Automata minimization and glueing of categories

Computability
in Europe 2017
June 15

Thomas Colcombet
joint work with Daniela Petrişan

INSTITUT
DE RECHERCHE
EN INFORMATIQUE
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AGENCE NATIONALE DE LA RECHERCHE

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[MFCS 2017] & [Informal presentation in SIGLOG column]

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Description of the situation

Automata

Automata

An **deterministic automaton** is

$$\langle Q, i, f, (\delta_a)_{a \in A} \rangle$$

where

Q is a set of **states**,

$i \colon 1 \to Q$ is the **initial map**

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Rabin & Scott

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It computes the language:

$$\begin{aligned} [\![\mathcal{A}]\!] \colon A^* &\to [1, 2] \\ u &\mapsto f \circ \delta_u \circ i \end{aligned}$$

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Example

$$L_{\texttt{Vec}}(u) = \begin{cases} 2^{|u|_a} & \text{if } |u|_b \text{ is even, and } |u|_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

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Is it possible to do better?

A better implementation

$$L_{\mathtt{Vec}}(u) = \begin{cases} 2^{|u|_a} & \text{if } |u|_b \text{ is even, and } |u|_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution in vector spaces

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Informally: use one bit for the parity to the number of b's.

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$$Q = (\{\texttt{odd}\} \times \mathbb{R}) \cup (\{\texttt{even}\} \times \mathbb{R})$$

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Why is it a better implementation?

Is there a good notion of such automata?

What are their properties (e.g. minimization) ?

A definition via categories

Categories

A category has objects and arrows

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$X, Y, Z \ldots$

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$X, Y, Z \ldots$ $f : X \to Y$

Categories

A category has **objects** and **arrows**

$X, Y, Z \ldots$

$$f\colon \underset{\text{source}}{X} \to \underset{\text{target}}{Y}$$

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A category has **objects** and **arrows**

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source

target

- There is an identity arrow for all object:

$$\mathtt{Id}_X : X \to X$$

Categories

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- There is an identity arrow for all object:

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- Arrows compose: for $f : X \to Y$ and $g : Y \to Z$ there is an arrow:

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+ some associatively axioms.

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+ some associatively axioms.

Set = (sets, maps)
Vec = (vector spaces, linear maps)
Aff = (affine spaces, affine maps)
Rel = (sets, binary relations)

Automata in a category

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Auto(L) is the category of **(C,I,F)-automata** for the **(C,I,F)-language** L.

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A **morphism** is an arrow

$$h \colon Q_{\mathcal{A}} \to Q_{\mathcal{B}}$$

such that tfdc:

$$\begin{tikzcd}
 & Q_{\mathcal{A}} \arrow[dd, "h"] \\
I \arrow[ur, "i_{\mathcal{A}}"] \arrow[dr, "i_{\mathcal{B}}"'] & \\
 & Q_{\mathcal{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_{\mathcal{A}} \arrow[r, "\delta_{\mathcal{A}}(a)"] \arrow[d, "h"'] & Q_{\mathcal{A}} \arrow[d, "h"] \\
Q_{\mathcal{B}} \arrow[r, "\delta_{\mathcal{B}}(a)"'] & Q_{\mathcal{B}}
\end{tikzcd}
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\begin{tikzcd}
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Rk: Morphisms preserve the language.

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Auto(L) is the category of (C,I,F)-automata for the (C,I,F)-language L.

- (Set,1,2)-automata are deterministic automata
- (Rel,1,1)-automata are non-deterministic automata
- (Vec,K,K)-automata are automata weighted over a field **K**. (more generally semi-modules)
- …

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Rk: Morphisms preserve the language.

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(free co-product completion of **Vec**)

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Remark: Vec is a subcategory of Duvs.

Duvs-automata

$$L_{\texttt{Vec}}(u) = \begin{cases} 2^{|u|_a} & \text{if } |u|_b \text{ is even, and } |u|_c = 0 \\ 0 & \text{otherwise} \end{cases}$$

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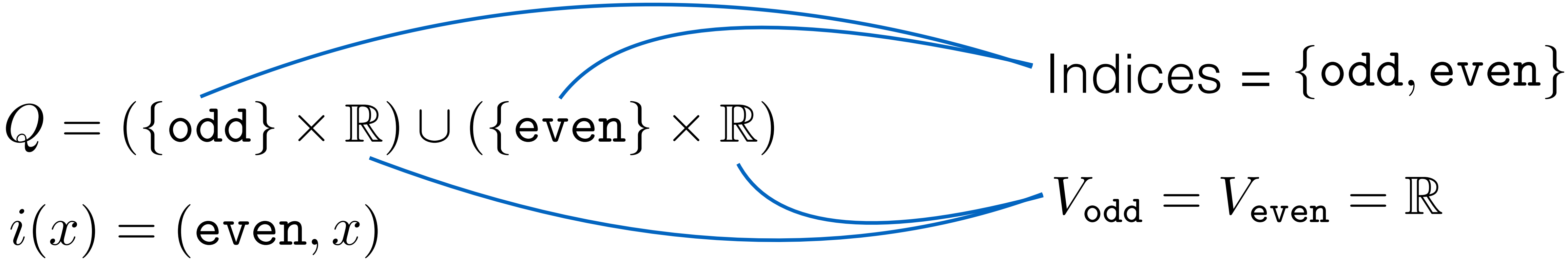

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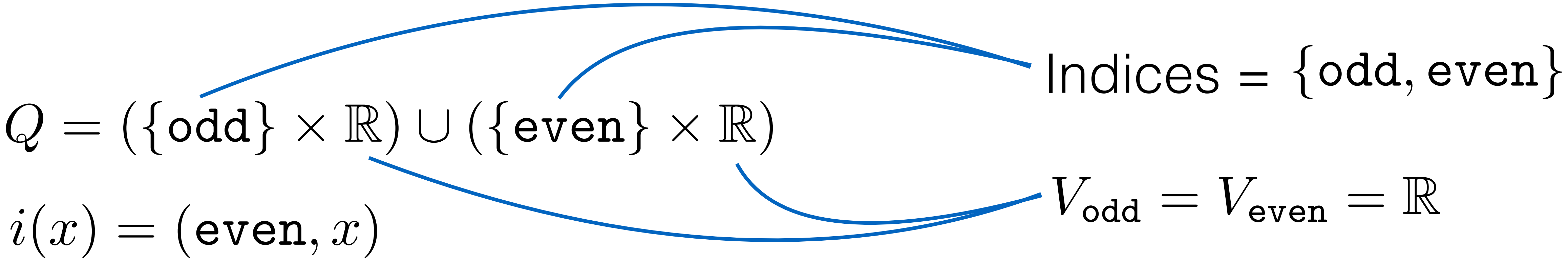

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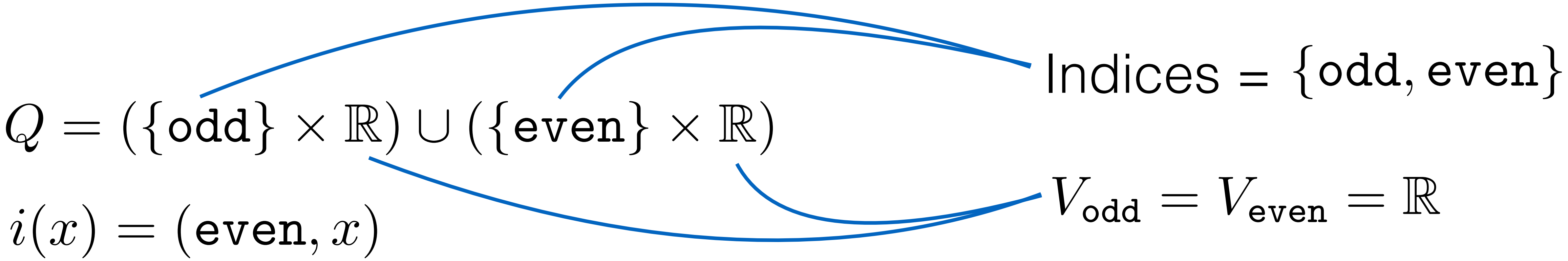

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What can be done?

Minimizing automata via categories

Ingredients for the existence of a minimal automaton

Questions:

Given a (C,I,F)-automaton,

- what does it mean to be minimal?
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Initial and final automata

In a category, an object is

- initial if there is one and exactly one arrow from it to every other object
- final if there is one and exactly one arrow to it from every other object

$$\begin{tikzcd}
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Remark: Initial and final automata exist as soon as the category has countable copowers and powers (works e.g. for Set, Vec, Aff,…).

