\Delta^-(a), v_\Delta^-(b))$ and $s = \max(v_\Delta^+(a), v_\Delta^+(b))$. Let $\Gamma_0(a)$ be the singleton $\{a\}$. Then, for $i \ge 1$, the i-th step consists in computing the set $\Gamma_i(a)$ of those elements which both are simply conjugate to an element of $\Gamma_{i-1}(a)$ and belong to $[r, s]$. The process stops at the i_0-th step, where i_0 is the least index satisfying $\Gamma_{i_0-1}(a) = \Gamma_{i_0}(a)$. As the interval $[r, s]$ is finite, such an index i_0 exists, and is less than $\mathrm{card}[r, s]$. Now, by Proposition 3.3, a and b are conjugate if and only if b belongs to $\Gamma_{i_0}(a)$. □

We describe now an improvement of the previous solution. In [11], starting from the Δ-normal form, Elrifai and Morton have introduced two operations, called cycling and reverse cycling, which map every element to a distinguished conjugate. Here, we consider similar operations, which enable us to show that every conjugacy class admits a nonempty subclass containing elements with both the maximal possible power and the minimal possible copower. We will see that at least one element in this distinguished subclass can be found using finitely many (reverse) cycling operations, and that the whole of the subclass is obtained from there by using finitely repeated simple conjugacy operations.

Cycling consists in moving the first simple element distinct from Δ involved in the Δ-normal form to the end, and reverse cycling in moving the last simple element to the beginning.

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. The *cycling* and the *reverse cycling* are the applications ϕ_+ and ϕ_- from G into itself defined by

$$\phi_+(a) = \Delta^p a_2 \dots a_\ell \widetilde{\partial}^{2p}(a_1), \quad \text{and} \quad \phi_-(a) = \Delta^p \partial^{2p}(a_\ell) a_1 \dots a_{\ell-1},$$

where $\Delta^p a_1 \dots a_\ell$ is the Δ-normal form of a. In particular, we have $\phi_+(\Delta^p) = \Delta^p = \phi_-(\Delta^p)$.

Lemma 3.5. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then, for every a in G, we have $v_\Delta^-(a) \le v_\Delta^-(\phi_\pm(a)) \le v_\Delta^-(a) + 1$ and $v_\Delta^+(a) - 1 \le v_\Delta^+(\phi_\pm(a)) \le v_\Delta^+(a)$.*

Proof. Let $a = \Delta^p a_1 \dots a_\ell$ be the Δ-normal form of a. By definition, we have $\phi_+(a) = \Delta^p a_2 \dots a_\ell \widetilde{\partial}^{2p}(a_1)$ and $\phi_-(a) = \Delta^p \partial^{2p}(a_\ell) a_1 \dots a_{\ell-1}$. We immediatly obtain $v_\Delta^-(a) \leq v_\Delta^-(\phi_\pm(a))$, and we deduce $v_\Delta^+(\phi_\pm(a)) \leq v_\Delta^+(a)$ from the equalities

$$\begin{aligned}&\phi_+(a)\widetilde{\partial}^{2p-1}(a_1)\partial^3(a_\ell)\dots\partial^{2\ell-1}(a_2)\\ &\qquad = \phi_-(a)\partial(a_{\ell-1})\dots\partial^{2\ell-3}(a_1)\partial^{2p+2l-1}(a_\ell) = \Delta^{p+\ell}.\end{aligned}$$

Finally, $v_\Delta^-(\phi_\pm(a)) \leq v_\Delta^-(a)+1$ and $v_\Delta^+(a)-1 \leq v_\Delta^+(\phi_\pm(a))$ follow from Lemma 2.5. □

Lemma 3.6. *Assume that M is a small Gaussian monoid. Then, for every a in M, we have $\|a\|=\|\widetilde{\partial}^2(a)\|=\|\partial^2(a)\|$.*

Proof. By Proposition 1.1, the automorphisms $\widetilde{\partial}^2$ and ∂^2 induce permutations of the atoms. On the other hand, the application $\|.\|$ associates to an element a the maximal number of atoms in a decomposition of a. □

The previous lemma is required for our subsequent inductive argument needed for establishing the next result, which appears in [11] in the case of braids.