Factorization systems

A pair of families of arrows $(\mathcal{E}, \mathcal{M})$ is a factorization system if:

Factorization systems

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« surjections »

« monomorphisms »
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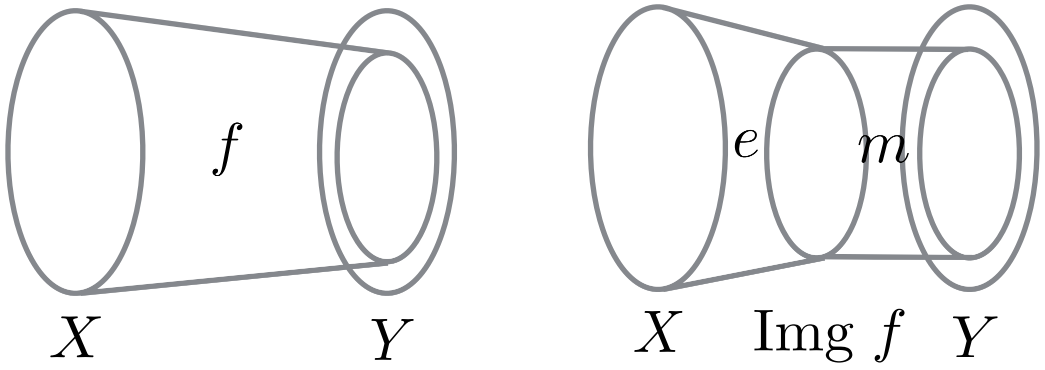

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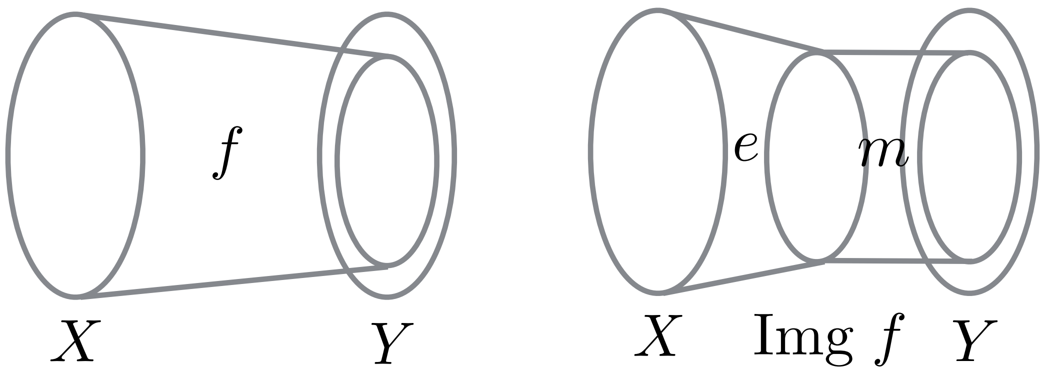

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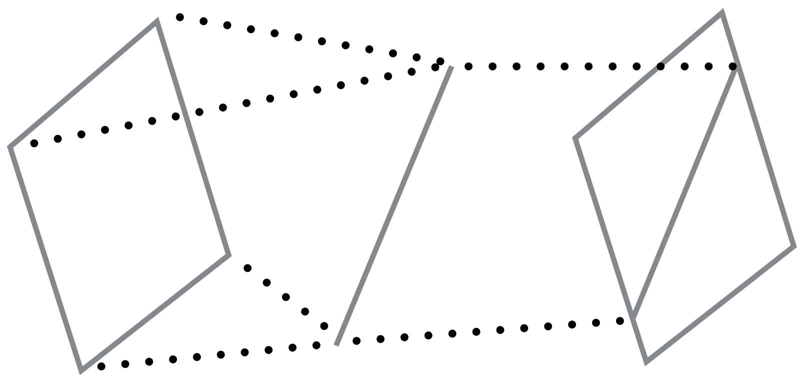

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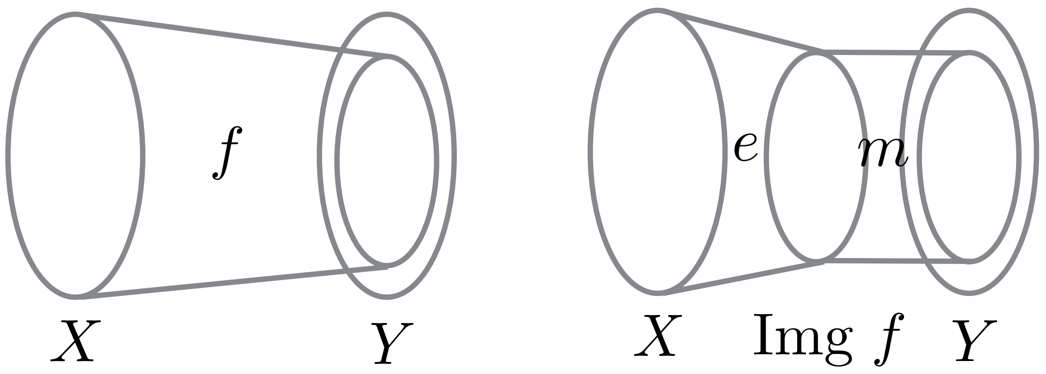

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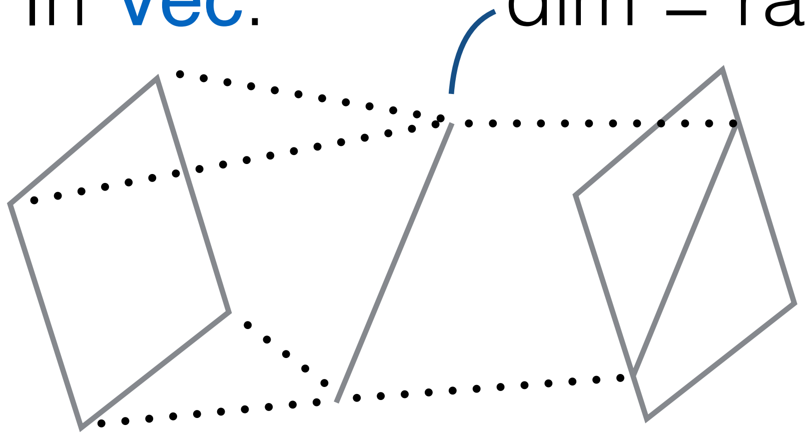
dim = rank f

Factorization system for automata

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Definition:

- an $\mathcal{M}$-subobject X of Y is such that there is an $\mathcal{M}$-arrow $m\colon X \to Y$,
- an $\mathcal{E}$-quotient X of Y is such that there is an $\mathcal{E}$-arrow $e\colon Y \to X$,
- X $(\mathcal{E}, \mathcal{M})$-divides Y if it is a $\mathcal{E}$-quotient of an $\mathcal{M}$-subobject of Y.

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Lemma: In a category with initial object, final object, and a factorization system $(\mathcal{E}, \mathcal{M})$ then:

- there exists an object $\mathtt{Min}$ that $(\mathcal{E}, \mathcal{M})$-divides all objects,
- furthermore $\mathtt{Min} \approx \mathtt{Obs}(\mathtt{Reach}(X)) \approx \mathtt{Reach}(\mathtt{Obs}(X))$ for all X,

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- $\mathtt{Reach}(X)$ is the factorization of the only arrow from I to X, and
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$$\begin{tikzcd}
 & & X \arrow[drr, bend left] & & \\
I \arrow[urr, bend left] \arrow[r, two heads] \arrow[drr, bend right, two heads] & \mathtt{Reach}(X) \arrow[ur, tail] \arrow[rr, two heads] & & \mathtt{Obs}(\mathtt{Reach}(X)) \arrow[r, tail] \arrow[dl, dashed, no head] & F \\
 & & \mathtt{Min} \arrow[urr, tail, bend right] & &
\end{tikzcd}$$

At this point…

We know that:

- C-automata and C-languages can be defined generally in a category C, yielding a

  category Auto(L) of « C-automata for the language L »

- for having a minimal object in a category, it is sufficient to have:
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But, what about minimizing duvs-automata?

Glueings

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$$Q = \mathbb{R}^2$$

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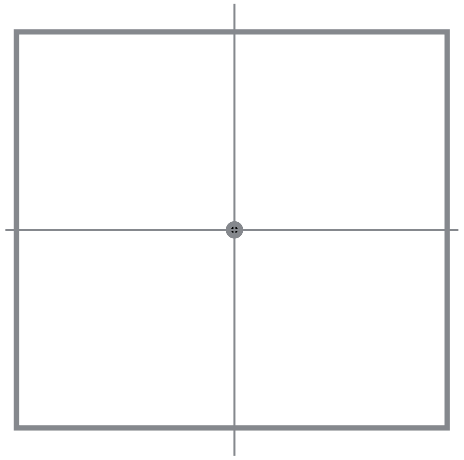

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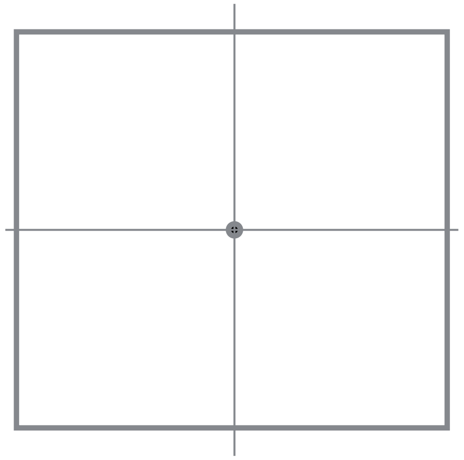

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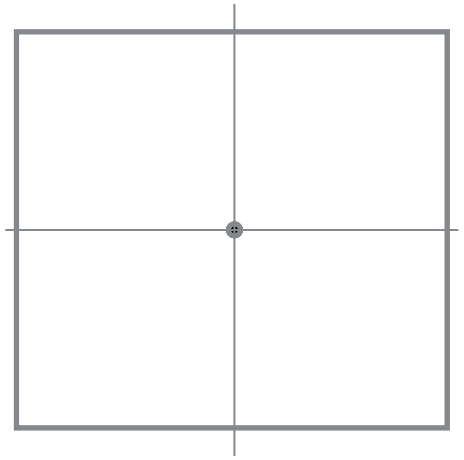

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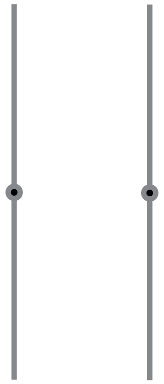

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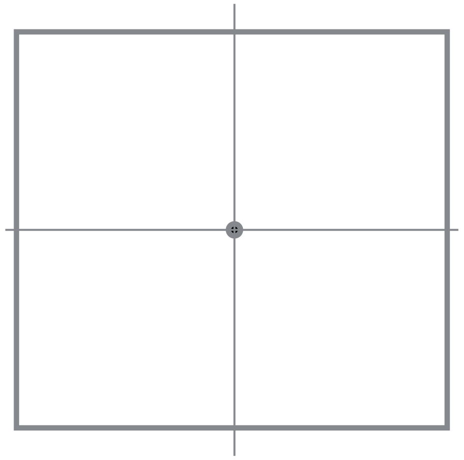

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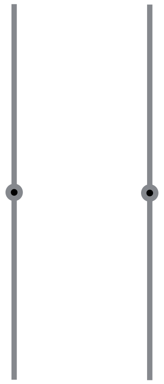

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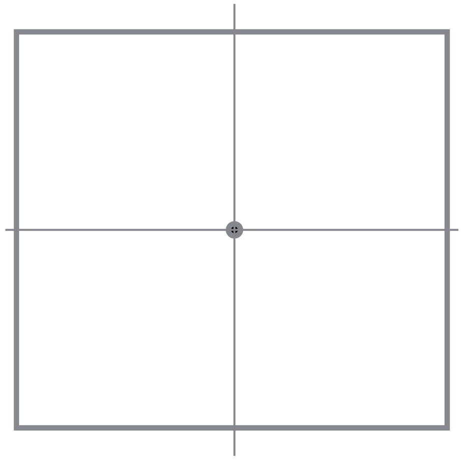

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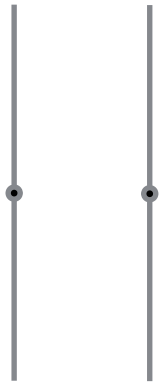

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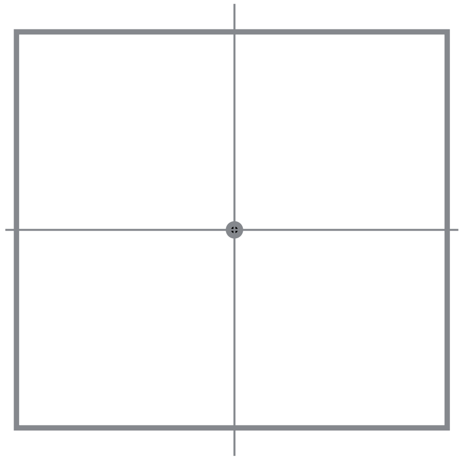

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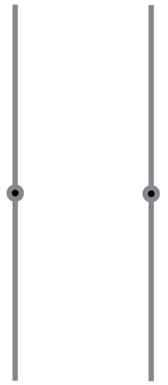

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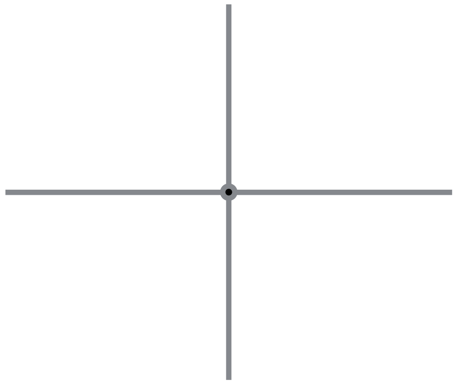

Defining **Glue(Vec)**

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A glueing of vector space is
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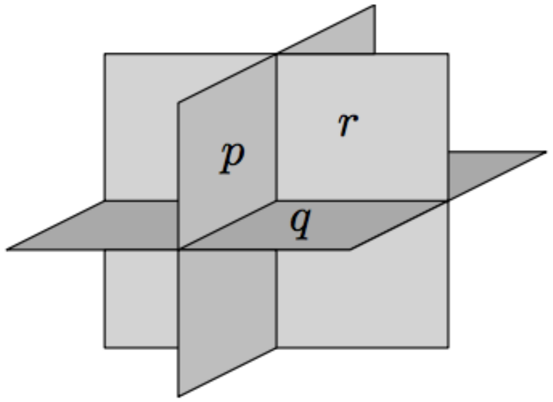

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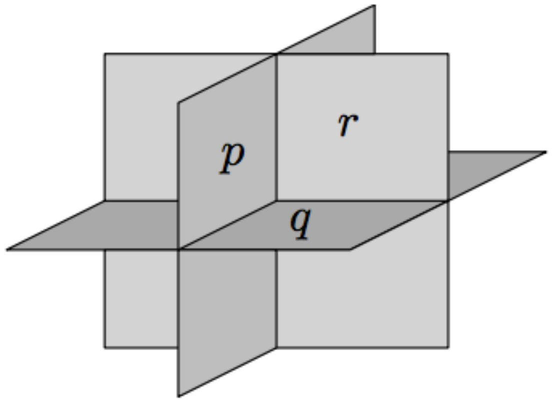

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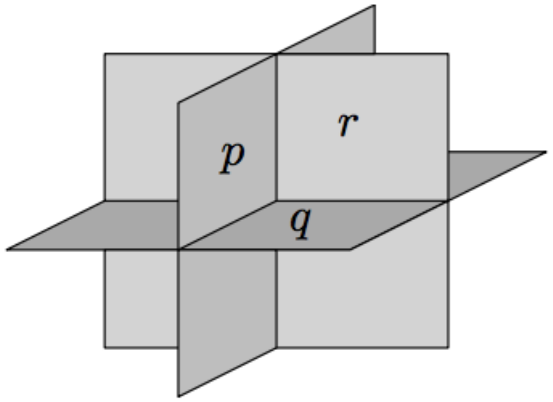

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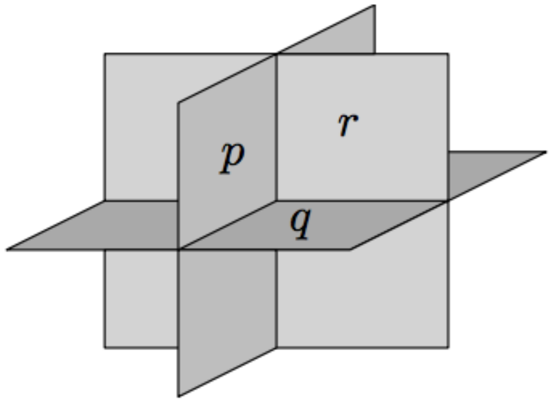

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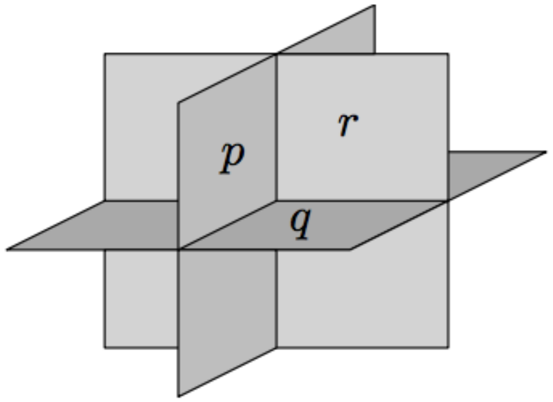

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The advantage is that the concepts are well known, definition properly stated, and this can be applied to other categories than Vec.

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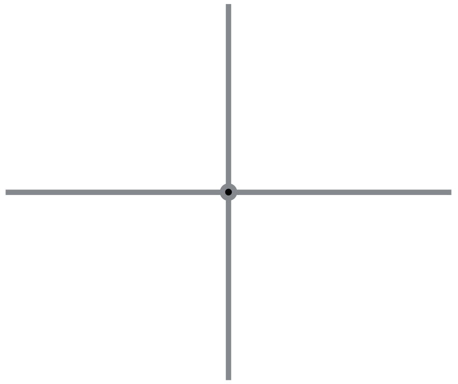

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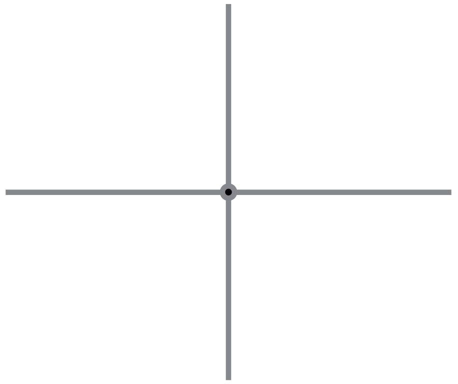

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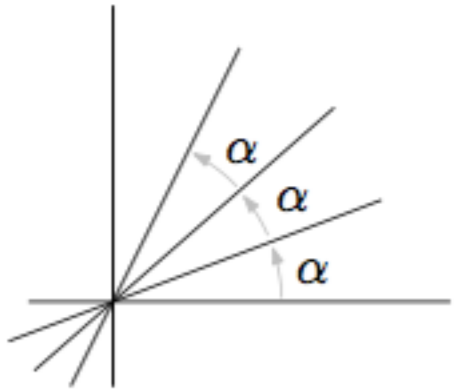