Proposition 3.7. *Assume that M is a small Gaussian monoid, G is its group of fractions, a belongs to G, and some conjugate b to a satisfies $v_\Delta^-(b) > v_\Delta^-(a)$. Then there exists an integer m satisfying $v_\Delta^-(\phi_+^m(a)) > v_\Delta^-(a)$.*

Proof. By Lemma 3.1, there exists an element d in M satisfying $a = d^{-1}bd$, hence

$$da = bd. \tag{3.2}$$

We prove the result of the proposition using induction on $\|d\|$. For $\|d\|= 0$, there is nothing to prove. Assume $\|d\| > 0$. Let $p = v_\Delta^-(a)$. There exist elements a', b' in M satisfying $a = \Delta^p a'$ and $b = \Delta^p b'$ with, by hypothesis, $v_\Delta^-(b') > 0$. From (3.2), we deduce $d\Delta^p a' = \Delta^p b' d$, hence $\partial^{2p}(d)a' = b'd$. Now, as $\Delta \leq b' \leq b'd$ holds, we have $\Delta \leq \partial^{2p}(d)a'$, and Lemma 2.3 implies $\Delta \leq \partial^{2p}(d)a_1$, where a_1 is $\Delta \wedge a'$. Applying the automorphism $\widetilde{\partial}^{2p}$, we obtain $\Delta \leq d\widetilde{\partial}^{2p}(a_1)$. There exists an element g in M satisfying

$$d\widetilde{\partial}^{2p}(a_1) = \Delta g. \tag{3.3}$$

Writing $\Delta = \widetilde{\partial}^{2p+1}(a_1)\widetilde{\partial}^{2p}(a_1)$, we deduce

$$d = \widetilde{\partial}^2(g)\widetilde{\partial}^{2p+1}(a_1).$$

In particular, this implies $\|\widetilde{\partial}^2(g)\|<\|d\|$, as $a_1 \neq \Delta$ implies $\widetilde{\partial}^{2p+1}(a_1) \neq 1$. Now, by definition of cycling, we have

$$\widetilde{\partial}^{2p}(a_1)\phi_+(a) = a\widetilde{\partial}^{2p}(a_1). \tag{3.4}$$

Gathering (3.2), (3.3), and (3.4), we obtain

$$b\Delta g = bd\widetilde{\partial}^{2p}(a_1) = da\widetilde{\partial}^{2p}(a_1) = d\widetilde{\partial}^{2p}(a_1)\phi_+(a) = \Delta g\phi_+(a).$$

We deduce
$$\partial^2(b)g = g\phi_+(a).$$
The element $\phi_+(a)$ is conjugate to $\partial^2(b)$ by the element g of M, and, by Lemma 3.6, g satisfies $\|g\|<\|d\|$. Therefore, we have either $v_\Delta^-(\phi_+(a)) > p$, and we are done, or $v_\Delta^-(\phi_+(a)) = p$, and in this case we have $v_\Delta^-(\partial^2(b)) > v_\Delta^-(\phi_+(a))$, and, applying the induction hypothesis to the elements $\partial^2(b)$ and $\phi_+(a)$, there exists an integer n satisfying $v_\Delta^-(\phi_+^n(\phi_+(a))) > v_\Delta^-(\phi_+(a))$. □

Lemma 3.8. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then, for every a in G with Δ-normal form $a = \Delta^p a_1 \dots a_\ell$, the Δ-normal form of a^{-1} is*
$$a^{-1} = \Delta^{-p-\ell} b_1 \dots b_\ell \quad \textit{with} \quad b_i = \widetilde{\partial}^{2p+2\ell-2i+1}(a_{\ell-i+1}).$$

Proof. First, we have
$$\begin{aligned}
a^{-1} &= a_\ell^{-1} \dots a_1^{-1}\Delta^{-p} = \Delta^{-1}\widetilde{\partial}(a_\ell)\dots\Delta^{-1}\widetilde{\partial}(a_2)\Delta^{-1}\widetilde{\partial}(a_1)\Delta^{-p}\\
&= \Delta^{-1}\widetilde{\partial}(a_\ell)\dots\Delta^{-1}\widetilde{\partial}(a_2)\Delta^{-1-p}\widetilde{\partial}^{2p+1}(a_1)\\
&= \dots = \Delta^{-p-\ell}\widetilde{\partial}^{2p+2\ell-1}(a_\ell)\dots\widetilde{\partial}^{2p+3}(a_2)\widetilde{\partial}^{2p+1}(a_1).
\end{aligned}$$