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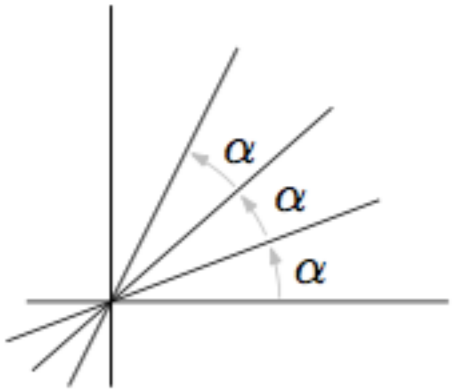

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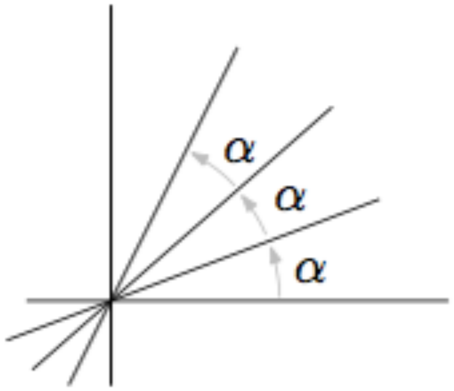

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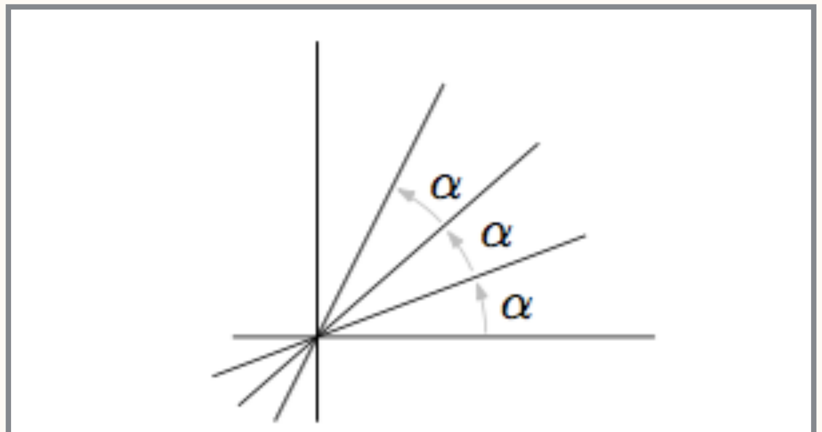

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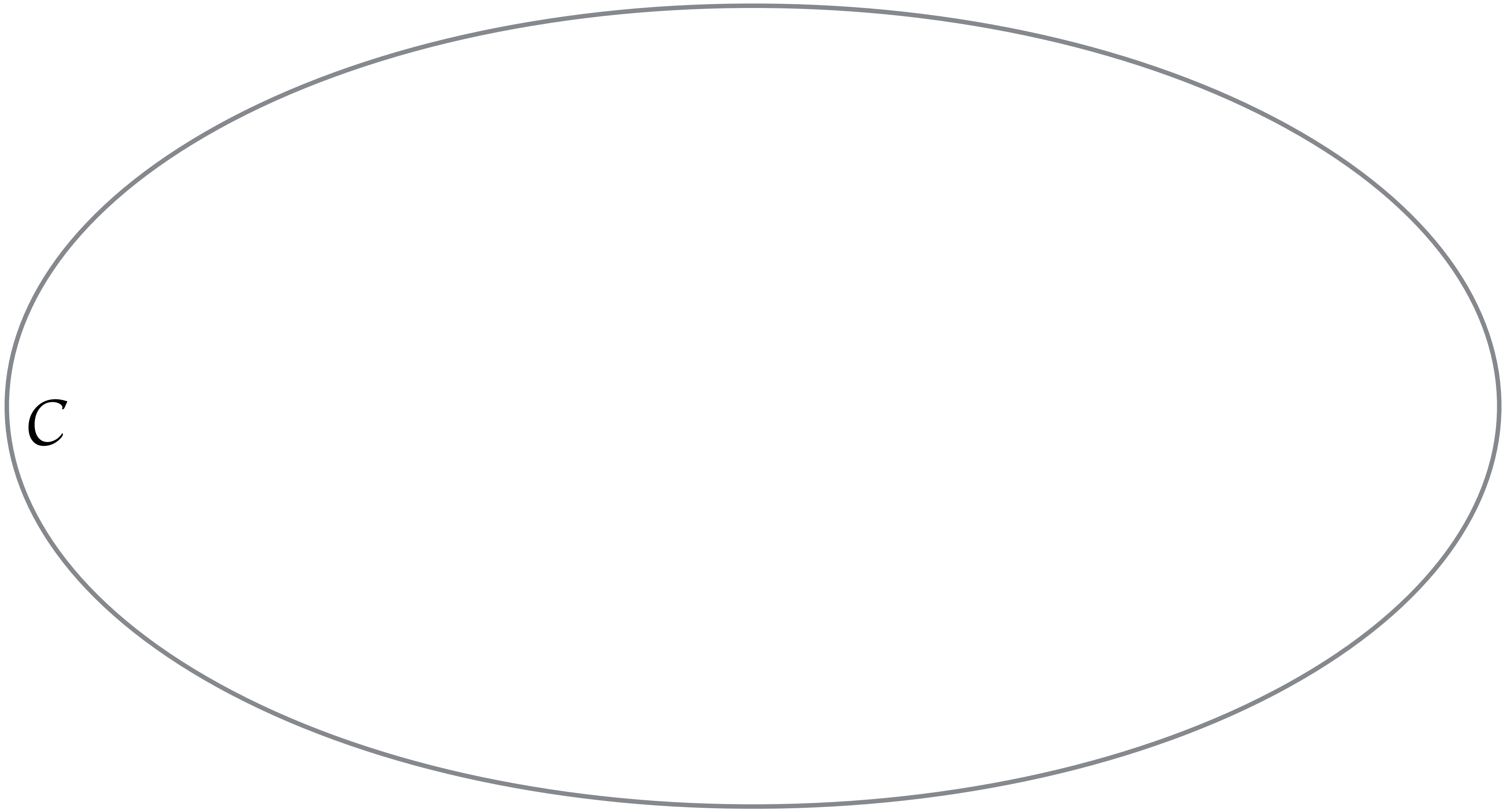

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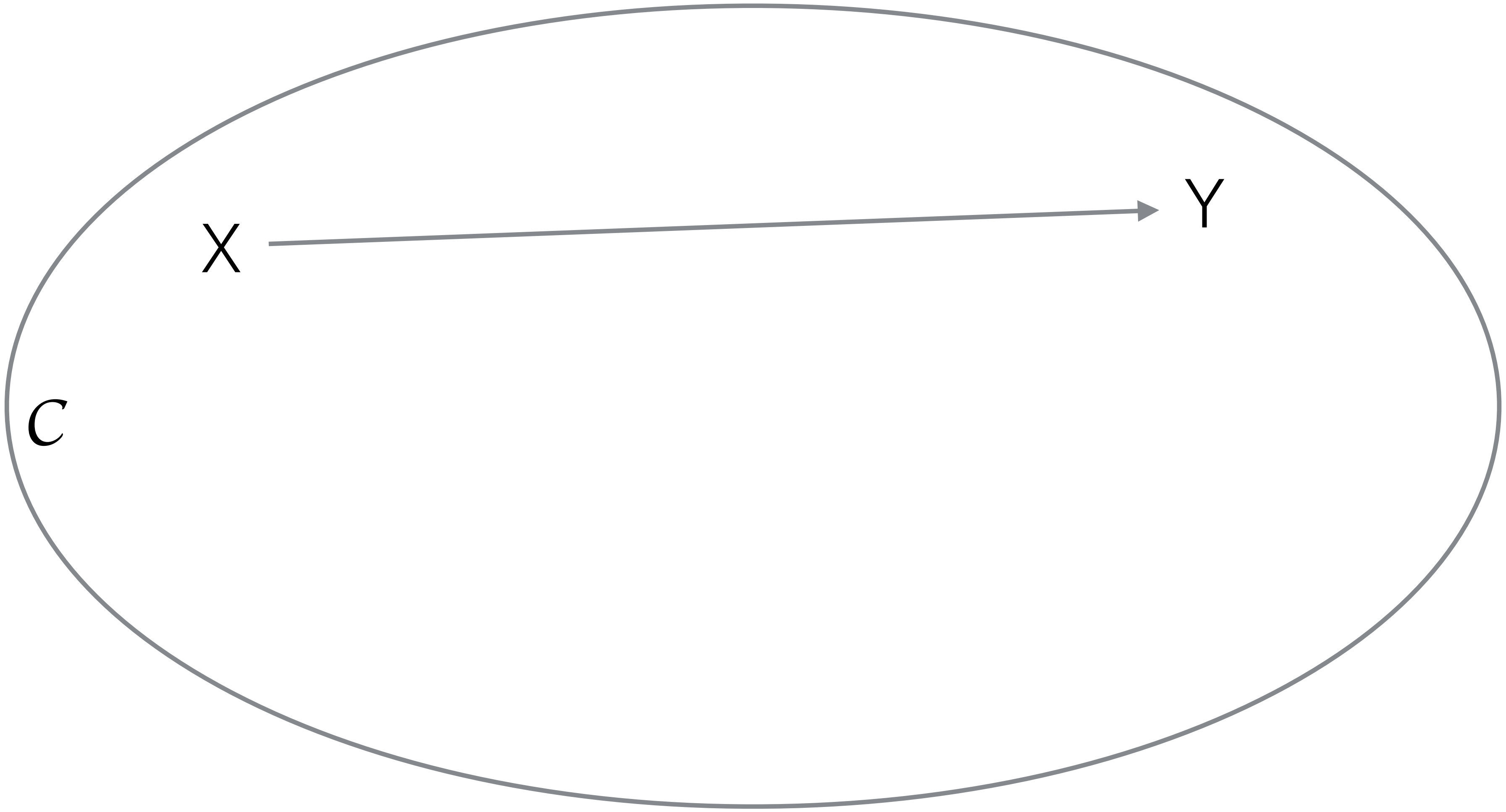