Now, we show the equality
$$\Delta \wedge (\widetilde{\partial}^{2p+2i-1}(a_i)\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1)) = \widetilde{\partial}^{2p+2i-1}(a_i),$$
for $1 \le i \le \ell$ by using induction on i. The result is trivial for $i = 1$. Assume $i > 1$. We have
$$\begin{aligned}
&\Delta \wedge (\widetilde{\partial}^{2p+2i-1}(a_i)\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1))\\
&\qquad = \widetilde{\partial}^{2p+2i-1}(a_i)(\widetilde{\partial}^{2p+2i-2}(a_i) \wedge (\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1))).
\end{aligned}$$

Applying the induction hypothesis, we obtain
$$\begin{aligned}
&\widetilde{\partial}^{2p+2i-2}(a_i) \wedge (\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1))\\
&\quad = (\widetilde{\partial}^{2p+2i-2}(a_i) \wedge (\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1))) \wedge \Delta\\
&\quad = \widetilde{\partial}^{2p+2i-2}(a_i) \wedge ((\widetilde{\partial}^{2p+2i-3}(a_{i-1})\dots\widetilde{\partial}^{2p+1}(a_1)) \wedge \Delta)\\
&\quad \overset{\text{(IH)}}{=} \widetilde{\partial}^{2p+2i-2}(a_i) \wedge \widetilde{\partial}^{2p+2i-3}(a_{i-1})\\
&\quad = \widetilde{\partial}^{2p+2i-2}(a_i \wedge \partial(a_{i-1})) = \widetilde{\partial}^{2p+2i-2}(a_i \wedge (a_{i-1}\backslash\Delta)) = \widetilde{\partial}^{2p+2i-2}(1) = 1,
\end{aligned}$$
which concludes the proof. □

Lemma 3.9. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then every element a in G satisfies $(\phi_-(a))^{-1} = \widetilde{\partial}^2(\phi_+(a^{-1}))$.*

Proof. Let $a = \Delta^p a_1 \dots a_\ell$ be the Δ-normal form of a, and let $q = v_\Delta^+(a)$, *i.e.*, $q = p + \ell$. Then, by Lemma 3.8, the Δ-normal form of a^{-1} is $a^{-1} = \Delta^{-q} b_1 \dots b_\ell$ with $b_i = \widetilde{\partial}^{2q-2i+1}(a_{\ell-i+1})$. Therefore, we have $\phi_+(a^{-1}) = \Delta^{-q} b_2 \dots b_\ell \partial^{2q}(b_1)$. From $\phi_-(a) = \Delta^p \partial^{2p}(a_\ell) a_1 \dots a_{\ell-1}$, we deduce

$$\phi_-(a)\widetilde{\partial}^2(\phi_+(a^{-1})) = \Delta^p \partial^{2p}(a_\ell) a_1 \dots a_{\ell-1} \Delta^{-q} \widetilde{\partial}^2(b_2) \dots \widetilde{\partial}^2(b_\ell) \partial^{2q-2}(b_1).$$

Now, for $1 \le i \le \ell - 1$, we have

$$\begin{aligned} a_{\ell-i} \Delta^{i-1-q} \widetilde{\partial}^2(b_{1+i}) &= a_{\ell-i} \widetilde{\partial}^{2+2i-2-2q}(\widetilde{\partial}^{2q-2i-1}(a_{\ell-i})) \Delta^{i-1-q} \\ &= a_{\ell-i} \partial(a_{\ell-i}) \Delta^{i-1-q} = \Delta^{i-q}. \end{aligned}$$

We obtain

$$\begin{aligned} \phi_-(a)\widetilde{\partial}^2(\phi_+(a^{-1})) &= \Delta^p \partial^{2p}(a_\ell) \Delta^{-p-1} \partial^{2q-2}(b_1) \\ &= \Delta^{-1} \widetilde{\partial}^2(a_\ell) \partial^{2q-2}(\widetilde{\partial}^{2q-1}(a_\ell)) \\ &= \Delta^{-1} \widetilde{\partial}^2(a_\ell) \widetilde{\partial}^2(\partial(a_\ell)) \\ &= \Delta^{-1} \widetilde{\partial}^2(a_\ell \partial(a_\ell)) = \Delta^{-1} \widetilde{\partial}^2(\Delta) = \Delta^{-1}\Delta = 1, \end{aligned}$$

which is the required result. □

We obtain the dual result of Proposition 3.7.