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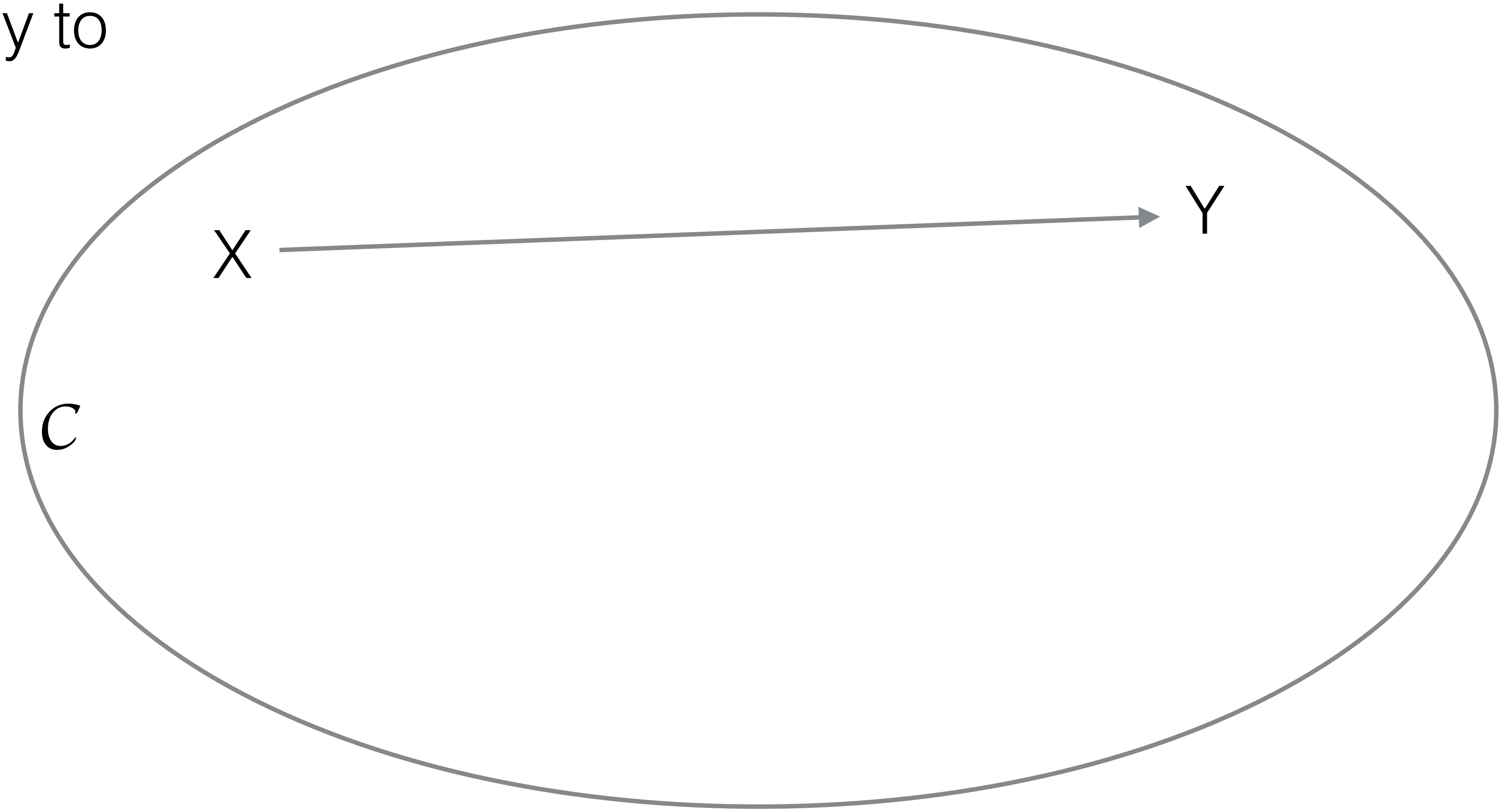

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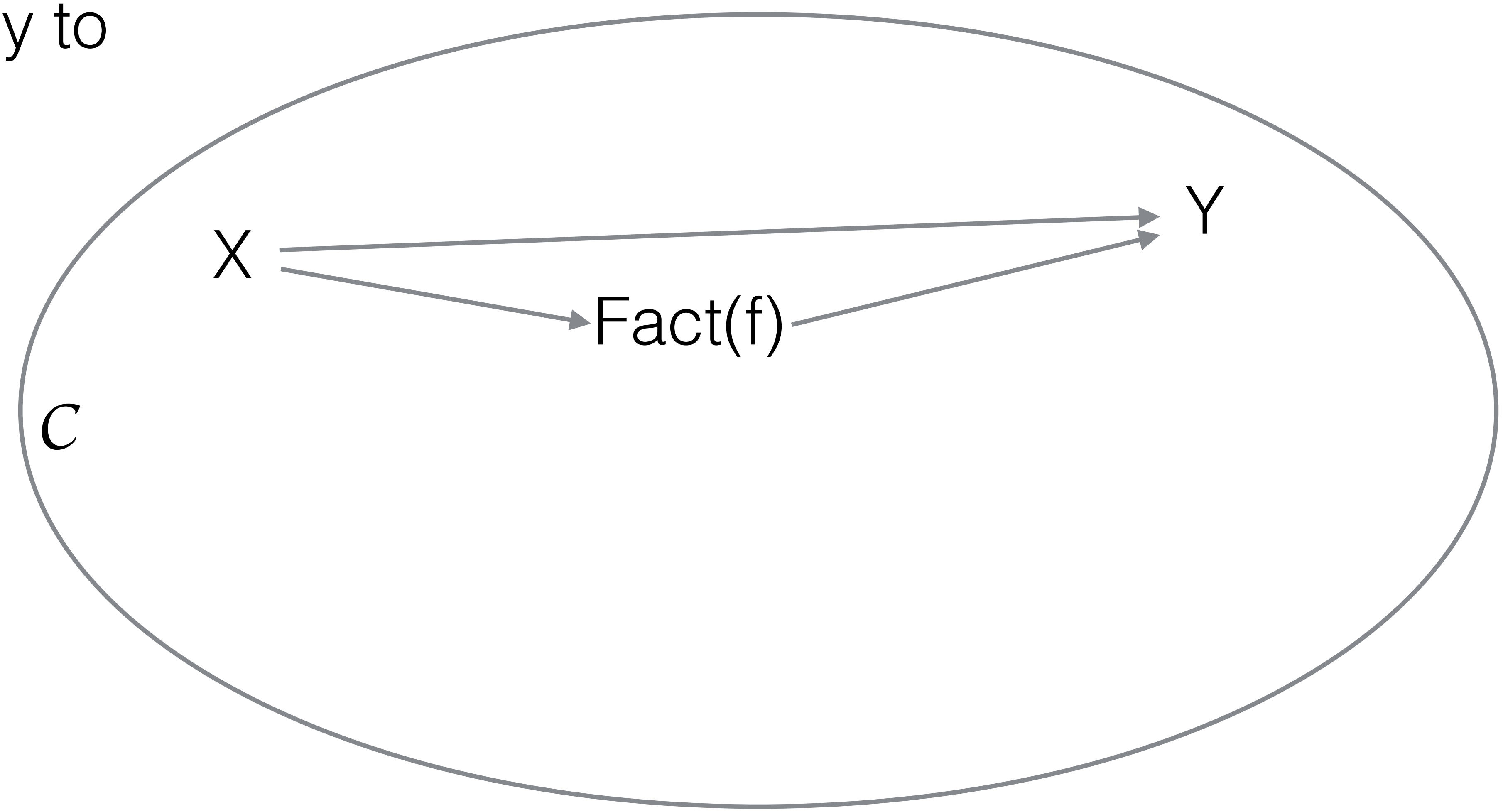