Proposition 3.10. *Assume that M is a small Gaussian monoid, G is its group of fractions, a belongs to G, and some conjugate b to a satisfies $v_\Delta^+(b) < v_\Delta^+(a)$. Then there exists an integer m satisfying $v_\Delta^+(\phi_-^m(a)) < v_\Delta^+(a)$.*

Proof. It suffices to apply Proposition 3.7 to a^{-1} and b^{-1}, and to use Lemma 3.9. □

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. For a in G, we denote the conjugacy class of a by $C(a)$, and we define the *summit power*, the *summit copower*, and the *summit gap* of a to be, respectively, the maximal power, the minimal copower, and the minimal gap in the class $C(a)$.

Proposition 3.11. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Then, for every a in G, those elements of $C(a)$ whose gap is the summit gap form a nonempty finite subclass of $C(a)$.*

Proof. By definition, the gap of an element in $C(a)$ is the summit gap if and only if its power is the summit power and its copower the summit copower. By Proposition 3.7, repeated cycling on a leads to some of those conjugates to a whose power is the summit power. Let $\phi_+^*(a)$ denote the set of the conjugates to a so obtained. By Proposition 3.10, repeated reverse cycling on the elements in $\phi_+^*(a)$ leads to some of those conjugates to a whose copower is the summit copower. Let $\phi_-^*(\phi_+^*(a))$ denote the set of the conjugates to a so obtained. Now, by Lemma 3.5, reverse cycling cannot decrease the power, so the power of elements in $\phi_-^*(\phi_+^*(a))$ is equal to the power of elements in $\phi_+^*(a)$, *i.e.*, is equal to the summit power. Therefore, the gap of elements in $\phi_-^*(\phi_+^*(a))$ is the summit gap. Then, we obtain all the elements of $C(a)$ whose gap is the summit gap by conjugating the elements in $\phi_-^*(\phi_+^*(a))$ by simple elements, and by keeping only those elements whose gap is the summit gap (see the proof of Corollary 3.4). $\square$

Definition. For a in G, the *summit class* $C^{\mathrm{sum}}(a)$ of a is defined to be the subclass of $C(a)$ containing those elements whose gap is the summit gap.

Proposition 3.12. *Assume that M is a small Gaussian monoid, and G is its group of fractions. Let a, b be elements in G. Finitely repeated cycling and reverse cycling on a (resp. on b) yields an element $\breve{a}$ in $C^{\mathrm{sum}}(a)$ (resp. an element $\breve{b}$ in $C^{\mathrm{sum}}(b)$), and finitely repeated simple conjugation on $\breve{a}$ yields the whole of $C^{\mathrm{sum}}(a)$. Then a, b are conjugate in G if and only if $\breve{b}$ belongs to $C^{\mathrm{sum}}(a)$.*

We obtain also simple criteria for proving non-conjugacy. Each of the following conditions:

(i) the shortest interval containing a and the one containing b are disjoint,

(ii) the summit powers of a and b are different,

(iii) the summit copowers of a and b are different,

(iv) the summit gaps of a and b are different,
implies that a and b are not conjugate.

4. An example

Here we describe an example of a group which is eligible for the previous approach, but which is significantly different from an Artin group.