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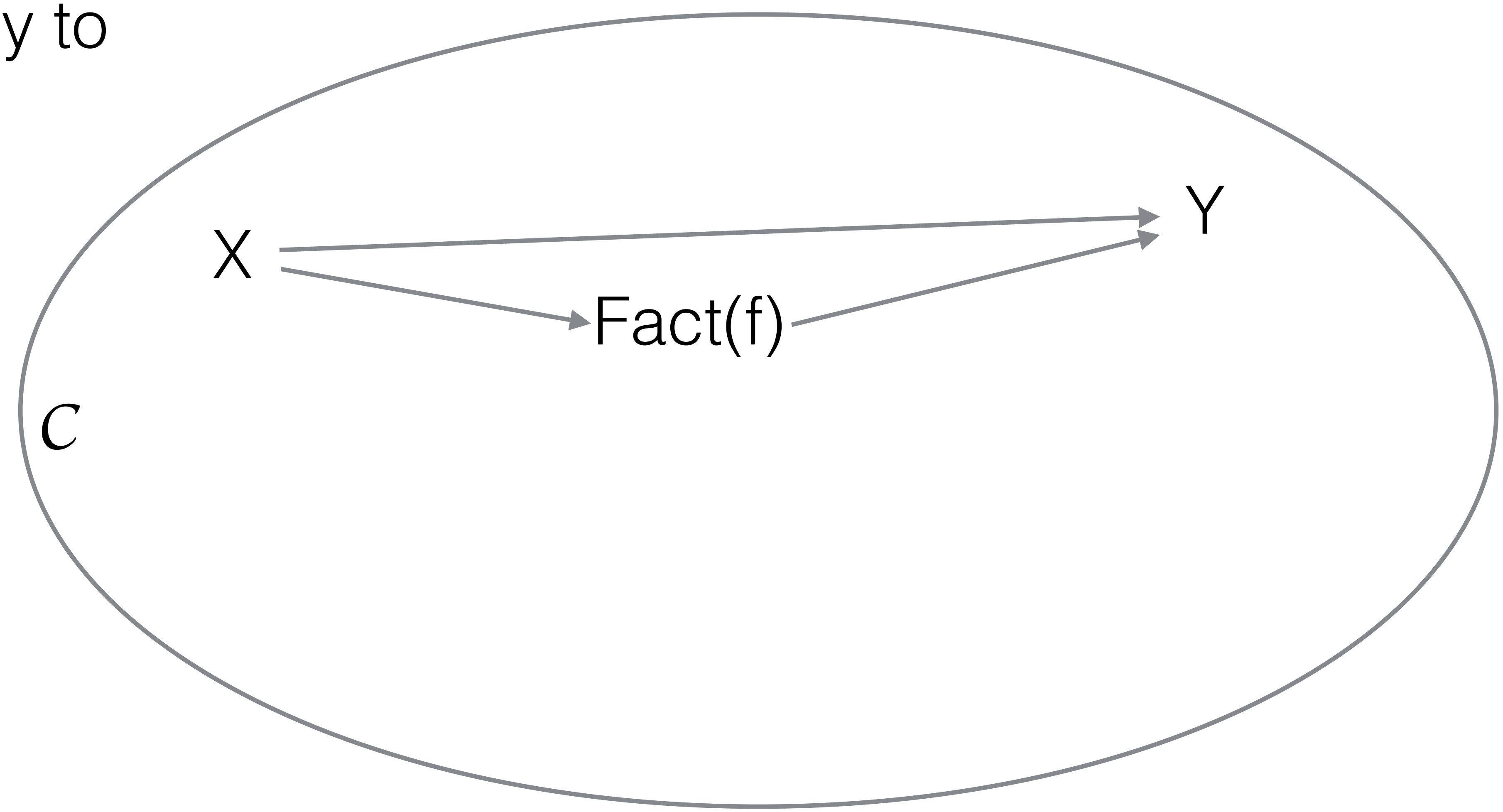

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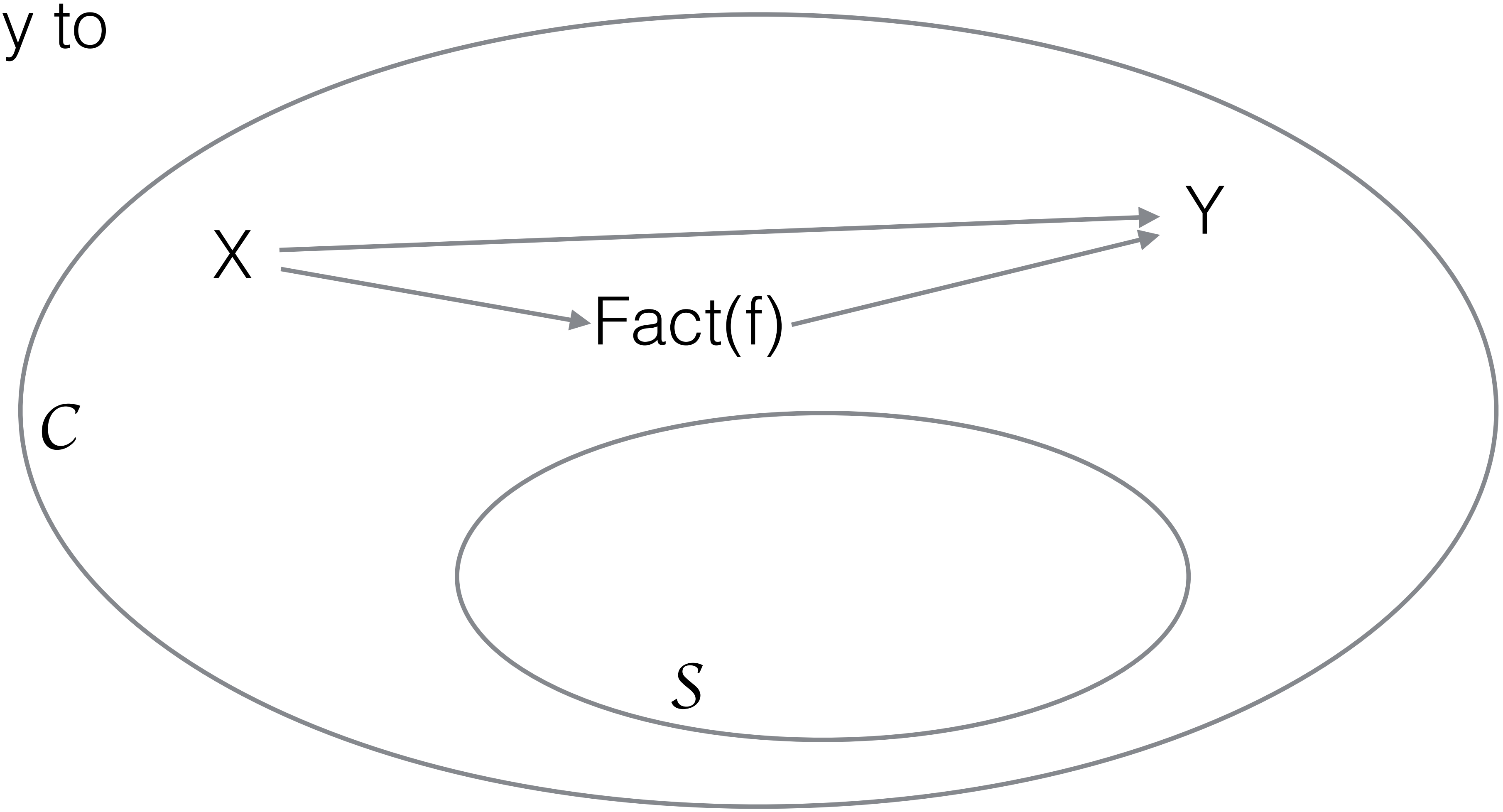

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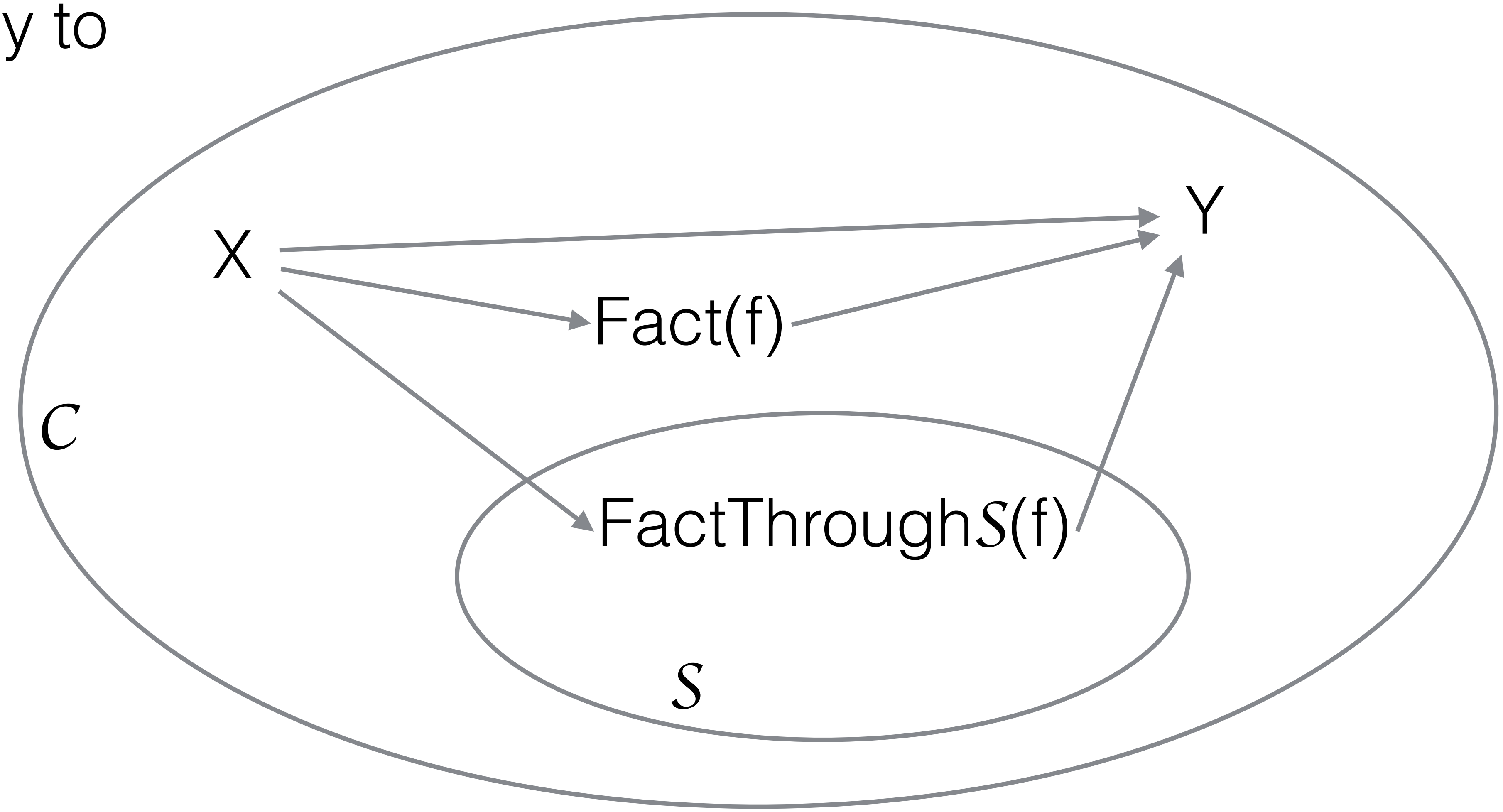

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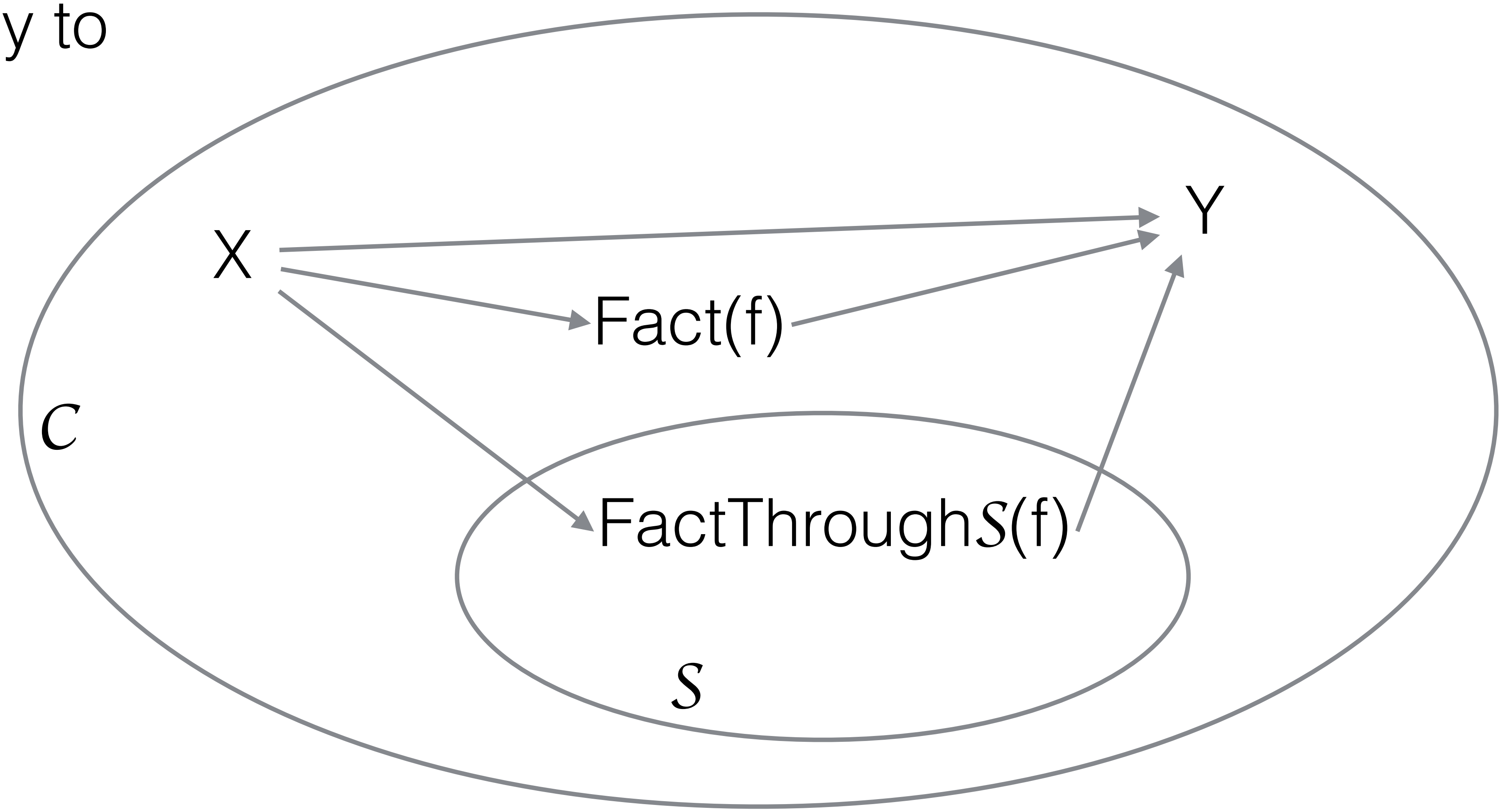

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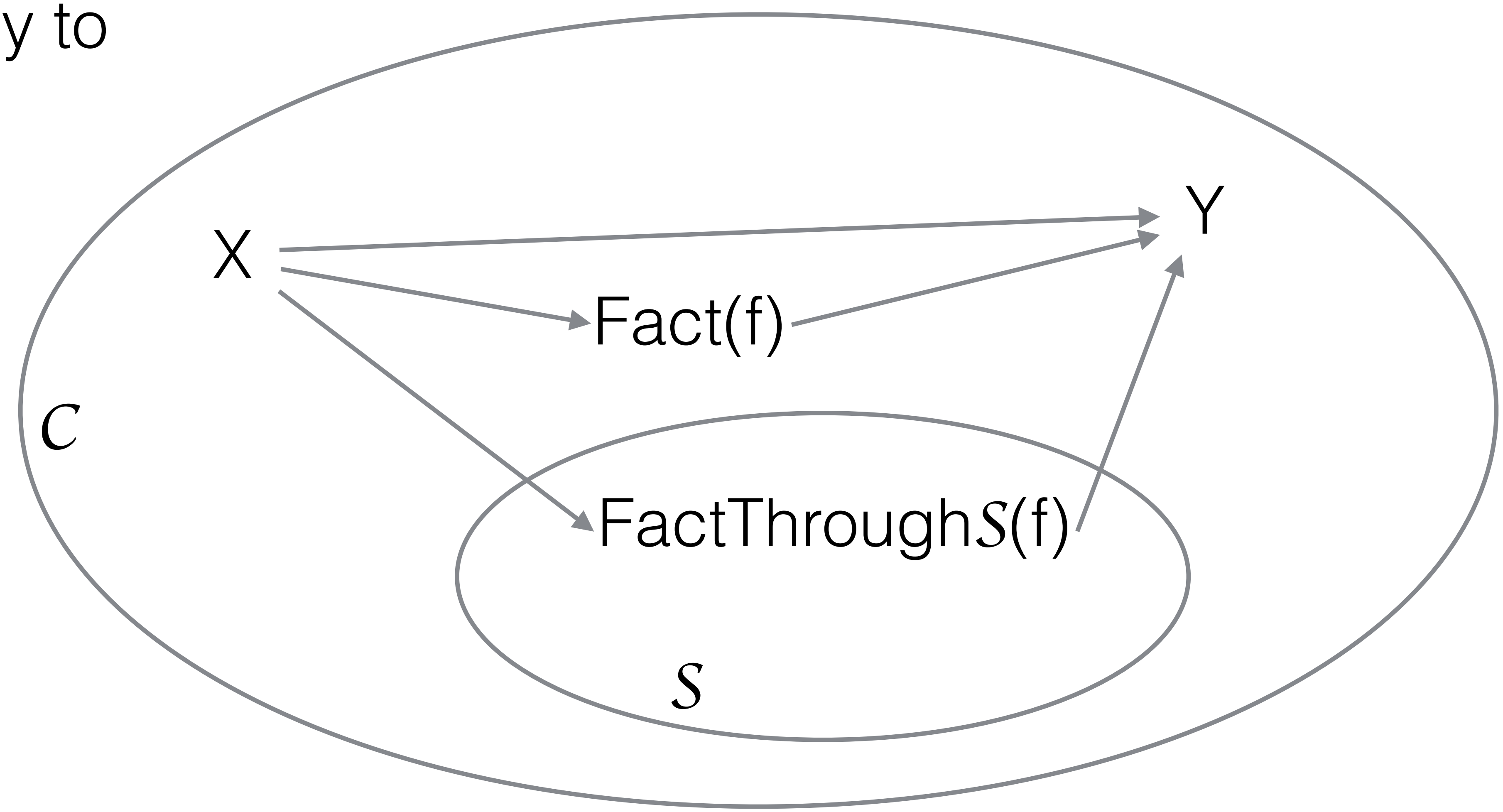

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(note that this distinction is not necessary for Set or Vec)