Let $M_\bullet$ be the monoid defined by the presentation

$$\langle\ x, y : xyxyx = yy\ \rangle.$$

Proving that $M_\bullet$ is a small Gaussian monoid amounts to proving that it is atomic, and that the closure of $\{x, y\}$ under $\backslash$ exists and is finite [7]. The latter condition is easily verified: the closure is

$$\{1, x, y, xy, yx, xyx, yxy, xyxy, yxyx, yxyxy\}.$$

The verification of the former condition is not trivial. There is no general method to prove the atomicity of a monoid given by a presentation. In good cases, one can exhibit a barycentric norm, *i.e.*, one such that $\mu(x_i)$ is a positive integer for every atom x_i, and $\mu(ab) = \mu(a) + \mu(b)$ holds for a, b in the monoid. For instance, in the monoid $\langle\, x, y : xyx = yy \,\rangle$, a barycentric norm can be defined by $\mu(x) = 1$, $\mu(y) = 2$ and $\mu(ab) = \mu(a) + \mu(b)$. In contradistinction, there exists no norm of the previous type for $M_\bullet$, nor does it either exist any norm satisfying $\mu(ab) = \mu(a) + \mu(b)$. Indeed, the equality $\mu(xyxyx) = \mu(yy)$ would imply $3\mu(x) + 2\mu(y) = 2\mu(y)$, hence $\mu(x) = 0$, contradicting the definition of a norm. However, the Knuth-Bendix algorithm [14] allows us to prove the atomicity of $M_\bullet$.

Lemma 4.1. *The monoid with presentation $\langle\, x, y : xyxyx = yy \,\rangle$ is atomic.*

Proof. Let us consider the reduction $xyxyx \rightharpoonup yy$. By closing the critical pairs [14], we obtain the system

$$xyxyx \Rightarrow yy, \tag{R_1}$$

$$xyyy \Rightarrow yyyx. \tag{R_2}$$

The reduction $\Rightarrow$ is confluent. First, we show that it is Noetherian. Every word w on $\{x, y\}$ can be written as $x^{m_1} y^{m_2} x^{m_3} \ldots y^{m_{2k}} x^{m_{2k+1}}$ with $m_1, m_{2k+1} \geq 0$ and $m_i > 0$ for $1 < i < 2k+1$. Now, let $G(w) = \sum_{i=1}^{k} (m_{2i} \sum_{j=1}^{i} m_{2j-1})$. Then $w_1 \Rightarrow w_2$ implies $G(w_1) > G(w_2)$, which proves that $\Rightarrow$ is Noetherian. In order to prove that $\Rightarrow$ is Artinian, we consider the system

$$yy \rightarrow xyxyx, \tag{S_1}$$

$$yyyx \rightarrow xyyy, \tag{S_2}$$

and we show that the new reduction $\rightarrow$ is Noetherian. We define $N(w) = \sum_{i=1}^{k} E(m_{2i}/3)$ where $E(n)$ is the integer part of n, and $P(w) = \sum_{i=1}^{k} d(m_{2i})$ where $d(n)$ is 1 if $n \equiv 2$ [3] holds, and 0 otherwise. Let $f_M(w) = 2N(w) + P(w)$. Then $w \rightarrow_{S_1} w'$ implies $f_M(w) > f_M(w')$, and $w \rightarrow_{S_2} w'$ implies $f_M(w) = f_M(w')$. Therefore, a sequence σ of reductions $\rightarrow$ starting from a word w contains a finite number n_1 of reductions $\rightarrow_{S_1}$, and $n_1 \leq f_M(w)$ holds. Then the maximal length λ of the words involved in σ satisfies $\lambda \leq |w| + 3n_1$. Finally, $w \rightarrow w'$ implies $w >_{lex(\lambda)} w'$, where $>_{lex(\lambda)}$ denotes the lexicographic order on words with length at most λ. So the number of reductions $\rightarrow$ in σ is finite—with 2^λ as an upper bound—which proves that $\rightarrow$ is Noetherian. $\square$

It is then easy to verify that $M_\bullet$ is a small Gaussian monoid. The closure of $\{x, y\}$ under $\backslash$ and $\vee$ is the finite set

$$S_\bullet = \{1, x, y, xy, yx, xyx, yxy, xyxy, xyxyx \equiv yy, \\ yxyx, yxyxy, yxyxyx \equiv yyy \equiv xyxyxy\}.$$

By definition, $S_\bullet$ is the set of simple elements of $M_\bullet$, and Δ is $(yx)^3 \equiv y^3 \equiv (xy)^3$, see Figure 2.

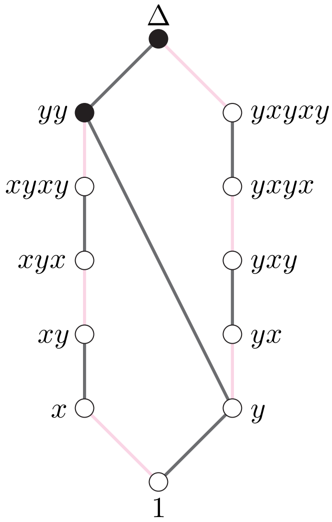

Figure 2. The lattice $(S_\bullet, \vee, \wedge)$ of simples in $M_\bullet$.

We now study an example of conjugacy problem in $G_\bullet$, the group of fractions of $M_\bullet$. Let a, b be the elements in $G_\bullet$ with respective Δ-normal forms

$$a = \Delta^{-3} \cdot xyxy \cdot yxyx \cdot xyxy \cdot yxy, \quad \text{and} \quad b = \Delta^{-4} \cdot x \cdot x \cdot x \cdot xyxy \cdot yxy \cdot yxyxy.$$

The question is to decide whether a and b are conjugate. First, a lies in $[-3, 1]$, while b lies in $[-4, 2]$. As the intervals intersect, we cannot conclude directly. Let us compute the summit class of a. First, repeated cycling on a gives

$$\begin{aligned} \phi_+(a) &= \Delta^{-2} \cdot yxyx \cdot xyxy \cdot y, \\ \phi_+^2(a) &= \Delta^{-1} \cdot xyx \cdot xyx = \phi_+^3(a). \end{aligned}$$

Let $\breve{a} = \Delta^{-1} \cdot xyx \cdot xyx$, we have $\phi_+^m(a) = \breve{a}$ for $m \geq 2$. Next, we have $\phi_-(\breve{a}) = \breve{a}$, so $\breve{a}$ belongs to $C^{\mathrm{sum}}(a)$. As $v_\Delta^-(\breve{a}) = -1$ and $v_\Delta^+(\breve{a}) = 1$, $C^{\mathrm{sum}}(a)$ is $C(a) \cap [-1, 1]$. Now, by Proposition 3.11, it remains to repeatedly conjugate $\breve{a}$ by simple elements, and to keep only the gap 2 elements. In the current case, the only element which both is simply conjugate to $\breve{a}$ and lies in $[-1, 1]$ is $\breve{a}$ itself. We conclude that $C^{\mathrm{sum}}(a)$ is the singleton $\{\Delta^{-1} \cdot xyx \cdot xyx\}$.

At this point, we cannot know whether the subclasses $C^{\mathrm{sum}}(a)$ and $C^{\mathrm{sum}}(b)$ are disjoint. Repeated cycling on b gives

$$\begin{aligned} \phi_+(b) &= \Delta^{-3} \cdot x \cdot x \cdot xyxy \cdot yxy, \\ \phi_+^2(b) &= \Delta^{-3} \cdot x \cdot xyxy \cdot yxyx, \\ \phi_+^3(b) &= \Delta^{-3} \cdot xyxy \cdot yxyx \cdot x, \\ \phi_+^4(b) &= \Delta^{-3} \cdot yxyx \cdot x \cdot xyxy, \\ \phi_+^5(b) &= \Delta^{-3} \cdot x \cdot xyxy \cdot yxyx = \phi_+^2(b). \end{aligned}$$

Next, we have

$$\begin{aligned}
\phi_-(\Delta^{-3}\cdot x\cdot xyxy\cdot yxyx) &= \Delta^{-3}\cdot yxyx\cdot x\cdot xyxy,\\
\phi_-(\Delta^{-3}\cdot yxyx\cdot x\cdot xyxy) &= \Delta^{-3}\cdot xyxy\cdot yxyx\cdot x,\\
\phi_-(\Delta^{-3}\cdot xyxy\cdot yxyx\cdot x) &= \Delta^{-3}\cdot x\cdot xyxy\cdot yxyx.
\end{aligned}$$

In particular, the element $\Delta^{-3}\cdot yxyx\cdot x\cdot xyxy$ belongs to $C^{\mathrm{sum}}(b)$, and we deduce that $C^{\mathrm{sum}}(b)$ is $C(b)\cap[-3,0]$. Therefore, we can conclude that the elements a and b are not conjugate, since they have distinct summit gaps.