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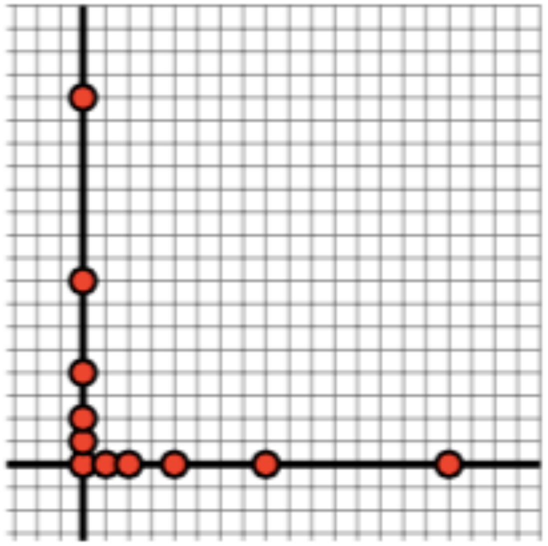

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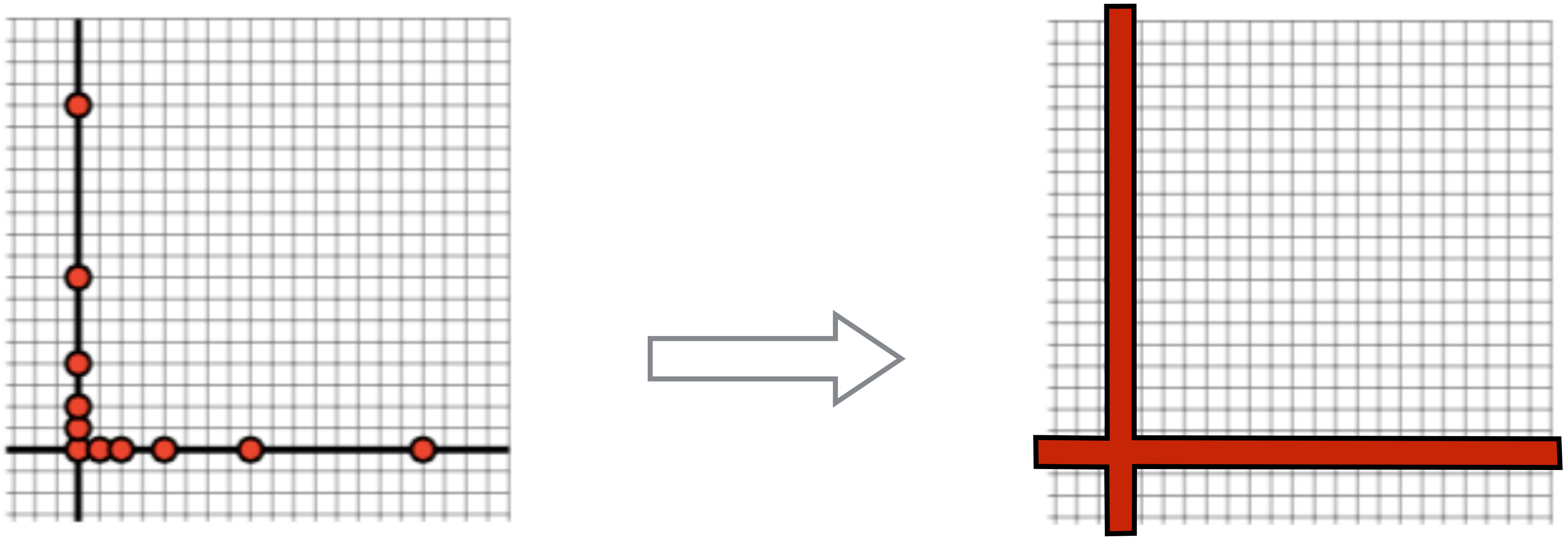

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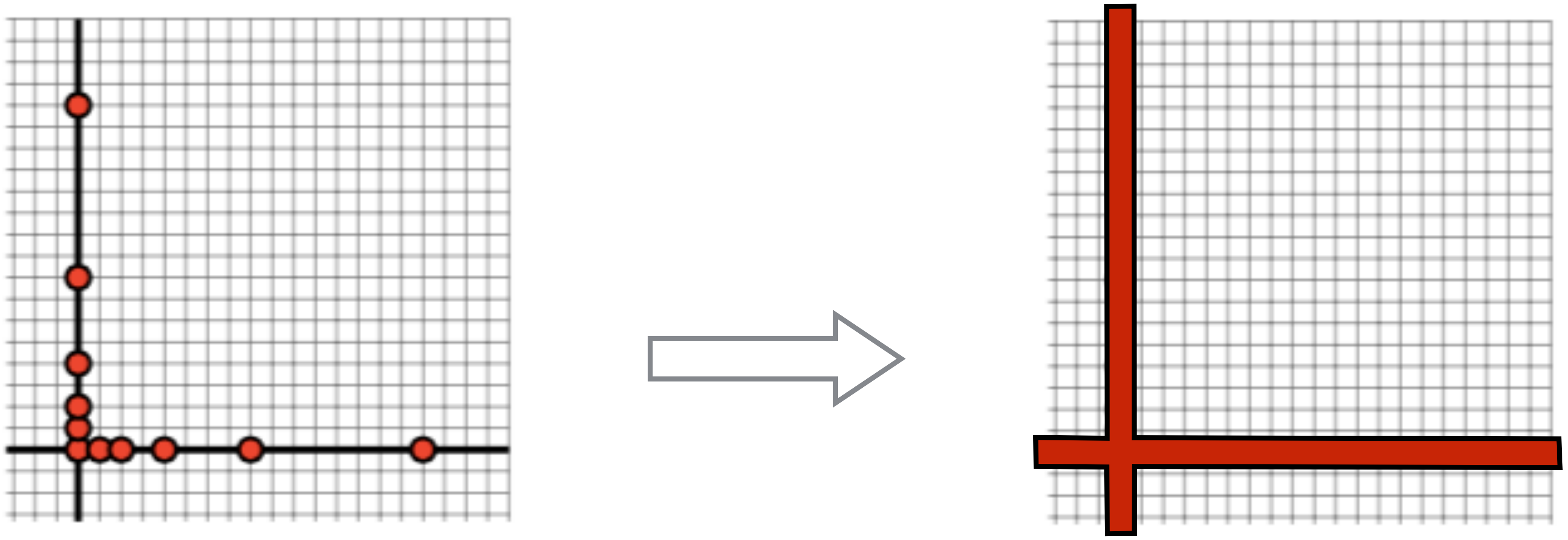

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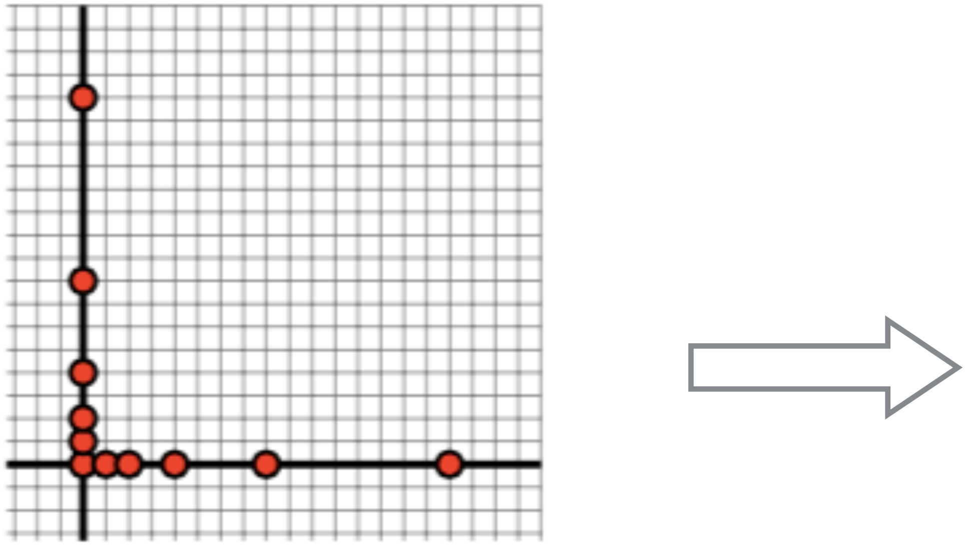

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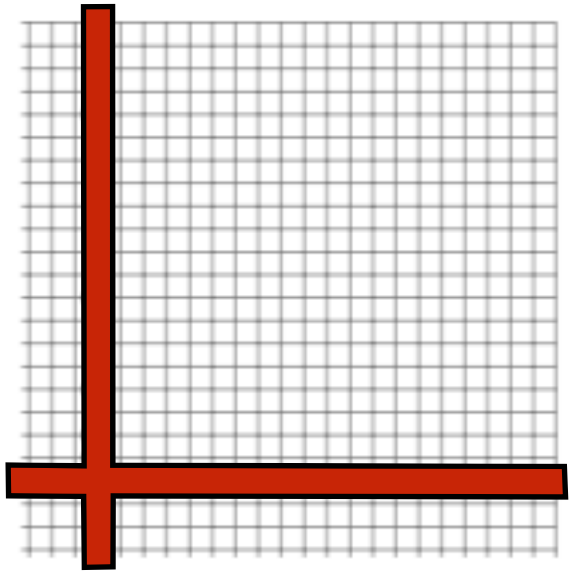

Subspace that can be
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Conclusion

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- new categorical concepts on the way,
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And then ?

- Make this construction effective… (generalization of sequencialization)
- tree automata
- algebras (monoids,…)
- infinite objects (omega-semigroup, o-semigroup, monads…).