Remark. In the case of $M_\bullet$, the automorphisms ∂^2 and $\widetilde{\partial}^2$ are the identity, which implies that the cycling operations are quite trivial. Non trivial cyclings may appear whenever the index of the automorphisms ∂^2 and $\widetilde{\partial}^2$ is large.

5. One way or another

As in [11], the normal form we have chosen corresponds to the one defined by Garside [13] in the case of braids, and it seems to be the most convenient as far as power and copower are involved. However, we can use fractional normal form explicitly defined in [1]—see also [10,11,12]—as well.

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. For c in G, the (left) *fractional normal form* of c is defined to be the unique decomposition

$$a_p^{-1}\dots a_1^{-1}b_1\dots b_q,$$

where a_p,.., b_q are simple elements all distinct from 1, and, writing $x\perp y$ for $x\wedge y=1$, we have $a_1\perp b_1$, $a_i\perp(a_{i-1}\backslash\Delta)$ for $2\le i\le p$, and $b_i\perp(b_{i-1}\backslash\Delta)$ for $2\le i\le q$.

With the notations above, we find

$$v_\Delta^-(c)=\max\{i : b_i=\Delta\}-p,\qquad\text{and}\qquad v_\Delta^+(c)=q-\max\{i : a_i=\Delta\}.$$

We have the following connection between the two normal forms.

Lemma 5.1. *Assume that M is a small Gaussian monoid, and G its group of fractions. Let c be an element in G with fractional normal form $a_p^{-1}\dots a_1^{-1}b_1\dots b_q$. Then the Δ-normal form of c is $\Delta^{-p}c_1\dots c_{p+q}$ with*

$$c_i=\begin{cases}\widetilde{\partial}^{2p-2i+1}(a_{p-i+1}) & \textit{for } 1\le i\le p,\\ b_{i-p} & \textit{for } p+1\le i\le p+q.\end{cases}$$

Proof. By definition, we have

$$\begin{aligned}\Delta^p c &= \Delta^p a_p^{-1} \dots a_1^{-1} b_1 \dots b_q \\ &= \Delta^p \Delta^{-1} \widetilde{\partial}(a_p) \dots \Delta^{-1} \widetilde{\partial}(a_1) b_1 \dots b_q \\ &= \widetilde{\partial}^{2p-1}(a_p) \dots \widetilde{\partial}(a_1) b_1 \dots b_q.\end{aligned}$$

Now, we show $\Delta \wedge (\widetilde{\partial}^{2i-1}(a_i) \dots \widetilde{\partial}(a_1) b_1 \dots b_q) = \widetilde{\partial}^{2i-1}(a_i)$ for $1 \le i \le p$ by induction on i. For $i = 1$, we have

$$\begin{aligned}\Delta \wedge (\widetilde{\partial}(a_1) b_1 \dots b_q) &= \widetilde{\partial}(a_1)(a_1 \wedge (b_1 \dots b_q)) \\ &= \widetilde{\partial}(a_1)((a_1 \wedge (b_1 \dots b_q)) \wedge \Delta) \\ &= \widetilde{\partial}(a_1)(a_1 \wedge ((b_1 \dots b_q) \wedge \Delta)) \\ &= \widetilde{\partial}(a_1)(a_1 \wedge b_1) = \widetilde{\partial}(a_1).\end{aligned}$$

Assume $i > 1$. We have

$$\begin{aligned}&\Delta \wedge (\widetilde{\partial}^{2i-1}(a_i) \dots \widetilde{\partial}(a_1) b_1 \dots b_q) \\ &\quad = \widetilde{\partial}^{2i-1}(a_i)(\widetilde{\partial}^{2i-2}(a_i) \wedge (\widetilde{\partial}^{2i-3}(a_{i-1}) \dots \widetilde{\partial}(a_1) b_1 \dots b_q)).\end{aligned}$$

Now, applying the induction hypothesis, we obtain

$$\begin{aligned}&\widetilde{\partial}^{2i-2}(a_i) \wedge (\widetilde{\partial}^{2i-3}(a_{i-1}) \dots \widetilde{\partial}(a_1) b_1 \dots b_q) \\ &\qquad = (\widetilde{\partial}^{2i-2}(a_i) \wedge (\widetilde{\partial}^{2i-3}(a_{i-1}) \dots \widetilde{\partial}(a_1) b_1 \dots b_q)) \wedge \Delta \\ &\qquad = \widetilde{\partial}^{2i-2}(a_i) \wedge ((\widetilde{\partial}^{2i-3}(a_{i-1}) \dots \widetilde{\partial}(a_1) b_1 \dots b_q) \wedge \Delta) \\ &\qquad \overset{\text{(IH)}}{=} \widetilde{\partial}^{2i-2}(a_i) \wedge \widetilde{\partial}^{2i-3}(a_{i-1}) = \widetilde{\partial}^{2i-2}(a_i \wedge \partial(a_{i-1})) \\ &\qquad = \widetilde{\partial}^{2i-2}(a_i \wedge (a_{i-1} \backslash \Delta)) = \widetilde{\partial}^{2i-2}(1) = 1.\end{aligned}$$

Therefore, we have $c_i = \widetilde{\partial}^{2p-2i+1}(a_{p-i+1})$ for $1 \le i \le p$, and $c_i = b_{i-p}$ for $p + 1 \le i \le p + q$. $\square$

We can define new cycling and reverse cycling with respect to the fractional normal form.

Definition. Assume that M is a small Gaussian monoid, and G is its group of fractions. The *θ-cycling* and the *reverse θ-cycling* are the applications θ_+ and θ_- from G into itself defined by

$$\theta_+(c) = a_{p-1}^{-1} \dots a_1^{-1} b_1 \dots b_q a_p^{-1}, \quad \text{and,} \quad \theta_-(c) = b_q a_p^{-1} \dots a_1^{-1} b_1 \dots b_{q-1},$$

with $c = a_p^{-1} \dots a_1^{-1} b_1 \dots b_q$ the fractional normal form of c.

The following result proves that cycling and θ-cycling are equivalent.

Proposition 5.2. *Assume that M is a small Gaussian monoid, and G its group of fractions. Then, for every c in G, we have $\theta_+(c) = \widetilde{\partial}^2(\phi_+(c))$ and $\theta_-(c) = \phi_-(c)$.*

Proof. First, using Lemma 5.1, we obtain

$$\begin{aligned}
\theta_+(c) &= a_{p-1}^{-1} \dots a_1^{-1} b_1 \dots b_q a_p^{-1} \\
&= a_{p-1}^{-1} \dots a_1^{-1} b_1 \dots b_q \Delta^{-1} \widetilde{\partial}(a_p) \\
&= \Delta^{-p} \widetilde{\partial}^{2p-1}(a_{p-1}) \dots \widetilde{\partial}^3(a_1) \widetilde{\partial}^2(b_1 \dots b_q) \widetilde{\partial}(a_p) \\
&= \widetilde{\partial}^2(\Delta^{-p} \widetilde{\partial}^{2p-3}(a_{p-1}) \dots \widetilde{\partial}(a_1) b_1 \dots b_q \partial(a_p)) \\
&= \widetilde{\partial}^2(\Delta^{-p} c_2 \dots c_{p+q} \partial^{2p}(c_1)) = \widetilde{\partial}^2(\phi_+(c)).
\end{aligned}$$

Next, by definition, we have

$$\begin{aligned}
\theta_+(c^{-1}) &= \theta_+(b_q^{-1} \dots b_1^{-1} a_1 \dots a_p) \\
&= b_{q-1}^{-1} \dots b_1^{-1} a_1 \dots a_p b_q^{-1} \\
&= (b_q a_p^{-1} \dots a_1^{-1} b_1 \dots b_{q-1})^{-1} = \theta_-(c)^{-1}.
\end{aligned}$$

From Lemma 3.9, we deduce $\theta_+(c^{-1}) = \widetilde{\partial}^2(\phi_+(c^{-1})) = \phi_-(c)^{-1}$, hence $\theta_-(c) = \phi_-(c)$. ∎

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